Trace Formulas of Higher Order

A Thesis

Submitted for the Degree of

Doctor of Philosophy

in the Faculty of Science

by

Arup Chattopadhyay

Theoretical Sciences Unit
Jawaharlal Nehru Centre for Advanced Scientific Research
(A Deemed University)
Bangalore – 560 064 , India.

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Dedicated to my grand mother

Yogmaya Ghosal (budi)

Declaration

I hereby declare that the work reported in this thesis entitled "**Trace Formulas of Higher Order** " is the result of investigations carried out by me at the Theoretical Sciences Unit, Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore, India under the supervision of Prof.K.B.Sinha and that this work has not been submitted elsewhere for the award of any other degree.

In keeping with the general practice in reporting scientific observations, due acknowledgement has been made whenever the work described is based on the findings of other investigators. Any omission, which might have occurred by oversight or misjudgment, is regretted.

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September 2012.

Certificate

I hereby certify that the matter embodied in this thesis entitled "**Trace Formulas of Higher Order** " has been carried out by Mr.Arup Chattopadhyay at the Theretical Sciences Unit, Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore, India under my supervision and it has not been submitted elsewhere for the award of any degree or diploma.

Prof.Kalyan B. Sinha
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Chapter 1

Introduction

In this chapter we give some basic definitions and facts and refer to ([18], [19], [1], [10]) for the details. Throughout this thesis, $\mathcal{H}$ will denote the separable Hilbert space over $\mathbb{C}$ unless otherwise mentioned.

1.1 Schatten p-ideals

Let A be an operator in a Hilbert space $\mathcal{H}$ with Dom(A) is the domain and Ran(A) is the range of the operator A. An operator A is said to be densely defined if Dom(A) is dense in $\mathcal{H}$. A densely defined operator A is said to be self-adjoint if $A = A^*$ (where A^* is the adjoint of the operator A). The set of all bounded and everywhere defined operators in $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$ is a Banach space with respect to operator norm $\|.\|$.

Definition 1.1.1. An operator $A \in \mathcal{B}(\mathcal{H})$ is called finite rank if the dimension of Ran(A) is finite (and hence closed). Then there are orthonormal vectors $\{h_1, h_2, ..., h_N\}$ in $\mathcal{H}$ with $N < \infty$ (spanning Ran(A)) and a orthogonal family $\{e_1, e_2, ..., e_N\}$ such that

$$Af = \sum_{i=1}^{N} \langle e_i, f\rangle h_i \quad \forall \;\; f \in \mathcal{H}.$$

The set of all finite rank operators is denoted by $\mathcal{B}_{00}(\mathcal{H})$.

Definition 1.1.2. A linear operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ is compact if $A(\{h : \|h\| \leq 1\})$ has compact closure in $\mathcal{H}$. The set of all compact operators is denoted by $\mathcal{B}_0(\mathcal{H})$.

The basic properties of $\mathcal{B}_0(\mathcal{H})$ are as follows:

(i) $\mathcal{B}_{00}(\mathcal{H}) \subseteq \mathcal{B}_0(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{B}_{00}(\mathcal{H})$ is dense in $\mathcal{B}_0(\mathcal{H})$ with respect to operator norm $\|.\|$.

(ii) $\mathcal{B}_0(\mathcal{H})$ is a *-ideal of $\mathcal{B}(\mathcal{H})$ i.e. if $S \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}_0(\mathcal{H})$, then AS and $SA \in \mathcal{B}_0(\mathcal{H})$ and if $A \in \mathcal{B}_0(\mathcal{H})$, then $A^* \in \mathcal{B}_0(\mathcal{H})$.

(iii) Let $A \in \mathcal{B}_0(\mathcal{H})$. Then there exist two orthonormal sets $\{e_k\}_{k=1}^{\infty}$ and $\{h_k\}_{k=1}^{\infty}$ such that for all $f \in \mathcal{H}$; $Af = \sum_{k=1}^{\infty} \lambda_k \langle e_k, f\rangle h_k$, where $\{\lambda_k\}_{k=1}^{\infty}$ are the singular values of A i.e. $\{\lambda_k\}_{k=1}^{\infty}$ are eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$ [see ([18], [1]) for definition of $(.)^{\frac{1}{2}}$], each repeated as often as its multiplicity and the infinite sum (viewed as the limit of a sequence of operators) converges in operator norm.

Definition 1.1.3. Let $\mathcal{H}$ be Hilbert space, $\{\phi_k\}_{k=1}^{\infty}$ an orthonormal basis. Then for any positive operator $A \in \mathcal{B}(\mathcal{H})$ (i.e. $\langle Ah, h\rangle \geq 0 \ \forall \ h \in \mathcal{H}$) we define Tr$A = \sum_{k=1}^{\infty} \langle \phi_k, A\phi_k\rangle$. The number Tr$A$ is called the **trace of** A and is independent of the orthonormal basis chosen. The trace has the following properties:

(i) Tr$(A + B) =$ Tr$A +$ TrB.

(ii) Tr$(\lambda A) = \lambda$TrA for all $\lambda \geq 0$.

(iii) Tr$(UAU^{-1}) =$ TrA for any unitary operator U.

(iv) If $0 \leq A \leq B$, then Tr$A \leq$ TrB.

Definition 1.1.4. An operator $A \in \mathcal{B}(\mathcal{H})$ is called trace class if and only if Tr$|A| < \infty$ and $\|A\|_1 =$ Tr$|A|$. The family of all trace class operators is denoted by $\mathcal{B}_1(\mathcal{H})$.

The basic properties of $\mathcal{B}_1(\mathcal{H})$ are given in the following:

(i) $\mathcal{B}_1(\mathcal{H})$ is a Banach space with norm $\|.\|_1$ and is a *-ideal in $\mathcal{B}(\mathcal{H})$.

(ii) Every $A \in \mathcal{B}_1(\mathcal{H})$ is compact. A compact operator A is in $\mathcal{B}_1(\mathcal{H})$ if and only if $\sum_{k=1}^{\infty} \lambda_k < \infty$, where $\{\lambda_k\}_{k=1}^{\infty}$ are the singular values of A, each repeated as often as its multiplicity.

(iii) The finite rank operators are $\|.\|_1$- dense in $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}_1(\mathcal{H})$ is not closed under the operator norm $\|.\|$.

(iv) If $A \in \mathcal{B}_1(\mathcal{H})$ and $\{\phi_k\}_{k=1}^{\infty}$ is any orthonormal basis, then $\sum_{k=1}^{\infty} \langle \phi_k, A\phi_k \rangle$ converges absolutely and the sum is independent of choice of basis.

On the basis of property (iv), we can extend the definition of trace to any $A \in \mathcal{B}_1(\mathcal{H})$ by $\text{Tr}A = \sum_{k=1}^{\infty} \langle \phi_k, A\phi_k \rangle$, where $\{\phi_k\}_{k=1}^{\infty}$ is any orthonormal basis. The following are some basic properties of the trace:

(i) $\text{Tr}(.)$ is linear.

(ii) $\text{Tr}A^* = \overline{\text{Tr}A}$.

(iii) $\text{Tr}AB = \text{Tr}BA$ if $A \in \mathcal{B}_1(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$.

Definition 1.1.5. An operator $A \in \mathcal{B}(\mathcal{H})$ is called Hilbert-Schmidt if and only if $\text{Tr}A^*A = \text{Tr}|A|^2 < \infty$. The family of all Hilbert-Schmidt operators is denoted by $\mathcal{B}_2(\mathcal{H})$.

The basic properties of $\mathcal{B}_2(\mathcal{H})$ are as follows:

(i) $\mathcal{B}_2(\mathcal{H})$ is a *-ideal in $\mathcal{B}(\mathcal{H})$.

(ii) $\mathcal{B}_2(\mathcal{H})$ with inner product $\langle A, B \rangle_2 = \text{Tr}\,(A^*B)$ $(A, B \in \mathcal{B}_2(\mathcal{H}))$ is a Hilbert space, which we shall denote here as $\tilde{\mathcal{H}}$.

(iii) If $\|A\|_2 = \langle A, A \rangle_2^{\frac{1}{2}} = (\text{Tr}\,(A^*A))^{\frac{1}{2}}$, then $\|A\| \leq \|A\|_2 \leq \|A\|_1$ and $\|A\|_2 = \|A^*\|_2$.

(iv) Every $A \in \mathcal{B}_2(\mathcal{H})$ is compact and a compact operator A is in $\mathcal{B}_2(\mathcal{H})$ if and only if $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$, where $\{\lambda_k\}$ are the singular values of A, each repeated as often as its multiplicity.

(v) The finite rank operators are $\|.\|_2$-dense in $\mathcal{B}_2(\mathcal{H})$.

(vi) $A \in \mathcal{B}_2(\mathcal{H})$ if and only if $\sum_{k=1}^{\infty} \|A\phi_k\|^2 < \infty$ for some orthonormal basis $\{\phi_k\}_{k=1}^{\infty}$.

(vii) $A \in \mathcal{B}_1(\mathcal{H})$ if and only if $A = BC$ with $B, C \in \mathcal{B}_2(\mathcal{H})$, and in such a case $\text{Tr}\,BC = \text{Tr}\,CB = \text{Tr}A$.

Definition 1.1.6. (The Schatten p-ideals) For $1 \leq p < \infty$, we define $\mathcal{B}_p(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : |A|^p \in \mathcal{B}_1(\mathcal{H})\}$ and set $\|A\|_p = (\text{Tr}|A|^p)^{\frac{1}{p}} = \left(\sum_{k=1}^{\infty} \lambda_k^p\right)^{\frac{1}{p}}$, where $\{\lambda_k\}$ are the singular values of

A, each repeated as often as its multiplicity. These spaces can be looked upon as non-commutative analogues of the Lebesgue space L^p (or ℓ^p). Here are some basic properties of the Schatten p-ideals:

(i) Let $1 \leq p < \infty$. Then $\mathcal{B}_p(\mathcal{H})$ is a Banach space with norm $\|.\|_p$ and is a *-ideal in $\mathcal{B}(\mathcal{H})$ with $\|A^*\|_p = \|A\|_p$; $A \in \mathcal{B}_p(\mathcal{H})$.

(ii) Let $1 \leq p, q, r < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $A \in \mathcal{B}_p(\mathcal{H})$ and $B \in \mathcal{B}_q(\mathcal{H})$, then $AB \in \mathcal{B}_r(\mathcal{H})$ and $\|AB\|_r \leq \|A\|_p \|B\|_q$.

(iii) Let $1 \leq p < \infty$ .Then $\mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}_p(\mathcal{H}) \subset \mathcal{B}_0(\mathcal{H})$ and $\mathcal{B}_p(\mathcal{H})$ is the closure of the finite rank operators in the norm $\|.\|_p$.

(iv) Let $1 \leq q \leq p < \infty$ and $A \in \mathcal{B}_q(\mathcal{H})$. Then $A \in \mathcal{B}_p(\mathcal{H})$ and $\|A\|_p \leq \|A\|^{(1-\frac{q}{p})} \|A\|_q^{\frac{q}{p}}$.

1.2 Spectral theorem

Definition 1.2.1. A orthogonal projections on $\mathcal{H}$-valued function E on $\mathbb{R}$ is called a spectral family if it satisfies the following properties:

(i) $E(\lambda)$ is non-decreasing i.e. $E(\lambda)E(\mu) = E(\min\{\lambda, \mu\})$.

(ii) $s - \lim\limits_{\lambda \to -\infty} E(\lambda) = 0$ and $s - \lim\limits_{\lambda \to \infty} E(\lambda) = I$.

A spectral family $E(\lambda)$ is said to be right continuous if $E(\lambda) = E(\lambda+0) = s - \lim\limits_{\eta \to 0^+} E(\lambda+\eta)$. In this thesis we shall require additionally the right continuous spectral family.

Definition 1.2.2. Let $\mathcal{H}$ be a given Hilbert space and Ω be a σ-algebra of subsets of a set X. Then a spectral measure for $(X, \Omega, \mathcal{H})$ is a function $E : \Omega \longrightarrow \mathcal{B}(\mathcal{H})$ such that

(i) for each $\Delta \in \Omega$, $E(\Delta)$ is an orthogonal projection,

(ii) $E(\emptyset) = 0$ and $E(X) = I$,

(iii) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for $\Delta_1,\ \Delta_2 \in \Omega$,

(iv) if $\{\Delta_k\}_{k=1}^{\infty}$ are pairwise disjoint sets in Ω, then $E\left(\bigcup\limits_{k=1}^{\infty} \Delta_k\right) = \sum\limits_{k=1}^{\infty} E(\Delta_k)$, where the infinite sum converges strongly.

Given a spectral family you can always construct a spectral measure on Borel subsets of $\mathbb{R}$ by defining on basic Borel sets and then extending it to arbitrary Borel sets by standard methods (for details see sec.5-2, [1]).

Let A be a self-adjoint operator in $\mathcal{H}$. Then there exists a unique spectral family $\{E_A(\lambda)\}$ such that for $f \in \text{Dom}(A)$ i.e. for $f \in \mathcal{H}$ such that $\int\limits_{-\infty}^{\infty} \lambda^2 \langle f, E_A(d\lambda)f\rangle < \infty$ and for any $g \in \mathcal{H}$, $\langle g, Af\rangle = \int\limits_{-\infty}^{\infty} \lambda \langle g, E_A(d\lambda)f\rangle$. Also each $E_A(\lambda)$ commutes with all bounded operators that commute with A.

Given a self-adjoint operator A in $\mathcal{H}$, the resolvent set $\rho(A)$ [the set of all $z \in \mathbb{C}$ such that $(A - zI)^{-1} \in \mathcal{B}(\mathcal{H})$] of A contains all $z \in \mathbb{C}$ such that $\text{Im}z \neq 0$; in fact for such z, $\left\|(A - zI)^{-1}\right\| \leq |\text{Im}z|^{-1}$. Thus $\sigma(A)$, the spectrum of A, defined by $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is a subset of $\mathbb{R}$ and in fact, is exactly the support of the spectral family E_A.

1.3 Double operator integrals on $\mathcal{B}_2(\mathcal{H})$

Double operator integrals theory was developed by Birman and Solomyak in a series of publications ([3], [4], [5], [6], [8]). In this section, we will give some brief idea of double operator integrals on $\mathcal{B}_2(\mathcal{H})$.

Let A and B be two (possibly unbounded) self-adjoint operators on a Hilbert space $\mathcal{H}$ with spectral family (measure) E_A and E_B respectively. Let $\mathcal{G}$ be the set function on $\mathbb{R}^2$ defined as follows:

$$\mathcal{G}(\delta \times \Delta) : X \longrightarrow E_A(\delta)XE_B(\Delta) \quad \text{for} \quad \delta, \Delta \in \text{Borel}(\mathbb{R});\ X \in \mathcal{B}_2(\mathcal{H}).$$

Then the values of $\mathcal{G}$ on the set of all Borel measurable rectangles $\delta \times \Delta \subset \mathbb{R}^2$ are orthogonal projections on the Hilbert space $\tilde{\mathcal{H}}$ and hence $\mathcal{G}$ can be extended to a spectral measure in the Hilbert space $\tilde{\mathcal{H}}$ on the Borel sets of $\mathbb{R}^2$. Thus integrals can be defined as a weak Riemann (or Lebesgue) integral with respect to this spectral measure. For example, if ϕ is a bounded measurable function on $\mathbb{R}^2$,

$$\mathcal{A}_{\mathcal{G}}(\cdot) \equiv \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(\lambda,\mu)\, E_A(d\lambda)(\cdot)E_B(d\mu) \text{ exists in } \mathcal{B}\left(\mathcal{B}_2(\mathcal{H})\right) \text{ and } \|\mathcal{A}_{\mathcal{G}}(X)\| \leq \|\phi\|_\infty \text{Var}\left(\mathcal{G}(.)X\right),$$

where

$$\text{Var}\left(\mathcal{G}(.)X\right) = \sup_{\left[\{\delta_i\times\Delta_j\}_{i,j=1}^n \;;\; \|Y\|_2\leq 1 \,:\, \{\delta_i\times\Delta_j\}_{i,j=1}^n \text{is a partition of } \mathbb{R}^2\right]} \sum_{i,j=1}^{n} |\langle Y, \mathcal{G}(\delta_i\times\Delta_j)X\rangle_2| \leq \|X\|_2.$$

Next for any $X, Y \in \mathcal{B}_2(\mathcal{H})$, we have

$$\left\langle Y, \left(\int_{\mathbb{R}^2} \phi\, \mathcal{G}(d\lambda\times d\mu)\right) X\right\rangle_2 = \int_{\mathbb{R}^2} \phi(\lambda,\mu)\, \langle Y, \mathcal{G}(d\lambda\times d\mu)X\rangle_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(\lambda,\mu)\, \langle Y, E_A(d\lambda)XE_B(d\mu)\rangle_2 .$$

i.e.

$$\text{Tr}\{Y^* \left(\int_{\mathbb{R}^2} \phi(\lambda,\mu)\, \mathcal{G}(d\lambda\times d\mu)\right) X\} = \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(\lambda,\mu)\text{Tr}\{Y^* E_A(d\lambda)XE_B(d\mu)\}.$$

Moreover, we have

$$\left\|\left(\int_{\mathbb{R}^2} \phi(\lambda,\mu)\, \mathcal{G}(d\lambda\times d\mu)\right) X\right\|_2^2 = \left\|\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(\lambda,\mu)\, E_A(d\lambda)XE_B(d\mu)\right\|_2^2$$
$$= \int_{\mathbb{R}^2} |\phi(\lambda,\mu)|^2 \langle X, \mathcal{G}(d\lambda\times d\mu)X\rangle_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} |\phi(\lambda,\mu)|^2 \text{Tr}\{X^* E_A(d\lambda)XE_B(d\mu)\} \leq \|\phi\|_\infty \|X\|_2^2. \quad (1.3.1)$$

Theorem 1.3.1. *Let A be a self-adjoint operator in $\mathcal{H}$ and V be a self-adjoint operator such that $V \in \mathcal{B}_2(\mathcal{H})$. Let ϕ be a Lipschitz function on $\mathbb{R}$ i.e. $|\phi(\lambda)-\phi(\mu)| \leq c(\phi)|\lambda-\mu| \quad \forall \quad \lambda,\ \mu \in \mathbb{R}$ (the least $c(\phi)$ satisfying the inequality is called the Lipschitz norm $\|\phi\|_{\text{Lip}}$ of ϕ). Then $\phi(A+V)-\phi(A) \in \mathcal{B}_2(\mathcal{H})$ and*

$$\phi(A+V)-\phi(A) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\phi(\lambda)-\phi(\mu)}{\lambda-\mu} E_{A+V}(d\lambda)VE_A(d\mu),$$

where $E_{A+V}(\lambda)$ and $E_A(\mu)$ are the spectral families (measures) of the self-adjoint operators $(A+V)$ and A respectively. Moreover,

$$\|\phi(A+V)-\phi(A)\|_2 \;\leq\; \sup_{\lambda\neq\mu}\left|\frac{\phi(\lambda)-\phi(\mu)}{\lambda-\mu}\right| \|V\|_2 = \|\phi\|_{\text{Lip}}\|V\|_2.$$

Proof. For the proof see ([5], [6], [8]). □

1.4 Krein's spectral shift function and associated trace formula

In this section we will discuss briefly about krein's theorem on the spectral shift function and the trace formula for a pair of self-adjoint operators (H, H_0).

The original proof of Krein ([15], [22]) uses properties of perturbation determinants and the integral representation of holomorphic functions on the upper half plane with a bounded imaginary part. In 1985, Voiculescu [24] approached the trace formula from a different direction and gave an alternative proof without using function theory for the case of bounded self-adjoint operators. Later Sinha and Mohapatra [22] extended Voiculescu's method to the unbounded self-adjoint and unitary cases [23]. There is also the interesting approach of Birman and Solomyak [7] using the theory of double operator integrals. Let us begin by stating the theorem in finite dimensional Hilbert space.

Theorem 1.4.1. *(Theorem 1.1. [22]) Let H and H_0 be two self-adjoint operators in a finite dimensional Hilbert space $\mathcal{H}$ with $E_H(\lambda)$ and $E_{H_0}(\lambda)$ are the spectral families of H and H_0 respectively. Then there exists a unique real-valued bounded function ξ such that*

(i) $\xi(\lambda) = \mathrm{Tr}\{E_{H_0}(\lambda) - E_H(\lambda)\}, \quad \lambda \in \mathbb{R},$

(ii) $\int_{\mathbb{R}} \xi(\lambda) d\lambda = \mathrm{Tr}\,(H - H_0)\,,$

(iii) for $\phi \in C^1(\mathbb{R})$ (set of all once continuously differentiable functions on $\mathbb{R}$),

$$\mathrm{Tr}\,[\phi(H) - \phi(H_0)] = \int_{\mathbb{R}} \phi'(\lambda)\xi(\lambda)d\lambda. \tag{1.4.1}$$

Furthermore, ξ is a constant in every real open interval in $\rho(H) \cap \rho(H_0)$ and has support in $[a, b]$, where $a = \min\{\inf \sigma(H), \inf \sigma(H_0)\}$, $b = \max\{\sup \sigma(H), \sup \sigma(H_0)\}$.

(iv) If $H - H_0 = \tau|g\rangle\langle g|$ with $\tau > 0$, $\|g\| = 1$ (we have used Dirac notation for rank one perturbations), then ξ is a $\{0, 1\}$ -valued function. More precisely, $\xi(\lambda) = \sum_{j=1}^{r} \chi_{\Delta_j}(\lambda)$ for r disjoint intervals $\Delta_j \subset \mathbb{R}$, $1 \leq r \leq n$.

Remark 1.4.2. The function ξ as in Theorem 1.4.1 is known as Krein's spectral shift function and the formula (1.4.1) is called Krein's trace formula in finite dimension. But in an infinite dimensional Hilbert space, the relation $\xi(\lambda) = \mathrm{Tr}\{E_{H_0}(\lambda) - E_H(\lambda)\}$ will not make sense in general because $E_{H_0}(\lambda) - E_H(\lambda)$ may not be trace class (for details see [22]), even if $H - H_0$ is rank one as in (iv).

Next we shall define Krein's spectral shift function and associated trace formula for a pair of self-adjoint operators in an infinite dimensional Hilbert space $\mathcal{H}$ such that their difference is trace-class. In fact, we will reproduce the proof of Krein's theorem given by Voiculescu [24] for bounded self-adjoint case as well as for unbounded self-adjoint case given by Sinha and Mohapatra [22]. We begin with a theorem which is adapted from Weyl-von Neumann theorem [13].

Theorem 1.4.3. *(Adaptation of Weyl-von Neumann theorem, [22]) Let A be a self-adjoint operator in $\mathcal{H}$, $f \in \mathcal{H}$ and $\epsilon > 0$, K a compact set in $\mathbb{R}$. Then there exist a finite rank projection P in $\mathcal{H}$ such that*

(i) $\|(I-P)AP\|_2 < \epsilon$ *and* $\left\|(I-P)\mathrm{e}^{\mathrm{i}tA}P\right\|_2 < \epsilon$ *uniformly for* $t \in K$.

(ii) $\|(I-P)f\| < \epsilon$.

Proof. Let E_A be the spectral measure associated with the self-adjoint operator A, and choose $a > 0$ such that $\|[I - E_A((-a,a])]f\| < \epsilon$. For each positive integer n and $1 \leq k \leq n$, set $E_k = E_A\left(\left(\frac{2k-2-n}{n}a, \frac{2k-n}{n}a\right]\right)$ so that

$$E_k E_j = \delta_{kj} E_j \quad \text{and} \quad \sum_{k=1}^{n} E_k = E_A\left((-a,a]\right).$$

We also set for $1 \leq k \leq n$,

$$g_k = \begin{cases} \frac{E_k f}{\|E_k f\|}, & \text{if } E_k f \neq 0. \\ 0, & \text{if } E_k f = 0. \end{cases}$$

Now

$$\int_{-\infty}^{\infty} \lambda^2 \|E_A(d\lambda) g_k\|^2 = \int_{\frac{2k-n-2}{n}a}^{\frac{2k-n}{n}a} \|E_A(d\lambda) g_k\|^2 < \infty,$$

and hence $\{g_k\}_{k=1}^{n} \subseteq \mathrm{Dom}(A)$. Again

$$Ag_k = A\left(\frac{E_k f}{\|E_k f\|}\right) = \frac{1}{\|E_k f\|} A(E_k f) = \frac{E_k A(E_k f)}{\|E_k f\|} = E_k(Ag_k) \in E_k\mathcal{H},$$

since E_k commutes with A. Let P be the orthogonal projection onto the subspace generated by $\{g_1, g_2, g_3, \ldots\ldots, g_n\}$ so that $\dim P\mathcal{H} \leq n$. Set $\lambda_k = \frac{2k-n-1}{n}a$, then

$$\|\left(A-\lambda_k\right) g_k\|^2 = \int_{\frac{2k-n-2}{n}a}^{\frac{2k-n}{n}a} (\lambda - \lambda_k)^2 \|E_A(d\lambda) g_k\|^2 \leq \left(\frac{a}{n}\right)^2.$$

Next $PAg_k \in P\mathcal{H}$ and hence

$$PAg_k = \sum_{j=1}^{n} \langle PAg_k, g_j\rangle g_j = \sum_{j=1}^{n} \langle Ag_k, g_j\rangle g_j = \langle Ag_k, g_k\rangle g_k \in E_k\mathcal{H},$$

since $Ag_k \in E_k\mathcal{H}$ and $g_j \in E_j\mathcal{H}$. Therefore, $(I-P)Ag_k = Ag_k - PAg_k \in E_k\mathcal{H}$. So for any $u \in \mathcal{H}$,

$$Pu = \sum_{j=1}^{n} \langle Pu, g_j\rangle g_j = \sum_{j=1}^{n} \langle u, g_j\rangle g_j \quad \text{and hence}$$

$$(I-P)APu = \sum_{j=1}^{n} \langle u, g_j\rangle (I-P)Ag_j = \sum_{j=1}^{\infty} \langle u, g_j\rangle (I-P)(A-\lambda_j)g_j. \quad \text{Therefore}$$

$$\|(I-P)APu\|^2 = \sum_{j=1}^{\infty} |\langle u, g_j\rangle|^2 \|(I-P)(A-\lambda_j)g_j\|^2 \leq \left(\frac{a}{n}\right)^2 \|u\|^2.$$

i.e. $\|(I-P)AP\| \leq \left(\frac{a}{n}\right)$ and hence $\|(I-P)AP\|_2 \leq \sqrt{n}\ \|(I-P)AP\| \leq \left(\frac{a}{\sqrt{n}}\right)$. Thus Thus again by the same calculation as in page 831 of [22] , it follows that

$$\alpha(t) \equiv \left\|(I-P)\mathrm{e}^{\mathrm{i}tA}P\right\|_2 = \left\|(I-P)\left(\mathrm{e}^{\mathrm{i}tA} - I\right)P\right\|_2 = \left\|(I-P)\int_0^t ds \frac{d}{ds}\left(\mathrm{e}^{\mathrm{i}sA}\right)P\right\|_2$$

$$= \left\|(I-P)\int_0^t ds\ \mathrm{e}^{\mathrm{i}sA}\ \mathrm{i}AP\right\|_2 \leq \int_0^t \{\left\|(I-P)\mathrm{e}^{\mathrm{i}sA}P\right\|_2 \|AP\| + \|(I-P)\mathrm{e}^{\mathrm{i}sA}(I-P)\| \|(I-P)AP\|_2\} ds$$

$$\leq 2\ a \int_0^t \alpha(s)ds + T\ \frac{a}{\sqrt{n}} \quad \text{for} \quad |t| \leq T, \tag{1.4.2}$$

solving this Gronwall-type inequality (1.4.2) leads to

$$\alpha(t) \equiv \left\|(I-P)\mathrm{e}^{\mathrm{i}tA}P\right\|_2 \ \leq \frac{(T\ a\ \mathrm{e}^{2at})}{\sqrt{n}} \ \leq \frac{\left(T\ a\ \mathrm{e}^{2aT}\right)}{\sqrt{n}}.$$

On the other hand $E_A((-a,a])f = (I-P)\sum_{k=1}^{n} E_k f = \sum_{k=1}^{n} \|E_k f\|(I-P)g_k = 0$, so that

$$\|(I-P)f\| = \|(I-P)\left[I - E_A\left((-a,a]\right)\right]f\| \ \leq \|\left[I - E_A\left((-a,a]\right)\right]f\| < \epsilon.$$

The result follows by choosing n sufficiently large. □

Lemma 1.4.4. *([24]) Let H and H_0 be two bounded self-adjoint operators in an infinite dimensional Hilbert space $\mathcal{H}$ such that $H - H_0 = \tau|g\rangle\langle g|$; $\tau \geq 0$ and $\|g\| = 1$. Then there is $\xi_{(H,H_0)} \in$*

$L^\infty(\mathbb{R})$ with $0 \leq \xi_{(H,H_0)} \leq 1$ and $\text{supp}\xi_{(H,H_0)} \subset [a,b]$, where $a = \inf\sigma(H_0)$; $\sup\sigma(H_0)+\tau$, such that

$$\text{Tr}\{p(H)-p(H_0)\} = \int_a^b p'(\lambda)\xi_{(H,H_0)}d\lambda, \tag{1.4.3}$$

for every polynomial $p(.)$ in $[a,b]$. Moreover $\left\|\xi_{(H,H_0)}\right\|_{L^1(\mathbb{R})} = \|\tau|g\rangle\langle g|\|_1 = \tau$ and $\tau = \text{Tr}(\tau|g\rangle\langle g|) = \int_{-\infty}^{\infty}\xi_{(H,H_0)}(\lambda)d\lambda$.

Proof. It will be sufficient to prove the lemma for $p(\lambda) = \lambda^r$. Note that for $r=0$, both sides of (1.4.3) is identically zero. Therefore without loss of generality we can assume that $r \geq 1$.

Applying Theorem 1.4.1 with $A = H_0$ and $f = g$, we get a sequence $\{P_n\}$ of finite rank projections such that $\left\|P_n^\perp H_0 P_n\right\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$ and $\|(I-P_n)g\| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence

$$\left\|P_n^\perp H P_n\right\|_2 \leq \left\|P_n^\perp H_0 P_n\right\|_2 + \tau\left\||P_n^\perp g\rangle\langle P_n g|\right\|_2 \leq \left\|P_n^\perp H_0 P_n\right\|_2 + \tau\left\|P_n^\perp g\right\|\|g\|,$$

which converges to 0 as $n \longrightarrow \infty$. Since $P_n^\perp H_0^k P_n = P_n^\perp H_0^{k-1}P_n^\perp H_0 P_n + P_n^\perp H_0^{k-1}P_n H_0 P_n$, then by mathematical induction we conclude that $\left\|P_n^\perp H_0^k P_n\right\|_2 \longrightarrow 0$ as $n \longrightarrow \infty \quad \forall\ k \in \mathbb{N}$. Similarly, $\left\|P_n^\perp H^k P_n\right\|_2 \longrightarrow 0$ as $n \longrightarrow \infty \quad \forall\ k \in \mathbb{N}$. Next consider the expression

$$\begin{aligned}
&\text{Tr}\left\{[H^r - H_0^r]\right\} - \text{Tr}\left\{P_n\left[(P_nHP_n)^r - (P_nH_0P_n)^r\right]P_n\right\}\\
&\quad = \text{Tr}\left\{[H^r - H_0^r] - P_n\left[(P_nHP_n)^r - (P_nH_0P_n)^r\right]P_n\right\}\\
&\quad = \text{Tr}\left\{\sum_{k=0}^{r-1} H^{r-k-1}\,\tau|g\rangle\langle g|\,H_0^k - \sum_{k=0}^{r-1} P_n\left[(P_nHP_n)^{r-k-1}\,P_n\tau|g\rangle\langle g|P_n\,(P_nH_0P_n)^k\right]P_n\right\}\\
&\quad = \tau\sum_{k=0}^{r-1}\text{Tr}\left\{H^{r-k-1}\,|g\rangle\langle g|\,H_0^k - P_n\left[(P_nHP_n)^{r-k-1}\,|P_ng\rangle\langle P_ng|\,(P_nH_0P_n)^k\right]P_n\right\}\\
&\quad = \tau\sum_{k=0}^{r-1}\text{Tr}\left\{\left[H^{r-k-1}P_n - (P_nHP_n)^{r-k-1}\right]P_n\,|g\rangle\langle g|\,H_0^k + H^{r-k-1}P_n^\perp\ |g\rangle\langle g|\,H_0^k\right.\\
&\qquad\qquad \left. + (P_nHP_n)^{r-k-1}P_n\,|g\rangle\langle g|\,P_n^\perp H_0^k + (P_nHP_n)^{r-k-1}P_n\,|g\rangle\langle g|\,P_n\left[P_nH_0^k - (P_nH_0P_n)^k\right]\right\}.
\end{aligned} \tag{1.4.4}$$

In the first term of the expression (1.4.4) :

$$\begin{aligned}
&\left\|\left[H^{r-k-1} - (P_nHP_n)^{r-k-1}\right]P_n\right\|_2 = \left\|\sum_{j=0}^{r-k-2} H^{r-k-j-2}[H - P_nHP_n](P_nHP_n)^jP_n\right\|_2\\
&\quad = \left\|\sum_{j=0}^{r-k-2} H^{r-k-j-2}P_n^\perp HP_n(P_nHP_n)^jP_n\right\|_2 \leq (r-k-1)\|H\|^{r-k-2}\left\|P_n^\perp HP_n\right\|_2\\
&\quad \leq r(1+\|H\|)^r\left\|P_n^\perp HP_n\right\|_2,
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ and hence

$$\begin{aligned}
&\left\| \left[H^{r-k-1}P_n - (P_nHP_n)^{r-k-1}\right] P_n \; |g\rangle\langle g| \; H_0^k \right\|_1 \\
&\qquad \leq \left\| \left[H^{r-k-1}P_n - (P_nHP_n)^{r-k-1}\right] P_n \right\|_2 \; \left\| |g\rangle\langle g| \; H_0^k \right\|_2 \\
&\qquad \leq r(1+\|H\|)^r \|H_0\|^k \|g\|^2 \left\| P_n^{\perp} H P_n \right\|_2 ,
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$. Similarly for the fourth term in (1.4.4), we note that

$$\begin{aligned}
\left\| P_n \left[H_0^k - (P_nH_0P_n)^k\right] \right\|_2 &\leq \left\| \sum_{j=0}^{k-1} P_n H_0^{k-j-1} P_n^{\perp} H_0 P_n (P_nH_0P_n)^j \right\|_2 \\
&\leq k\|H_0\|^{k-1} \left\| P_n^{\perp} H_0 P_n \right\|_2 \leq k(1+\|H_0\|)^k \left\| P_n^{\perp} H_0 P_n \right\|_2 ,
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ and hence

$$\begin{aligned}
&\left\| (P_nHP_n)^{r-k-1} P_n \; |g\rangle\langle g| \; P_n \left[P_nH_0^k - (P_nH_0P_n)^k\right] \right\|_1 \\
&\qquad \leq \left\| (P_nHP_n)^{r-k-1} P_n \; |g\rangle\langle g| \right\|_2 \left\| P_n \left[P_nH_0^k - (P_nH_0P_n)^k\right] \right\|_2 \\
&\qquad \leq k(1+\|H_0\|)^k \|H\|^{r-k-1} \|g\|^2 \left\| P_n^{\perp} H_0 P_n \right\|_2 ,
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$. For the second term in (1.4.4), we have

$$\begin{aligned}
&\left\| H^{r-k-1} P_n^{\perp} \;\; |g\rangle\langle g| \; H_0^k \right\|_1 = \left\| H^{r-k-1} \; |P_n^{\perp} g\rangle\langle g| \; H_0^k \right\|_1 \\
&\qquad \leq \|H\|^{r-k-1} \|H_0\|^k \left\| |P_n^{\perp} g\rangle\langle g| \right\|_1 \leq \|H\|^{r-k-1} \|H_0\|^k \|(I-P_n)g\| \|g\|,
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ and for the third term in (1.4.4), we have the following estimate

$$\begin{aligned}
&\left\| (P_nHP_n)^{r-k-1} P_n \; |g\rangle\langle g| \; P_n^{\perp} H_0^k \right\|_1 = \left\| (P_nHP_n)^{r-k-1} \; |P_n g\rangle\langle P_n^{\perp} g| \; H_0^k \right\|_1 \\
&\qquad \leq \|H\|^{r-k-1} \|H_0\|^k \left\| |P_n g\rangle\langle P_n^{\perp} g| \right\|_1 \leq \|H\|^{r-k-1} \|H_0\|^k \|P_n g\| \|(I-P_n)g\|,
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$. Therefore the right hand side of (1.4.4) converges to 0 as $n \longrightarrow \infty$ and hence we have proved that

$$\text{Tr}\left\{H^r - H_0^r\right\} = \lim_{n \longrightarrow \infty} \text{Tr}\left\{P_n \left[(P_nHP_n)^r - (P_nH_0P_n)^r\right] P_n\right\}. \tag{1.4.5}$$

But on the other hand

$$\text{Tr}\left\{P_n \left[(P_nHP_n)^r - (P_nH_0P_n)^r\right] P_n\right\} = \int_a^b r\lambda^{r-1} \xi_n(\lambda) \, d\lambda,$$

by applying Theorem 1.4.1 to P_nHP_n and $P_nH_0P_n$ on the finite dimensional Hilbert space $P_n\mathcal{H}$ and $a = \inf \sigma(H_0)$; $b = \sup \sigma(H_0) + \tau$. Using part (iv) of Theorem 1.4.1, we conclude that

$0 \leq \xi_n \leq 1 \ \forall \ n \in \mathbb{N}$ and hence $\{\xi_n\}$ is in the unit ball of $L^\infty([a,b])$. But $L^\infty([a,b]) = L^1([a,b])^*$ and hence Banach-Alaoglu's theorem we conclude the unit ball of $L^1([a,b])^*$ is w*-compact. Also the unit ball of $L^1([a,b])^*$ is w*-metrizable, since $L^1([a,b])$ is separable (polynomials are dense in $L^1([a,b])$). Therefore compactness and sequential compactness are equivalent in the unit ball of $L^1([a,b])^*$ with respect to w*-topology. Hence there exist a subsequence $\{\xi_{n_k}\}$ and a function $\xi_{(H,H_0)} \in L^\infty([a,b])$ with $0 \leq \xi_{(H,H_0)} \leq 1$ such that $\xi_{n_k} \longrightarrow \xi_{(H,H_0)}$ in w*-sense i.e.

$$\lim_{k\longrightarrow\infty} \int_a^b f(\lambda)\xi_{n_k}(\lambda)\, d\lambda = \int_a^b f(\lambda)\xi_{(H,H_0)}(\lambda)\, d\lambda \quad \text{for all} \quad f \in L^1([a,b]).$$

Hence

$$\text{Tr}\,\{H^r - H_0^r\} = \lim_{k\longrightarrow\infty} \int_a^b r\lambda^{r-1}\xi_{n_k}(\lambda)\, d\lambda = \int_a^b r\lambda^{r-1}\xi_{(H,H_0)}(\lambda)\, d\lambda, \tag{1.4.6}$$

where $\xi_{(H,H_0)} \in L^\infty(\mathbb{R})$ with $0 \leq \xi_{(H,H_0)} \leq 1$ and $\text{supp}\xi_{(H,H_0)} \subseteq [a,b]$. In particular for $r = 1$, equation (1.4.6) gives

$$\int_a^b \xi_{(H,H_0)}(\lambda)\, d\lambda = \text{Tr}\,\{H - H_0\} = \tau \quad i.e. \quad \|\xi_{(H,H_0)}\|_{L^1(\mathbb{R})} = \tau = |\tau|.$$

□

Remark 1.4.5. Let H and H_0 be two bounded self-adjoint operators in an infinite dimensional Hilbert space $\mathcal{H}$ such that $H - H_0 = \tau|g\rangle\langle g|;\ \ \tau < 0$ and $\|g\| = 1$. Hence $H_0 - H = -\tau|g\rangle\langle g|;$ $-\tau > 0$ and therefore by applying Lemma 1.4.4 for the pair (H_0, H) we get that

$$\text{Tr}\,\{p(H_0) - p(H)\} = \int_a^b p'(\lambda)\xi_{(H_0,H)}(\lambda)\, d\lambda \quad \text{i.e.}$$

$$\text{Tr}\,\{p(H) - p(H_0)\} = \int_a^b p'(\lambda)\left[-\xi_{(H_0,H)}(\lambda)\right]\, d\lambda = \int_a^b p'(\lambda)\xi_{(H,H_0)}(\lambda)\, d\lambda,$$

for every polynomial $p(.)$; $\ \xi_{(H,H_0)}(\lambda) \equiv -\xi_{(H_0,H)}(\lambda);\ \ a = \inf\sigma(-H_0);\ \ b = \sup\sigma(-H_0) - \tau;$ $0 \leq \left|\xi_{(H,H_0)}\right| \leq 1;\ \int_{\mathbb{R}} \xi_{(H,H_0)}(\lambda)\, d\lambda = \tau$ and $\left\|\xi_{(H,H_0)}\right\|_{L^1} = \left\|\xi_{(H_0,H)}\right\|_{L^1} = |\tau| = -\tau.$

Next theorem is the M.G.Krein's trace formula for general trace class perturbation in bounded self-adjoint case.

Theorem 1.4.6. *([24]) Let H and H_0 be two bounded self-adjoint operators in an infinite dimensional Hilbert space $\mathcal{H}$ such that $H - H_0 \equiv V \in \mathcal{B}_1(\mathcal{H})$. Then there exists a unique real-valued function $\xi \in L^1(\mathbb{R})$ such that*

$$\text{Tr}\,\{p(H) - p(H_0)\} = \int_a^b p'(\lambda)\xi(\lambda)\, d\lambda, \tag{1.4.7}$$

for every polynomial $p(.)$, *where* $a = \inf \sigma(H_0) - \|V\|$; $\quad b = \sup \sigma(H_0) + \|V\|$ *and* $\int_{\mathbb{R}} \xi(\lambda)\, d\lambda = \mathrm{Tr}(V)$ *;* $\|\xi\|_{L^1} \leq \|V\|_1$.

Proof. It will be sufficient to prove the theorem for $p(\lambda) = \lambda^r$. Since $V \in \mathcal{B}_1(\mathcal{H})$, then

$$V = \sum_{j=1}^{\infty} \tau_j |g_j\rangle\langle g_j| \quad \text{with} \quad \sum_{j=1}^{\infty} |\tau_j| < \infty; \quad \|g_j\| = 1 \quad \text{for each} \quad j.$$

Set $V_k \equiv \sum\limits_{j=1}^{k} \tau_j |g_j\rangle\langle g_j|$ and $H_k \equiv H_0 + V_k$ for $k = 1, 2, 3, \ldots\ldots$ Then $\|V - V_k\|_1 = \sum\limits_{j=k+1}^{\infty} |\tau_j| \longrightarrow 0$ as $k \longrightarrow \infty$, since $\sum\limits_{j=1}^{\infty} |\tau_j| < \infty$ and

$$\begin{aligned} \|H^r - H_k^r\|_1 &= \left\| \sum_{l=0}^{r-1} H^{r-l-1}(H - H_k) H_k^l \right\|_1 = \left\| \sum_{l=0}^{r-1} H^{r-l-1}(V - V_k) H_k^l \right\|_1 \\ &\leq \sum_{l=0}^{r-1} \|H\|^{r-l-1} \|V - V_k\|_1 \|H_k\|^l \leq \|V - V_k\|_1 \left(\sum_{l=0}^{r-1} \|H\|^{r-l-1} (\|H_0\| + \|V\|)^l \right), \end{aligned}$$

which converges to 0 as $k \longrightarrow \infty$. But on the other hand $H_k^r - H_0^r = \sum_{m=1}^{k} \left(H_m^r - H_{m-1}^r \right)$ and hence

$$\mathrm{Tr}\ \{H_k^r - H_0^r\} = \sum_{m=1}^{k} \mathrm{Tr}\ \{H_m^r - H_{m-1}^r\} = \sum_{m=1}^{k} \int_a^b r \lambda^{r-1} \xi_{(H_m, H_{m-1})}(\lambda)\, d\lambda, \tag{1.4.8}$$

by applying Lemma 1.4.4 and Remark 1.4.5 accordingly for the pair (H_m, H_{m-1}) and

$$\int_{\mathbb{R}} \xi_{(H_m, H_{m-1})}(\lambda)\, d\lambda = \mathrm{Tr}\ (\tau_m |g_m\rangle\langle g_m|) = \tau_m; \quad \|\xi_{(H_m, H_{m-1})}\|_{L^1} = |\tau_m|;$$

$a = \inf \sigma(H_0) - \|V\|$; $\quad b = \sup \sigma(H_0) + \|V\|$; $\quad$ supp $\xi_{(H_m, H_{m-1})} \subseteq [a, b]\ \forall\ m \in \mathbb{N}$. Define

$$\xi(\lambda) = \sum_{k=1}^{\infty} \xi_{(H_m, H_{m-1})}(\lambda) \tag{1.4.9}$$

Set $S_k(\lambda) \equiv \sum\limits_{m=1}^{k} \xi_{(H_m, H_{m-1})}(\lambda)$, then $S_k \in L^1(\mathbb{R})$ for each $k \in \mathbb{N}$ and for $k', k \in \mathbb{N}$ such that $k' > k$, we have

$$\begin{aligned} \|S_k - S_{k'}\|_{L^1} &= \int_{\mathbb{R}} |S_k(\lambda) - S_{k'}(\lambda)|\ d\lambda = \int_{\mathbb{R}} \left| \sum_{m=k+1}^{k'} \xi_{(H_m, H_{m-1})}(\lambda) \right| d\lambda \\ &\leq \sum_{m=k+1}^{k'} \int_{\mathbb{R}} \left| \xi_{(H_m, H_{m-1})}(\lambda) \right|\ d\lambda = \sum_{m=k+1}^{k'} |\tau_m|, \end{aligned}$$

which converges to 0 as $k \longrightarrow \infty$, proving that the right hand side of (1.4.9) converges in L^1-norm and hence $\xi \in L^1(\mathbb{R})$. Also ξ is real valued since each $\xi_{(H_m,H_{m-1})}$ is real valued and supp $\xi \subseteq [a,b]$ since supp $\xi_{(H_m,H_{m-1})} \subseteq [a,b]$ for each m. Moreover by using Fubini's theorem we have

$$\|\xi\|_{L^1} = \int_{\mathbb{R}} |\xi(\lambda)|\, d\lambda \leq \sum_{m=1}^{\infty} \int_{\mathbb{R}} |\xi_{(H_m,H_{m-1})}(\lambda)|\, d\lambda = \sum_{m=1}^{\infty} |\tau_m| = \|V\|_1 \quad \text{and}$$

$$\int_{\mathbb{R}} \xi(\lambda)\, d\lambda = \sum_{m=1}^{\infty} \int_{\mathbb{R}} \xi_{(H_m,H_{m-1})}(\lambda)\, d\lambda = \sum_{m=1}^{\infty} \tau_m = \text{Tr}(V).$$

Finally by taking limit as $k \longrightarrow \infty$ on both sides of (1.4.8) and using Fubini's theorem we conclude that

$$\begin{aligned}\text{Tr}\,\{H^r - H_0^r\} &= \lim_{k\longrightarrow\infty} \text{Tr}\,\{H_k^r - H_0^r\} = \sum_{m=1}^{\infty} \int_a^b r\lambda^{r-1} \xi_{(H_m,H_{m-1})}(\lambda)\, d\lambda \\ &= \int_a^b r\lambda^{r-1} \sum_{m=1}^{\infty} \xi_{(H_m,H_{m-1})}(\lambda)\, d\lambda = \int_a^b r\lambda^{r-1}\xi(\lambda)\, d\lambda.\end{aligned}$$

For uniqueness, let us assume that there exists $\xi_1,\ \xi_2 \in L^1([a,b])$ such that

$$\text{Tr}\,[\, p(H) - p(H_0)\,] = \int_a^b p'(\lambda)\xi_j(\lambda) d\lambda,$$

where $p(.)$ is a polynomial and $j = 1, 2$. Therefore

$$\int_a^b p'(\lambda)\, \xi(\lambda)\, d\lambda = 0 \ \forall \quad \text{polynomials} \quad p(.) \quad \text{and} \quad \xi \equiv \xi_1 - \xi_2 \in L^1([a,b]),$$

which together with the fact that $\int_a^b \xi_1(\lambda)\, d\lambda = \int_a^b \xi_2(\lambda)\, d\lambda = \text{Tr}(V)$ (which one can easily arrive at by setting $p(\lambda) = \lambda$ in the above formula), implies that

$\displaystyle\int_a^b \lambda^r \xi(\lambda) d\lambda = 0 \ \forall \ r \geq 0.$ Hence by an application of Fubini's theorem, we get that

$$\int_{-\infty}^{\infty} e^{-\text{i}t\lambda}\, \xi(\lambda)\, d\lambda \ = \ \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} (-\text{i}t\lambda)^n\, \xi(\lambda)\, d\lambda = 0.$$

Hence

$$\int_{-\infty}^{\infty} \text{e}^{-\text{i}t\lambda}\, \xi(\lambda)\, d\lambda = 0 \,\forall\, t \in \mathbb{R}.$$

Therefore ξ is an $L^1([a,b])$- function whose Fourier transform $\hat{\xi}(t)$ vanishes identically, implying that $\xi = 0$ or $\xi_1 = \xi_2$ a.e. □

The function ξ, which is obtained in Theorem 1.4.6 is called Krein's spectral shift function and the formula (1.4.7) is known as Krein's trace formula in bounded self-adjoint case. Next we will start with a lemma which reduce the Krein's theorem in finite dimension in an unbounded self-adjoint case.

Lemma 1.4.7. *([22]) Let H and H_0 be two self-adjoint operators in $\mathcal{H}$ such that $H - H_0 = \tau|g\rangle\langle g|; \quad \tau > 0$ and $\|g\| = 1$. Then there exists a sequence $\{P_n\}$ of finite rank projections in $\mathcal{H}$ such that $P_n g \longrightarrow g$ as $n \longrightarrow \infty$ and for any $T > 0$*

$$\text{Tr}\{\text{e}^{\text{i}tH} - \text{e}^{\text{i}tH_0}\} = \lim_{n\longrightarrow\infty} \text{Tr}\{P_n \left[\text{e}^{\text{i}tP_nHP_n} - \text{e}^{\text{i}tp_nH_0p_n}\right] P_n\}, \tag{1.4.10}$$

uniformly for all t with $|t| \leq T$.

Proof. Applying Theorem 1.4.3 with $A = H_0$ and $f = g$, we get a sequence $\{P_n\}$ of finite rank projections in $\mathcal{H}$ such that $\left\|P_n^{\perp} H_0 P_n\right\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$; $\quad \left\|P_n^{\perp} \text{e}^{\text{i}tH_0} P_n\right\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$ and $\|(I - P_n)g\| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence

$$\left\|P_n^{\perp} H P_n\right\|_2 \leq \left\|P_n^{\perp} H_0 P_n\right\|_2 + \tau \left\||P_n^{\perp} g\rangle\langle P_n g|\right\|_2 \leq \left\|P_n^{\perp} H_0 P_n\right\|_2 + \tau \left\|(I - P_n)g\right\| \|g\|,$$

which converges to 0 as $n \longrightarrow \infty$. Thus

$$\begin{aligned}
&\text{Tr}\{\text{e}^{\text{i}tH} - \text{e}^{\text{i}tH_0}\} - \text{Tr}\{P_n \left[\text{e}^{\text{i}tP_nHP_n} - \text{e}^{\text{i}tP_nH_0P_n}\right] P_n\} \\
&= \text{Tr}\{\int_0^1 d\alpha \, \frac{d}{d\alpha} \left(\text{e}^{\text{i}t\alpha H} . \text{e}^{\text{i}t(1-\alpha)H_0}\right)\} - \text{Tr}\{P_n \int_0^1 d\alpha \, \frac{d}{d\alpha} \left(\text{e}^{\text{i}t\alpha P_nHP_n} . \text{e}^{\text{i}t(1-\alpha)P_nH_0P_n}\right) P_n\} \\
&= \text{Tr}\{\int_0^1 d\alpha \, \text{e}^{\text{i}t\alpha H} \text{i}t \, \tau|g\rangle\langle g| \, \text{e}^{\text{i}t(1-\alpha)H_0}\} \\
&\qquad - \text{Tr}\{\int_0^1 d\alpha \, P_n \text{e}^{\text{i}t\alpha P_nHP_n} \text{i}t \, P_n \tau|g\rangle\langle g|P_n \, \text{e}^{\text{i}t(1-\alpha)P_nH_0P_n} P_n\} \\
&= \tau \, \text{i}t \int_0^1 d\alpha \, \text{Tr}\{\text{e}^{\text{i}t\alpha H} \, |g\rangle\langle g| \, \text{e}^{\text{i}t(1-\alpha)H_0} - P_n \text{e}^{\text{i}t\alpha P_nHP_n} \, P_n|g\rangle\langle g|P_n \, \text{e}^{\text{i}t(1-\alpha)P_nH_0P_n} P_n\} \\
&= \tau \, \text{i}t \int_0^1 d\alpha \, \text{Tr}\{\left[\text{e}^{\text{i}t\alpha H} - \text{e}^{\text{i}t\alpha P_nHP_n}\right] P_n|g\rangle\langle g| \, \text{e}^{\text{i}t(1-\alpha)H_0} + \, \text{e}^{\text{i}t\alpha H} P_n^{\perp} \, |g\rangle\langle g| \, \text{e}^{\text{i}t(1-\alpha)H_0} \\
&\qquad + \, \text{e}^{\text{i}t\alpha P_nHP_n} \, P_n|g\rangle\langle g|P_n^{\perp} \, \text{e}^{\text{i}t(1-\alpha)H_0} + \, \text{e}^{\text{i}t\alpha P_nHP_n} \, P_n|g\rangle\langle g|P_n \, \left[\text{e}^{\text{i}t(1-\alpha)H_0} - \text{e}^{\text{i}t(1-\alpha)P_nH_0P_n}\right]\}
\end{aligned} \tag{1.4.11}$$

In the first term of the expression (1.4.11) :

$$\begin{aligned}
\left\|\left[\text{e}^{\text{i}t\alpha H} - \text{e}^{\text{i}t\alpha P_nHP_n}\right] P_n\right\|_2 &= \left\|\text{i}t \, \alpha \int_0^1 d\beta \, \text{e}^{\text{i}t\alpha\beta H} [H - P_nHP_n] \text{e}^{\text{i}t\alpha(1-\beta)P_nHP_n} P_n\right\|_2 \\
&= \left\|\text{i}t \, \alpha \int_0^1 d\beta \, \text{e}^{\text{i}t\alpha\beta H} P_n^{\perp} H P_n \text{e}^{\text{i}t\alpha(1-\beta)P_nHP_n} P_n\right\|_2 \leq T \left\|P_n^{\perp} H P_n\right\|_2,
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$, uniformly for $|t| \leq T$ and hence

$$\begin{aligned}&\left\|\left[\mathrm{e}^{\mathrm{i}t\alpha H} - \mathrm{e}^{\mathrm{i}t\alpha P_n H P_n}\right] P_n|g\rangle\langle g|\ \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1 \\ &\quad \leq \left\|\left[\mathrm{e}^{\mathrm{i}t\alpha H} - \mathrm{e}^{\mathrm{i}t\alpha P_n H P_n}\right] P_n\right\|_2 \left\||g\rangle\langle g|\ \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_2 \leq T\ \left\|P_n^{\perp} H P_n\right\|_2\ \|g\|^2,\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$, uniformly for $|t| \leq T$. Similarly for the fourth term in (1.4.11), we note that

$$\begin{aligned}&\left\|P_n\ \left[\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} - \mathrm{e}^{\mathrm{i}t(1-\alpha)P_n H_0 P_n}\right]\right\|_2 \\ &= \left\|\mathrm{i}t\ (1-\alpha)\int_0^1 d\beta\ P_n \mathrm{e}^{\mathrm{i}t(1-\alpha)\beta H_0} P_n^{\perp} H_0 P_n \mathrm{e}^{\mathrm{i}t(1-\alpha)(1-\beta)P_n H_0 P_n}\right\|_2 \leq T\left\|P_n^{\perp} H_0 P_n\right\|_2,\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$, uniformly for $|t| \leq T$ and hence

$$\begin{aligned}&\left\|\mathrm{e}^{\mathrm{i}t\alpha P_n H P_n}\ P_n|g\rangle\langle g|P_n\ \left[\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} - \mathrm{e}^{\mathrm{i}t(1-\alpha)P_n H_0 P_n}\right]\right\|_1 \\ &\quad \leq \left\|\mathrm{e}^{\mathrm{i}t\alpha P_n H P_n}\ P_n|g\rangle\langle g|\right\|_2 \left\|P_n\ \left[\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} - \mathrm{e}^{\mathrm{i}t(1-\alpha)P_n H_0 P_n}\right]\right\|_2 \\ &\qquad \leq \left\|\mathrm{e}^{\mathrm{i}t\alpha P_n H P_n}\ |P_n g\rangle\langle g|\right\|_2\ T\left\|P_n^{\perp} H_0 P_n\right\|_2 \leq\ T\left\|P_n^{\perp} H_0 P_n\right\|_2 \|g\|^2,\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$, uniformly for $|t| \leq T$. For the second term in (1.4.11), we have

$$\left\|\mathrm{e}^{\mathrm{i}t\alpha H} P_n^{\perp}\ |g\rangle\langle g|\ \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1 = \left\|\mathrm{e}^{\mathrm{i}t\alpha H}\ |P_n^{\perp} g\rangle\langle g|\ \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1 \leq \left\||P_n^{\perp} g\rangle\langle g|\right\|_1 \leq \|(I-P_n)g\|\|g\|,$$

which converges to 0 as $n \longrightarrow \infty$, uniformly for $|t| \leq T$ and for the third term in (1.4.11), we have the following estimate

$$\begin{aligned}&\left\|\mathrm{e}^{\mathrm{i}t\alpha P_n H P_n}\ P_n|g\rangle\langle g|P_n^{\perp}\ \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1 \\ &= \left\|\mathrm{e}^{\mathrm{i}t\alpha P_n H P_n}\ |P_n g\rangle\langle P_n^{\perp} g|\ \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1 \leq \left\||P_n g\rangle\langle P_n^{\perp} g|\right\|_1 \leq \|g\|\|(I-P_n)g\|,\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$, uniformly for $|t| \leq T$. Therefore the right hand side of (1.4.11) converges to 0 as $n \longrightarrow \infty$, uniformly for $|t| \leq T$ and hence we have proved that

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\} = \lim_{n\longrightarrow\infty} \mathrm{Tr}\{P_n\left[\mathrm{e}^{\mathrm{i}tP_n H P_n} - \mathrm{e}^{\mathrm{i}tp_n H_0 p_n}\right] P_n\},$$

uniformly for all t with $|t| \leq T$. □

Next theorem is the Krein's theorem for unbounded self-adjoint case.

Theorem 1.4.8. *([22]) Let H and H_0 be two self-adjoint operators in $\mathcal{H}$ such that $H - H_0 \equiv V \in \mathcal{B}_1(\mathcal{H})$. Then there exists a unique real-valued function $\xi \in L^1(\mathbb{R})$ such that*

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\} = (\mathrm{i}t)\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\ \xi(\lambda)\ d\lambda.$$

Moreover,

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\,\xi(\lambda)\,d\lambda = \mathrm{Tr}(V) \quad \textit{and} \quad \|\xi\|_{L^1(\mathbb{R})} \leq \|V\|_1.$$

Proof. At first we let $V \equiv \tau|g\rangle\langle g|; \quad \tau > 0$ and $\|g\| = 1$. Hence by Lemma 1.4.7 , we conclude that, there exists a sequence $\{P_n\}$ of finite rank projections such that $P_n g \longrightarrow g$ as $n \longrightarrow \infty$ and

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\} = \lim_{n\longrightarrow\infty} \mathrm{Tr}\{P_n\left[\mathrm{e}^{\mathrm{i}tH_n} - \mathrm{e}^{\mathrm{i}tH_{0,n}}\right]P_n\},$$

where $H_n = P_n H P_n$ and $H_{0.n} = P_n H_0 P_n$, and the convergence is uniform in t for $|t| \leq T$. Note that by construction $P_n\mathcal{H} \subseteq \mathrm{Dom}(H_0) = \mathrm{Dom}(H)$ (see the proof of Theorem 1.4.3) and hence both H_n and $H_{0,n}$ are self-adjoint operators in the finite dimensional Hilbert space $P_n\mathcal{H}$. By Theorem 1.4.1 (iv), we get a $\{0,1\}$-valued $L^1(\mathbb{R})$- function ξ_n such that

$$\mathrm{Tr}\{P_n\left[\mathrm{e}^{\mathrm{i}tH_n} - \mathrm{e}^{\mathrm{i}tH_{0,n}}\right]P_n\} = \mathrm{i}t\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\,\xi_n(\lambda)\,d\lambda, \tag{1.4.12}$$

and hence

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\} = \mathrm{i}t\lim_{n\longrightarrow\infty}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\,\xi_n(\lambda)\,d\lambda, \tag{1.4.13}$$

the convergence being uniform in t for $|t| \leq T$. Since $t \longrightarrow \mathrm{e}^{\mathrm{i}tH_n},\ \mathrm{e}^{\mathrm{i}tH_{0,n}}$ are norm continuous in $P_n\mathcal{H}$ and P_nVP_n is rank one, then using bounded convergence theorem, we have from (1.4.12) that

$$\begin{aligned}
\int_{\mathbb{R}} \xi_n(\lambda)\,d\lambda = \lim_{t\longrightarrow 0}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\,\xi_n(\lambda)\,d\lambda &= \lim_{t\longrightarrow 0}\frac{1}{\mathrm{i}t}\,\mathrm{Tr}\left\{P_n\left[\mathrm{e}^{\mathrm{i}tH_n} - \mathrm{e}^{\mathrm{i}tH_{0,n}}\right]P_n\right\} \\
&= \lim_{t\longrightarrow 0}\frac{1}{\mathrm{i}t}\,\mathrm{Tr}\left\{P_n\int_0^t ds\,\frac{d}{ds}\left(\mathrm{e}^{\mathrm{i}sH_n}.\mathrm{e}^{\mathrm{i}(t-s)H_{0,n}}\right)P_n\right\} \\
&= \lim_{t\longrightarrow 0}\frac{1}{t}\int_0^t ds\,\mathrm{Tr}\{P_n\mathrm{e}^{\mathrm{i}sH_n}\,P_nVP_n\,\mathrm{e}^{\mathrm{i}(t-s)H_{0,n}}P_n\} \\
&= \mathrm{Tr}\{P_nVP_n\}.
\end{aligned} \tag{1.4.14}$$

Thus

$$\int_{\mathbb{R}} \xi_n(\lambda)\,d\lambda = \mathrm{Tr}\{\tau|P_ng\rangle\langle P_ng|\} = \tau\|P_ng\|^2 = \tau(1 - \|P_n^{\perp}g\|^2) > \tau(1-\epsilon^2),$$

since $\|P_n^{\perp}g\| \longrightarrow 0$ as $n \longrightarrow \infty$ i.e. given $\epsilon > 0,\ \exists$ a natural number $N \in \mathbb{N}$ such that $\|P_n^{\perp}g\| < \epsilon$ $\forall\ n \geq N$. Set

$$\mu_n(\Delta) = \frac{1}{\tau\|P_ng\|^2}\int_{\Delta} \xi_n(\lambda)\,d\lambda$$

for every Borel set $\Delta \subseteq \mathbb{R}$ and hence we have a family $\{\mu_n\}$ of probability measure by (1.4.14). Also note that

$$\hat{\mu}_n(t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\, \mu_n(d\lambda) = \frac{1}{\tau\|P_n g\|^2} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\, \xi_n(\lambda)\, d\lambda,$$

which by (1.4.13) and the fact that $\|P_n g\|^2 \longrightarrow \|g\|^2 = 1$ as $n \longrightarrow \infty$, converges to

$$\frac{1}{\mathrm{i}t\tau}\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\} \equiv \hat{\mu}(t) \quad \text{uniformly in } t \text{ in compact sets in } \mathbb{R}\backslash\{0\}.$$

On the other hand $\hat{\mu}(0) = \frac{1}{\tau\|P_n g\|^2}\int_{\mathbb{R}} \xi_n(\lambda)\, d\lambda = 1$ for all $n \in \mathbb{N}$. Again

$$\begin{aligned}
\lim_{t\longrightarrow 0} \hat{\mu}(t) &= \lim_{t\longrightarrow 0} \frac{1}{\mathrm{i}t\tau}\, \mathrm{Tr}\left\{\left[\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\right]\right\} = \lim_{t\longrightarrow 0} \frac{1}{\mathrm{i}t\tau}\, \mathrm{Tr}\left\{\int_0^t ds\, \frac{d}{ds}\left(\mathrm{e}^{\mathrm{i}sH}.\mathrm{e}^{\mathrm{i}(t-s)H_0}\right)\right\} \\
&= \lim_{t\longrightarrow 0} \frac{1}{t\tau} \int_0^t ds\, \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}sH}\, V\, \mathrm{e}^{\mathrm{i}(t-s)H_0}\} = \frac{1}{\tau}\, \mathrm{Tr}\{V\} = 1 \equiv \hat{\mu}(0),
\end{aligned}$$

by definition. Thus by Levy-Cramer continuity theorem [16], there exists a probability measure μ on $\mathbb{R}$ such that $\mu_n \longrightarrow \mu$ weakly i.e.

$$\int_{\mathbb{R}} \phi(\lambda)\, \mu_n(d\lambda) \longrightarrow \int_{\mathbb{R}} \phi(\lambda)\, \mu(d\lambda) \quad \text{for every bounded continuous function} \quad \phi.$$

Let $\Delta = (a, b] \subseteq \mathbb{R}$ and let $\{\phi_n\}$ be a sequence of smooth functions of support in $(a - \frac{1}{n}, b + \frac{1}{n}]$ such that $0 \leq \phi_n \leq 1$ and $\|\chi_\Delta - \phi_n\|_1 \longrightarrow 0$ as $n \longrightarrow \infty$, where χ_Δ is the characteristic function of Δ. Choosing a subsequence if necessary and using the bounded convergence theorem, we have

$$\lim_{n\longrightarrow\infty} \lim_{m\longrightarrow\infty} \int_{\mathbb{R}} \phi_n(\lambda)\, \mu_m(d\lambda) = \lim_{n\longrightarrow\infty} \int_{\mathbb{R}} \phi_n(\lambda)\, \mu(d\lambda) = \mu(\Delta).$$

Thus

$$\begin{aligned}
\mu(\Delta) &= \lim_{n\longrightarrow\infty} \lim_{m\longrightarrow\infty} \frac{1}{\tau\|P_m g\|^2} \int_{\mathbb{R}} \phi_n(\lambda)\, \xi_m(\lambda)\, (d\lambda) \\
&= \frac{1}{\tau} \lim_{n\longrightarrow\infty} \lim_{m\longrightarrow\infty} \int_{\mathbb{R}} \phi_n(\lambda)\, \xi_m(\lambda)\, (d\lambda) \\
&\leq \tau^{-1} \lim_{n\longrightarrow\infty} \int_{a-\frac{1}{n}}^{b+\frac{1}{n}} \phi_n(\lambda)\, d\lambda \\
&\leq \tau^{-1}\left(b - a + \frac{2}{n}\right) = \tau^{-1}(b - a),
\end{aligned}$$

since $0 \leq \xi_m(\lambda) \leq 1$ for all m and all λ. This shows that μ is absolutely continuous and we set $\xi(\lambda) = \tau\frac{\mu(d\lambda)}{d\lambda} \equiv \tau\frac{d\mu(\lambda)}{d\lambda}$. Then ξ is a non-negative L^1-function and we have that

$$\hat{\mu}(t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\, \mu(d\lambda) = \tau^{-1} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\, \xi(\lambda)\, d\lambda \quad \text{and hence}$$

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\} = (\mathrm{i}t) \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\, \xi(\lambda)\, d\lambda. \tag{1.4.15}$$

Also dividing both sides of (1.4.15) by $\mathrm{i}t$ and taking limit as $t \longrightarrow 0$, we get that

$$\begin{aligned}
\int_{\mathbb{R}} \xi(\lambda)\, d\lambda &= \lim_{t\longrightarrow 0} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\, \xi(\lambda)\, d\lambda = \lim_{t\longrightarrow 0} \frac{1}{\mathrm{i}t}\, \mathrm{Tr}\left\{\left[\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\right]\right\} \\
&= \lim_{t\longrightarrow 0} \frac{1}{\mathrm{i}t}\, \mathrm{Tr}\left\{\int_0^t ds\, \frac{d}{ds}\left(\mathrm{e}^{\mathrm{i}sH}.\mathrm{e}^{\mathrm{i}(t-s)H_0}\right)\right\} \\
&= \lim_{t\longrightarrow 0} \frac{1}{t} \int_0^t ds\, \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}sH}\, V\, \mathrm{e}^{\mathrm{i}(t-s)H_0}\} = \mathrm{Tr}\{V\} = \tau \geq 0.
\end{aligned}$$

i.e. $\|\xi\|_{L^1} = |\tau|$, since ξ is non-negative and $\tau \geq 0$.

Now if V is rank one and negative i.e. $V = \tau|g\rangle\langle g|; \quad \tau < 0$ and $\|g\| = 1$. Then $H_0 - H = -\tau|g\rangle\langle g|; \quad -\tau > 0$ and $\|g\| = 1$ and obtain as above a non-negative L^1-function η such that

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH_0} - \mathrm{e}^{\mathrm{i}tH}\} = (\mathrm{i}t) \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\, \eta(\lambda)\, d\lambda \text{ and } \int_{\mathbb{R}} \eta(\lambda)\, d\lambda = \mathrm{Tr}\left(-\tau|g\rangle\langle g|\right) = -\tau.$$

Defining $\xi(\lambda) = -\eta(\lambda)$, we get

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\} = (\mathrm{i}t) \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\, \xi(\lambda)\, d\lambda \text{ and } \int_{\mathbb{R}} \xi(\lambda)\, d\lambda = \tau; \ \ \|\xi\|_{L^1} = \|\eta\|_{L^1} = |\tau|.$$

hence the relation (1.4.15) is valid for all V rank one with some real-valued L^1-function ξ such that $\int_{\mathbb{R}} \xi(\lambda)\, d\lambda = \mathrm{Tr}(V)$ and $\|\xi\|_{L^1} \leq \|V\|_1$.

Now let $V \in \mathcal{B}_1(\mathcal{H})$, and write $V = \sum\limits_{k=1}^{\infty} \tau_k|g_k\rangle\langle g_k|$ with $\sum\limits_{k=1}^{\infty} |\tau_k| < \infty; \quad \|g_k\| = 1$ for each $k \in \mathbb{N}$. Set $V_k \equiv \sum\limits_{j=1}^{k} \tau_j|g_j\rangle\langle g_j|$ and $H_k \equiv H_0 + V_k$ for $k = 1, 2, 3.....$ Then $\|V - V_k\|_1 \longrightarrow 0$ as $k \longrightarrow \infty$ and hence $\|H - H_k\|_1 = \|V - V_k\|_1 \longrightarrow 0$ as $k \longrightarrow \infty$, since $\sum\limits_{k=1}^{\infty} |\tau_k| < \infty$. Therefore

$$\left\|\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_k}\right\|_1 = \left\|(\mathrm{i}t) \int_0^1 d\alpha\, \mathrm{e}^{\mathrm{i}t\alpha H}[H - H_k]\mathrm{e}^{\mathrm{i}t(1-\alpha)H_k}\right\|_1 \leq |t|\|H - H_k\|_1,$$

which converges to 0 as $k \longrightarrow \infty$, uniformly in t for $|t| \leq T$. But on the other hand

$$\mathrm{e}^{\mathrm{i}tH_k} - \mathrm{e}^{\mathrm{i}tH_0} = \sum_{m=1}^{k} \left(\mathrm{e}^{\mathrm{i}tH_m} - \mathrm{e}^{\mathrm{i}tH_{m-1}}\right) \quad \text{and hence}$$

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH_k} - \mathrm{e}^{\mathrm{i}tH_0}\} = \sum_{m=1}^{k} \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH_m} - \mathrm{e}^{\mathrm{i}tH_{m-1}}\} = \sum_{m=1}^{k} (\mathrm{i}t) \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\, \xi_m(\lambda)\, d\lambda, \tag{1.4.16}$$

where $\xi_m(\lambda)$ is a real-valued L^1 function as obtained in (1.4.15) corresponding to the pair (H_m, H_{m-1}) such that $\int_{\mathbb{R}} \xi_m(\lambda)\, d\lambda = \tau_m$ and $\|\xi_m\|_{L^1} = |\tau_m|$. Define

$$\xi(\lambda) = \sum_{k=1}^{\infty} \xi_m(\lambda) \tag{1.4.17}$$

Set $S_k(\lambda) \equiv \sum\limits_{m=1}^{k} \xi_m(\lambda)$, then $S_k \in L^1(\mathbb{R})$ for each $k \in \mathbb{N}$ and for $k', k \in \mathbb{N}$ such that $k' > k$, we have

$$\begin{aligned}\|S_k - S_{k'}\|_{L^1} &= \int_{\mathbb{R}} |S_k(\lambda) - S_{k'}(\lambda)|\ d\lambda = \int_{\mathbb{R}} \left| \sum_{m=k+1}^{k'} \xi_m(\lambda) \right|\ d\lambda \\ &\leq \sum_{m=k+1}^{k'} \int_{\mathbb{R}} |\xi_m(\lambda)|\ d\lambda = \sum_{m=k+1}^{k'} |\tau_m|,\end{aligned}$$

which converges to 0 as $k \longrightarrow \infty$, since $\sum\limits_{m=1}^{\infty} |\tau_m| < \infty$ and hence the right hand side of (1.4.17) converges in L^1-norm. Therefore $\xi \in L^1(\mathbb{R})$ and also ξ is real valued since each ξ_m is real valued. Moreover by using Fubini's theorem we have

$$\|\xi\|_{L^1} = \int_{\mathbb{R}} |\xi(\lambda)|\, d\lambda \leq \sum_{m=1}^{\infty} \int_{\mathbb{R}} |\xi_m(\lambda)|\, d\lambda = \sum_{m=1}^{\infty} |\tau_m| = \|V\|_1 \quad \text{and}$$

$$\int_{\mathbb{R}} \xi(\lambda)\, d\lambda = \sum_{m=1}^{\infty} \int_{\mathbb{R}} \xi_m(\lambda)\, d\lambda = \sum_{m=1}^{\infty} \tau_m = \mathrm{Tr}(V).$$

Finally by taking limit as $k \longrightarrow \infty$ on both sides of (1.4.16) and using Fubini's theorem we conclude that

$$\begin{aligned}\mathrm{Tr}\, \{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\} &= \lim_{k \longrightarrow \infty} \mathrm{Tr}\, \{\mathrm{e}^{\mathrm{i}tH_k} - \mathrm{e}^{\mathrm{i}tH_0}\} = \sum_{m=1}^{\infty} \mathrm{i}t \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda} \xi_m(\lambda)\, d\lambda \\ &= \mathrm{i}t \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda} \sum_{m=1}^{\infty} \xi_m(\lambda)\, d\lambda = \mathrm{i}t \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda} \xi(\lambda)\, d\lambda.\end{aligned}$$

For uniqueness, let us assume that there exists $\xi_1, \xi_2 \in L^1(\mathbb{R})$ such that

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\} = (\mathrm{i}t) \int_{\mathbb{R}} \mathrm{e}^{it\lambda} \xi_j(\lambda) d\lambda,$$

for $j = 1, 2$ and hence

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda} \left[\xi_1(\lambda) - \xi_2(\lambda)\right] d\lambda = 0 \ \ \forall \ \ 0 \neq t \in \mathbb{R} \quad \text{and} \quad \xi_1 - \xi_2 \in L^1(\mathbb{R}).$$

Also for $t = 0$, $\int_{\mathbb{R}} \xi_1(\lambda) d\lambda = \int_{\mathbb{R}} \xi_2(\lambda) d\lambda = \text{Tr}(V)$ and hence

$$\int_{\mathbb{R}} e^{it\lambda} \left[\xi_1(\lambda) - \xi_2(\lambda)\right] d\lambda = 0 \;\; \forall \;\; t \in \mathbb{R} \quad \text{and} \quad \xi_1 - \xi_2 \in L^1(\mathbb{R}).$$

Then by Fourier Inversion Theorem we conclude that $\xi_1 = \xi_2$ a.e. ☐

1.5 Koplienko Formula

In 1982, Koplienko extended Krein's trace formula (1.4.1) to the next order when $H - H_0 \equiv V \in \mathcal{B}_2(\mathcal{H})$ (where H and H_0 are two self-adjoint operators in $\mathcal{H}$). He proved that there exists a unique non-negative function $\eta \in L^1(\mathbb{R})$ such that

$$\text{Tr}\{\phi(H) - \phi(H_0) - D^{(1)}\phi(H_0) \bullet V\} = \int_{-\infty}^{\infty} \phi''(\lambda)\, \eta(\lambda)\, d\lambda, \tag{1.5.1}$$

where $D^{(1)}\phi(H_0)$ denotes the first order Frechet derivative of ϕ at H_0 (see [2]) and $D^{(1)}\phi(H_0)(V)$ is the first order Frechet derivative of the function ϕ at the self-adjoint operator H_0 acting on V and ϕ is a rational function with poles off $\mathbb{R}$. The proof of Koplienko in [14] is based on the theory of multiple integrals and is not complete. The formula (1.5.1) is known as Koplienko trace formula and the function η is called Koplienko spectral shift function. Later in 1993, Boyadzhiev [9] obtained Koplienko trace formula in a setting of unital C^*- algebra and the method is entirely different from that of Koplienko. He proved the following theorem in [9].

Theorem 1.5.1. *Let $\mathcal{A}$ be a unital C^*- algebra and τ- a positive linear trace on $\mathcal{A}$. Then for any two self-adjoint elements $A, B \in \mathcal{A}$, there exists a non-negative function $\eta \in L^1(\mathbb{R})$ such that*

$$\tau\{\phi(A) - \phi(B) - (A - B)\phi'(B)\} = \int_{-\infty}^{\infty} \phi''(\lambda)\, \eta(\lambda)\, d\lambda \tag{1.5.2}$$

for every $\phi \in C^2([a, b])$ (set of twice continuously differentiable functions on [a,b]), where $\sigma(A), \sigma(B) \subseteq [a, b]$. Moreover the function η is supported on the smallest interval containing $\sigma(A), \sigma(B)$ and

$$\eta(\lambda) = \frac{1}{\pi}\tau\{(A - \lambda)\left[\tan^{-1}\left(\frac{A - \lambda}{\epsilon}\right) - \tan^{-1}\left(\frac{B - \lambda}{\epsilon}\right)\right]\} \quad \textit{a.e.}$$

Also by substituting $\phi(\lambda) = \lambda^2$ in (1.5.2),

$$\frac{1}{2}\tau\{(A - B)^2\} = \int_{-\infty}^{\infty} \eta(\lambda)\, d\lambda = \|\eta\|_{L^1}.$$

The proof of the above theorem is based on the following inequality

$$\tau\{f(A) - f(B) - (A - B)f'(B)\} \geq 0,$$

where A, B are two self-adjoint elements in $\mathcal{A}$ and f is an arbitrary continuously differentiable convex function on some interval containing $\sigma(A) \bigcup \sigma(B)$ via the Riesz representation theorem for positive continuous functionals. Boyadzhiev also stated that the results remain valid if we replace C^*-algebra by Jordan Banach algebra. Recently, Gesztesy, Pushnitski and Simon [12] gave an alternative proof of the formula (1.5.1) for the bounded case. In fact they showed in [12] that

(i) Given two bounded self-adjoint operators H, H_0 in $\mathcal{H}$ such that $H - H_0 \equiv V \in \mathcal{B}_2(\mathcal{H})$, then there exists a unique non-negative function $\eta_{(H,H_0)} \in L^1(\mathbb{R})$ supported on $[-\max\{\|H\|, \|H_0\|\}, \max\{\|H\|, \|H_0\|\}]$ such that $\phi(H) - \phi(H_0) - D^{(1)}\phi(H_0)(V) \in \mathcal{B}_1(\mathcal{H})$ and

$$\text{Tr}\{\phi(H) - \phi(H_0) - D^{(1)}\phi(H_0) \bullet V\} = \int_{-\infty}^{\infty} \phi''(\lambda)\, \eta_{(H,H_0)}(\lambda)\, d\lambda, \tag{1.5.3}$$

for any $g \in C^{\infty}(\mathbb{R})$(set of all infinitely differentiable functions on $\mathbb{R}$).

(ii) Moreover,

$$\|\eta_{(H,H_0)}\|_{L^1} = \int_{-\infty}^{\infty} \eta_{(H,H_0)}(\lambda)\, d\lambda = \frac{1}{2}\|V\|_2^2$$

and for any bounded self-adjoint operators H, H_1, H_0 in $\mathcal{H}$ such that $H - H_0,\ H_1 - H_0 \in \mathcal{B}_2(\mathcal{H})$, then

$$\int_{-\infty}^{\infty} \left|\eta_{(H,H_0)} - \eta_{(H_1,H_0)}\right| d\lambda \leq \|H - H_1\|_2 \left[\frac{1}{2}\|H - H_1\|_2 + \|H_1 - H_0\|_2\right] \tag{1.5.4}$$

In the proof of the above results, they first established the results for a trace class perturbation and then proved the formula (1.5.3) for a Hilbert-Schmidt perturbation by approximating it through trace class operators and using the estimate (1.5.4). In the next chapter we revisit the proof of Koplienko formula for bounded case and prove the unbounded case [20], using the idea of finite dimensional approximation as in the works of Voiculescu [24], Sinha and Mohapatra [22] and using the idea contained in the proof of the above results.

1.6 Higher Order Trace Formula

In 2009, Dykema and Skripka extended Koplienko trace formula in [11]. In fact they showed the existence of higher order spectral shift function when the unperturbed self-adjoint operator is bounded and the perturbation is Hilbert-Schmidt and the results are obtained in the semi-finite von Neumann algebra. Given a self-adjoint operator A (possibly unbounded) and a self-adjoint operator $V \in \mathcal{B}_2(\mathcal{H})$, Dykema and Skripka proved that for $2 < p \in \mathbb{N}$,

(i) there exists a unique finite real-valued measure ν_p on $\mathbb{R}$ such that the trace formula

$$\mathrm{Tr}\{\phi(A+V) - \sum_{j=0}^{p-1} \frac{1}{j!} D^{(j)}\phi(A) \bullet (\underbrace{V, V, \ldots, V}_{j-times})\} = \int_{-\infty}^{\infty} \phi^{(p)}(\lambda)\, d\nu_p(\lambda), \tag{1.6.1}$$

holds for suitable functions ϕ, where $D^{(j)}\phi(A) \bullet (\underbrace{V, V, \ldots, V}_{j-times})$ is the j-th order Frechet derivative of ϕ at A acting on $(\underbrace{V, V, \ldots, V}_{j-times})$ and $\phi^{(p)}$ is the p-th order derivative of ϕ. The total variation of ν_p is bounded by $\frac{1}{p!}\|V\|_2^p$.

(ii) If, in addition, A is bounded, then ν_p is absolutely continuous. The density η_p of ν_p is called p-th order spectral shift function and η_p can be expressed recursively via lower order ones.

The proof of the above results is based on multiple operator integrals, some properties of divided differences and splines and some techniques of free probability. They obtained η_p by analyzing the Cauchy transform of the measure η_p. In chapter 3, we give an alternative proof of the formula (1.6.1) for $p = 3$ and show the existence of η_3 in both cases when the unperturbed operator A is bounded or unbounded, but bounded below [21]. Our method of obtaining the formula (1.6.1) for $p = 3$ and showing the existence of η_3 is entirely different from that of Dykema and Skripka. We prove L^1- convergence in an appropriate L^1-space.

Recently, Potapov, Skripka and Sukochev in their preprint [17], have discussed more about higher order trace formula.

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Chapter 2

Koplienko Trace Formula

Let H and H_0 be two self-adjoint operators in $\mathcal{H}$ with $\sigma(H)$, $\sigma(H_0)$ as their spectra and $E_H(\lambda)$, $E_{H_0}(\lambda)$ the spectral families such that $H - H_0 \equiv V \in \mathcal{B}_2(\mathcal{H})$.

In this chapter first we discuss Koplienko formula in finite dimension and then we prove Koplienko formula for both bounded and unbounded self-adjoint cases via finite dimensional approximation [5].

2.1 Koplienko formula in finite dimension

Theorem 2.1.1. *Let H and H_0 be two bounded self-adjoint operators in a Hilbert space $\mathcal{H}$ such that $H - H_0 \equiv V$ and let $p(\lambda) = \lambda^r (r \geq 2)$.*

(i) Then $D^{(1)}p(H_0) \bullet X = \sum_{j=0}^{r-1} H_0^{r-j-1} X H_0^j$ *and* $\frac{d}{ds}(p(H_s)) = \sum_{j=0}^{r-1} H_s^{r-j-1} V H_s^j$*, where $H_s = H_0 + sV \ (0 \leq s \leq 1)$ and $X \in \mathcal{B}(\mathcal{H})$.*

(ii) If furthermore $dim\mathcal{H} < \infty$, then there exists a unique non-negative $L^1(\mathbb{R})$-function η such that

$$\mathrm{Tr}\{p(H) - p(H_0) - D^{(1)}p(H_0) \bullet V\} = \int_a^b p''(\lambda)\eta(\lambda)d\lambda, \tag{2.1.1}$$

for some $-\infty < a < b < \infty$.

Moreover,

$$\eta(\lambda) = \int_0^1 \text{Tr}\{V\,[E_{H_0}(\lambda) - E_{H_s}(\lambda)]\}ds, \tag{2.1.2}$$

where $E_{H_s}(.)$ is the spectral family of the self-adjoint operator H_s , and

$$\|\eta\|_1 = \frac{1}{2}\|V\|_2^2. \tag{2.1.3}$$

(iii) For $dim\mathcal{H} < \infty$,

$$\text{Tr}\{\text{e}^{\text{i}tH} - \text{e}^{\text{i}tH_0} - D^{(1)}(\text{e}^{\text{i}tH_0}) \bullet V\} = (\text{i}t)^2 \int_a^b \text{e}^{\text{i}t\lambda}\eta(\lambda)d\lambda, \tag{2.1.4}$$

for some $-\infty < a < b < \infty$, $t \in \mathbb{R}$ and η is given by (2.1.2) *.*

Proof. (i) For $p(\lambda) = \lambda^r (r \geq 2)$ and $X \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned} p(H_0+X) - p(H_0) = (H_0+X)^r - H_0^r &= \sum_{j=0}^{r-1} (H_0+X)^{r-j-1} X H_0^j \\ &= \sum_{j=0}^{r-1} H_0^{r-j-1} X H_0^j + \sum_{j=0}^{r-1} \left[(H_0+X)^{r-j-1} - H_0^{r-j-1}\right] X H_0^j \\ &= \sum_{j=0}^{r-1} H_0^{r-j-1} X H_0^j + \sum_{j=0}^{r-2} \sum_{k=0}^{r-j-2} (H_0+X)^{r-j-k-2} X H_0^k X H_0^j, \end{aligned} \tag{2.1.5}$$

and hence

$$\left\|p(H_0+X) - p(H_0) - \sum_{j=0}^{r-1} H_0^{r-j-1} X H_0^j\right\| \leq \sum_{j=0}^{r-2} \sum_{k=0}^{r-j-2} \|H_0+X\|^{r-j-k-2}\|X\|\|H_0\|^k\|X\|\|H_0\|^j,$$

proving that $D^{(1)}p(H_0) \bullet X = \sum_{j=0}^{r-1} H_0^{r-j-1} X H_0^j$. Similarly,

$$\begin{aligned} \frac{H_{s+h}^r - H_s^r}{h} = \sum_{j=0}^{r-1} H_{s+h}^{r-j-1} V H_s^j &= \sum_{j=0}^{r-1} H_s^{r-j-1} V H_s^j + \sum_{j=0}^{r-1} \left[H_{s+h}^{r-j-1} - H_s^{r-j-1}\right] V H_s^j \\ &= \sum_{j=0}^{r-1} H_s^{r-j-1} V H_s^j + h \sum_{j=0}^{r-2} \sum_{k=0}^{r-j-2} H_{s+h}^{r-j-k-2} V H_s^k V H_s^j \end{aligned}$$

and hence

$$\left\| \frac{H_{s+h}^r - H_s^r}{h} - \sum_{j=0}^{r-1} H_s^{r-j-1} V H_s^j \right\| \leq |h| \sum_{j=0}^{r-2} \sum_{k=0}^{r-j-2} \|H_{s+h}\|^{r-j-k-2} \|V\| \|H_s\|^k \|V\| \|H_s\|^j,$$

proving that every polynomial in H_s is norm-differentiable and that $\frac{d}{ds}(p(H_s)) = \sum_{j=0}^{r-1} H_s^{r-j-1} V H_s^j$.

(ii) By using the cyclicity of trace and noting that the trace now is a finite sum, we have that for $p(\lambda) = \lambda^r$ ($r \geq 2$),

$$\begin{aligned}
\text{Tr}\{p(H) - p(H_0) - D^{(1)}p(H_0) \bullet V\} &= \text{Tr}\ \{p(H) - p(H_0)\} - \text{Tr}\ \{D^{(1)}p(H_0) \bullet V\} \\
&= \text{Tr}\left(\int_0^1 \frac{d}{ds}\left(p(H_s)\right) ds\right) - \text{Tr}\left(\sum_{j=0}^{r-1} H_0^{r-j-1} V H_0^j\right) \\
&= \int_0^1 ds \text{Tr}\left(\sum_{j=0}^{r-1} H_s^{r-j-1} V H_s^j\right) - \int_0^1 ds \text{Tr}\left(\sum_{j=0}^{r-1} H_0^{r-j-1} V H_0^j\right) \\
&= \int_0^1 r \text{Tr}\left(V H_s^{r-1}\right) ds - \int_0^1 r \text{Tr}\left(V H_0^{r-1}\right) ds \\
&= \text{Tr}\left[rV \int_0^1 ds \int_a^b \lambda^{r-1}\{E_{H_s}(d\lambda) - E_{H_0}(d\lambda)\}\right].
\end{aligned}$$

It is easy to see that there exists $a, b \in \mathbb{R}$ ($-\infty < a < b < +\infty$) such that supp$E_{H_s}(.) \subseteq [a, b]$ for all $s \in [0, 1]$. By integrating by-parts and noting that $E_{H_s}(.) - E_{H_0}(.) = 0$ for $\lambda = a, b$, we have that

$$\begin{aligned}
&\text{Tr}\{p(H) - p(H_0) - D^{(1)}p(H_0) \bullet V\} \\
&\quad = \text{Tr}\left[rV \int_0^1 ds \left(\lambda^{r-1}\{E_{H_s}(\lambda) - E_{H_0}(\lambda)\} \mid_a^b - \int_a^b (r-1)\lambda^{r-2}\{E_{H_s}(\lambda) - E_{H_0}(\lambda)\}\right)\right] \\
&\quad = \int_a^b r(r-1)\lambda^{r-2} \left(\int_0^1 \text{Tr}\{V[E_{H_0}(\lambda) - E_{H_s}(\lambda)]ds\right) d\lambda \\
&\quad = \int_a^b p^{''}(\lambda)\eta(\lambda) d\lambda, \quad \text{where we have set} \quad \eta(\lambda) = \int_0^1 \text{Tr}\{V\,[E_{H_0}(\lambda) - E_{H_s}(\lambda)]\} ds.
\end{aligned}$$

To prove the positivity of $\eta(\lambda)$, we use the idea of double spectral integrals, introduced by Birman-Solomyak ([1], [2]). For fixed λ, and $\epsilon > 0$ define a smooth non-increasing function $\phi_{\epsilon,\lambda}$ such that

$$\phi_{\epsilon,\lambda}(\alpha) = \begin{cases} 0, & \text{if } \alpha \geq \lambda + \epsilon. \\ 1, & \text{if } a \leq \alpha \leq \lambda. \end{cases}$$

Therefore

$$\phi_{\epsilon,\lambda}(H_0)-\phi_{\epsilon,\lambda}(H_s)=\int_a^b \phi_{\epsilon,\lambda}(\alpha)E_{H_0}(d\alpha)-\int_a^b \phi_{\epsilon,\lambda}(\beta)E_{H_s}(d\beta)=\int_a^b\int_a^b \left[\phi_{\epsilon,\lambda}(\alpha)-\phi_{\epsilon,\lambda}(\beta)\right]E_{H_0}(d\alpha)E_{H_s}(d\beta)$$

$$=-s\int_a^b\int_a^b \frac{\phi_{\epsilon,\lambda}(\alpha)-\phi_{\epsilon,\lambda}(\beta)}{\alpha-\beta}E_{H_0}(d\alpha)VE_{H_s}(d\beta)=-s\int_{[a,b]\times[a,b]}\frac{\phi_{\epsilon,\lambda}(\alpha)-\phi_{\epsilon,\lambda}(\beta)}{\alpha-\beta}\mathcal{G}(d\alpha\times d\beta)V, \tag{2.1.6}$$

where $\mathcal{G}(\Delta\times\delta)X=E_{H_0}(\Delta)XE_{H_s}(\delta)$ ($X\in\mathcal{B}_2(\mathcal{H})$ and $\Delta\times\delta\subseteq\mathbb{R}\times\mathbb{R}$) extends to a spectral measure on $\mathbb{R}^2$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$ and its (weak-) total variation is less than or equal to $\|X\|_2$ (see section 1.3 of Chapter 1 for details). Thus

$$\begin{aligned}\operatorname{Tr}\{V\left[\phi_{\epsilon,\lambda}(H_0)-\phi_{\epsilon,\lambda}(H_s)\right]\} &= -s\int_a^b\int_a^b \frac{\phi_{\epsilon,\lambda}(\alpha)-\phi_{\epsilon,\lambda}(\beta)}{\alpha-\beta}\operatorname{Tr}\{VE_{H_0}(d\alpha)VE_{H_s}(d\beta)\}\\ &= -s\int_a^b\int_a^b \frac{\phi_{\epsilon,\lambda}(\alpha)-\phi_{\epsilon,\lambda}(\beta)}{\alpha-\beta}\langle V,\mathcal{G}(d\alpha\times d\beta)V\rangle_2\end{aligned} \tag{2.1.7}$$

Since by construction, $\phi_{\epsilon,\lambda}$ is a non-increasing function, the integrand in (2.1.7) is non-positive and hence

$$\operatorname{Tr}\{V\left[\phi_{\epsilon,\lambda}(H_0)-\phi_{\epsilon,\lambda}(H_s)\right]\}\geq 0 \qquad \forall\ \lambda,\epsilon>0. \tag{2.1.8}$$

Furthermore, for $f\in\mathcal{H}$,

$$\begin{aligned}\phi_{\epsilon,\lambda}(H_0)f=\int_a^b \phi_{\epsilon,\lambda}(\alpha)E_{H_0}(d\alpha)f &= \int_a^\lambda E_{H_0}(d\alpha)f+\int_\lambda^{\lambda+\epsilon}\phi_{\epsilon,\lambda}(\alpha)E_{H_0}(d\alpha)f\\ &= E_{H_0}(\lambda)f+\int_\lambda^{\lambda+\epsilon}\phi_{\epsilon,\lambda}(\alpha)E_{H_0}(d\alpha)f\end{aligned}$$

and hence

$$\left\|\left[\phi_{\epsilon,\lambda}(H_0)-E_{H_0}(\lambda)\right]f\right\|^2=\int_\lambda^{\lambda+\epsilon}|\phi_{\epsilon,\lambda}(\alpha)|^2\left\|E_{H_0}(d\alpha)f\right\|^2\leq\left\|\left[E_{H_0}(\lambda+\epsilon)-E_{H_0}(\lambda)\right]f\right\|^2,$$

which converges to 0 as $\epsilon\longrightarrow 0$ (spectral family is right continuous in our definition), proving that $\phi_{\epsilon,\lambda}(H_0)$ converges strongly to $E_{H_0}(\lambda)$. Similarly we can conclude that $\phi_{\epsilon,\lambda}(H_s)$ converges strongly to $E_{H_s}(\lambda)$ and hence by letting $\epsilon\downarrow 0$ in (2.1.8), we have

$$\operatorname{Tr}\{V\left[E_{H_0}(\lambda)-E_{H_s}(\lambda)\right]\}\geq 0 \quad \text{for} \quad 0\leq s\leq 1.$$

Therefore $\eta(\lambda)\geq 0$ for all $\lambda\in[a,b]$. The conclusion (2.1.3) is a consequence of the fact that

$$\|\eta\|_1=\int_a^b\eta(\lambda)d\lambda=\frac{1}{2}\int_a^b p''(\lambda)\eta(\lambda)d\lambda \quad (\text{where} \quad p(\lambda)=\lambda^2)$$

$$= \frac{1}{2}\text{Tr}\{H^2 - H_0^2 - D^{(1)}(H_0^2) \bullet V\} = \frac{1}{2}\text{Tr}\{(H_0+V)^2 - H_0^2 - (H_0V + VH_0)\} = \frac{1}{2}\|V\|_2^2.$$

(iii) For $X \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}t(H_0+X)} - \mathrm{e}^{\mathrm{i}tH_0} &= \int_0^1 d\alpha\, \frac{d}{d\alpha}\left[\mathrm{e}^{\mathrm{i}t\alpha(H_0+X)}.\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right] = \mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha(H_0+X)}X\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\\
&= \mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_0}X\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} + \mathrm{i}t\int_0^1 d\alpha\; \left[\mathrm{e}^{\mathrm{i}t\alpha(H_0+X)} - \mathrm{e}^{\mathrm{i}t\alpha H_0}\right]X\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\\
&= \mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_0}X\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} + (\mathrm{i}t)^2\int_0^1 \alpha d\alpha\int_0^1 d\beta\; \mathrm{e}^{\mathrm{i}t\alpha\beta(H_0+X)}X\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}X\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0},
\end{aligned}$$

and hence

$$\left\|\mathrm{e}^{\mathrm{i}t(H_0+X)} - \mathrm{e}^{\mathrm{i}tH_0} - \mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_0}X\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\| \le \frac{1}{2}t^2\|X\|^2,$$

proving that $D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet X = \mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_0}X\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}$. Similarly,

$$\begin{aligned}
\frac{\mathrm{e}^{\mathrm{i}tH_{s+h}} - \mathrm{e}^{\mathrm{i}tH_s}}{h} &= \frac{1}{h}\int_0^1 d\alpha\, \frac{d}{d\alpha}\left[\mathrm{e}^{\mathrm{i}t\alpha H_{s+h}}.\mathrm{e}^{\mathrm{i}t(1-\alpha)H_s}\right] = \mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_{s+h}}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_s}\\
&= \mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_s}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_s} + \mathrm{i}t\int_0^1 d\alpha\left[\mathrm{e}^{\mathrm{i}t\alpha H_{s+h}} - \mathrm{e}^{\mathrm{i}t\alpha H_s}\right]V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_s}\\
&= \mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_s}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_s} + (\mathrm{i}t)^2h\int_0^1 \alpha d\alpha\int_0^1 d\beta\; \mathrm{e}^{\mathrm{i}t\alpha\beta H_{s+h}}V\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_s}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_s},
\end{aligned}$$

and hence

$$\left\|\frac{\mathrm{e}^{\mathrm{i}tH_{s+h}} - \mathrm{e}^{\mathrm{i}tH_s}}{h} - \mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_s}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_s}\right\| \le \frac{1}{2}|h|t^2\|V\|^2,$$

proving that

$$\frac{d}{ds}\left(\mathrm{e}^{\mathrm{i}tH_s}\right) = \mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_s}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_s}.$$

Again by integrating by-parts and noting that $E_{H_s}(\lambda) - E_{H_0}(\lambda) = 0$ for $\lambda = a, b$ [where $[a,b] \supseteq \bigcup_s \text{supp} E_{H_s}(.)$], we have that

$$\begin{aligned}
&\text{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0} - D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V\} = \text{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0}\} - \text{Tr}\{D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V\}\\
&\quad= \int_0^1 ds\; \text{Tr}\frac{d}{ds}\left(\mathrm{e}^{\mathrm{i}tH_s}\right) - \int_0^1 ds\; \text{Tr}\{D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V\}\\
&\quad= \int_0^1 ds\; \text{Tr}\left(\mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_s}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_s}\right) - \int_0^1 ds\; \text{Tr}\left(\mathrm{i}t\int_0^1 d\alpha\; \mathrm{e}^{\mathrm{i}t\alpha H_0}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right)
\end{aligned}$$

$$= \mathrm{i}t \int_0^1 ds \int_0^1 d\alpha \,\mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}t\alpha H_s} V \mathrm{e}^{\mathrm{i}t(1-\alpha)H_s}\right) - \mathrm{i}t \int_0^1 ds \int_0^1 d\alpha \,\mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}t\alpha H_0} V \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right)$$
$$= \mathrm{i}t \int_0^1 ds \,\mathrm{Tr}\{V \left(\mathrm{e}^{\mathrm{i}tH_s} - \mathrm{e}^{\mathrm{i}tH_0}\right)\} = \mathrm{i}t \int_0^1 ds \int_a^b \mathrm{e}^{\mathrm{i}t\lambda} \,\mathrm{Tr}\{V[E_{H_s}(d\lambda) - E_{H_0}(d\lambda)]\}$$
$$= \mathrm{i}t \int_0^1 ds \left[\mathrm{e}^{\mathrm{i}t\lambda} \,\mathrm{Tr}\{V[E_{H_s}(\lambda) - E_{H_0}(\lambda)]\}|_{\lambda=a}^{b} - \mathrm{i}t \int_a^b \mathrm{e}^{\mathrm{i}t\lambda} \,\mathrm{Tr}\{V[E_{H_s}(\lambda) - E_{H_0}(\lambda)]\} d\lambda \right]$$
$$= (\mathrm{i}t)^2 \int_a^b \mathrm{e}^{\mathrm{i}t\lambda} \, d\lambda \int_0^1 ds \,\mathrm{Tr}\{V[E_{H_0}(\lambda) - E_{H_s}(\lambda)]\} = (\mathrm{i}t)^2 \int_a^b \mathrm{e}^{\mathrm{i}t\lambda} \eta(\lambda) d\lambda.$$

□

2.2 Reduction to finite dimension

We begin with a proposition collecting some results, following from the Weyl-von Neumann type construction (see [4] , [6]).

Proposition 2.2.1. *Let A be a self-adjoint operator (possibly unbounded) in a separable infinite dimensional Hilbert space $\mathcal{H}$ and let $\{f_l\}_{1\leq l\leq L}$ be a set of normalized vectors in $\mathcal{H}$ and $\epsilon > 0$.*

(i) Then there exists a finite rank projection P such that $\|(I-P)f_l\| < \epsilon$ for $1 \leq l \leq L$.

(ii) Furthermore, $(I-P)AP \in \mathcal{B}_2(\mathcal{H})$, $\|(I-P)AP\|_2 < \epsilon$ and $\|(I-P)\mathrm{e}^{\mathrm{i}tA}P\|_2 < \epsilon$ uniformly for t with $|t| \leq T$.

Proof. Let $E_A(.)$ be the spectral measure associated with the self-adjoint operator A, and choose $a_l > 0$ such that

$$\| \left[I - E_A\left((-a_l, a_l]\right)\right] f_l \| < \epsilon \quad \text{for} \quad 1 \leq l \leq L.$$

If we set $a = \max\{a_l : \quad 1 \leq l \leq L\}$, then

$$\| \left[I - E_A\left((-a, a]\right)\right] f_l \| \leq \| \left[I - E_A\left((-a_l, a_l]\right)\right] f_l \| < \epsilon \quad \text{for} \quad 1 \leq l \leq L.$$

Again as in the proof of Theorem 1.4.3, set $E_k = E_A\left(\left(\frac{2k-2-n}{n}a, \frac{2k-n}{n}a\right]\right)$ for each positive integer n and $1 \leq k \leq n$. We also set for $1 \leq k \leq n$ and $1 \leq l \leq L$,

$$g_{kl} = \begin{cases} \frac{E_k f_l}{\|E_k f_l\|}, & \text{if } E_k f_l \neq 0. \\ 0, & \text{if } E_k f_l = 0. \end{cases}$$

Let P be the orthogonal projection onto the subspace generated by $\{g_{kl} : 1 \leq k \leq n\,;\, 1 \leq l \leq L\}$ and hence dim$P\mathcal{H} \leq nL$. For fixed k $(1 \leq k \leq n)$, consider the set $\{g_{kl} : 1 \leq l \leq L\}$. By Gram - Schmidt process, let $\{h_{kl} : 1 \leq l \leq m(k) \leq L\}$ be the orthonormal set made out of $\{g_{kl} : 1 \leq l \leq L\}$. Hence $\{h_{kl} : 1 \leq k \leq n\,;\, 1 \leq l \leq m(k) \leq L\}$ is an orthonormal basis for $P\mathcal{H}$ and span$\{h_{kl} : 1 \leq k \leq n\,;\, 1 \leq l \leq m(k) \leq L\}$ = span$\{g_{kl} : 1 \leq k \leq n\,;\, 1 \leq l \leq L\}$. By definition for each fixed k $(1 \leq k \leq n)$, $\{g_{kl} : 1 \leq l \leq L\} \subseteq E_k\mathcal{H}$ and since $E_k\mathcal{H}$ is a linear space we have $\{h_{kl} : 1 \leq l \leq m(k) \leq L\} \subseteq E_k\mathcal{H}$. Clearly $\{g_{kl} : 1 \leq k \leq n\,;\, 1 \leq l \leq L\} \subseteq$ Dom$A$ and since Dom$A$ is a linear manifold we have

$$\text{span}\{h_{kl} : 1 \leq k \leq n\,;\, 1 \leq l \leq m(k) \leq L\} = \text{span}\{g_{kl} : 1 \leq k \leq n\,;\, 1 \leq l \leq L\} \subseteq \text{Dom}A.$$

Moreover $Ah_{kl}\,,\ PAh_{kl} \in E_k\mathcal{H}$ for each k and l, since $h_{kl} \in E_k\mathcal{H}$ and E_k commutes with A for each k and l. A simple calculation as in the proof of Theorem 1.4.3 in Chapter 1, shows that for $\lambda_k = \frac{2k-n-1}{n}a$,

$$\|\,(A-\lambda_k)\,h_{kl}\|^2 \leq \left(\frac{a}{n}\right)^2 \quad \text{for} \quad 1 \leq l \leq L, \quad \text{and therefore}$$

using Cauchy-Schwartz inequality we conclude that for $u \in \mathcal{H}$,

$$\|(I-P)APu\|^2 = \left\|\sum_{k=1}^{n}\sum_{l=1}^{m(k)}\langle u, h_{kl}\rangle(I-P)Ah_{kl}\right\|^2 = \left\|\sum_{k=1}^{n}\sum_{l=1}^{m(k)}\langle u, h_{kl}\rangle(I-P)(A-\lambda_k)h_{kl}\right\|^2$$

$$= \sum_{k=1}^{n}\left\|\sum_{l=1}^{m(k)}\langle u, h_{kl}\rangle(I-P)(A-\lambda_k)h_{kl}\right\|^2 \leq \sum_{k=1}^{n}\left(\sum_{l=1}^{m(k)}|\langle u, h_{kl}\rangle|\|(I-P)(A-\lambda_k)h_{kl}\|\right)^2$$

$$\leq \left(\frac{a}{n}\right)^2\sum_{k=1}^{n}\left(\sum_{l=1}^{m(k)}|\langle u, h_{kl}\rangle|\right)^2 \leq \left(\frac{a}{n}\right)^2\sum_{k=1}^{n}\left[\left(\sum_{l=1}^{m(k)}1^2\right)^{\frac{1}{2}}\left(\sum_{l=1}^{m(k)}|\langle u, h_{kl}\rangle|^2\right)^{\frac{1}{2}}\right]^2$$

$$= \left(\frac{a}{n}\right)^2\sum_{k=1}^{n}m(k)\sum_{l=1}^{m(k)}|\langle u, h_{kl}\rangle|^2 \leq \left(\frac{a}{n}\right)^2 L\sum_{k=1}^{n}\sum_{l=1}^{m(k)}|\langle u, h_{kl}\rangle|^2 \leq \left(\frac{a}{n}\right)^2 L\,\|u\|^2.$$

The operators $PA(I-P)$ and $(I-P)AP$ are finite rank operators with rank less than or equal to nL. Hence, using the above estimate we get that

$$\|(I-P)AP\|_2 = \|PA(I-P)\|_2 \leq \sqrt{\text{dim}(P)}\;\|(I-P)AP\| \leq \sqrt{nL}\;\left(\frac{a}{n}\right)\;\sqrt{L} = L\;\left(\frac{a}{\sqrt{n}}\right).$$

Thus again by the same calculation as in the proof of Theorem 1.4.3 in Chapter 1, it follows that

$$\alpha(t) \equiv \left\|(I-P)\mathrm{e}^{\mathrm{i}tA}P\right\|_2 \;\leq\; \frac{\left(T\;L\;a\;\mathrm{e}^{a\sqrt{L}t}\right)}{\sqrt{n}} \;\leq\; \frac{\left(T\;L\;a\;\mathrm{e}^{a\sqrt{L}T}\right)}{\sqrt{n}}.$$

Since $(I-P)F\left((-a,a]\right)f_l = 0$ for $1 \leq l \leq L$,

$$\|(I-P)f_l\| = \|(I-P)\left[I - F\left((-a,a]\right)\right] f_l\| \ \leq \|\left[I-F\left((-a,a]\right)\right] f_l\| < \epsilon \quad \text{for} \quad 1 \leq l \leq L.$$

The proof concludes by choosing n sufficiently large. □

Lemma 2.2.2. *Let H and H_0 be two self-adjoint operators in a separable infinite dimensional Hilbert space $\mathcal{H}$ such that $H - H_0 \equiv V \in \mathcal{B}_2(\mathcal{H})$. Then given $\epsilon > 0$, there exists a projection P of finite rank such that for all t with $|t| \leq T$,*

(i) $\|(I-P)H_0P\|_2 < \epsilon$, $\qquad \|(I-P)\mathrm{e}^{\mathrm{i}tH_0}P\|_2 < \epsilon$,

(ii) $\|(I-P)V\|_2 < 2\epsilon$, $\qquad \|(I-P)HP\|_2 < 3\epsilon$.

Proof. Let $V = \sum\limits_{l=1}^{\infty} \tau_l |f_l\rangle\langle f_l|$ be the canonical form of V with $\sum\limits_{l=1}^{\infty} \tau_l^2 < \infty$ and choose L in $V_L \equiv \sum\limits_{l=1}^{L} \tau_l |f_l\rangle\langle f_l|$ so that $\|V - V_L\|_2 = \sqrt{\sum\limits_{l=L+1}^{\infty} \tau_l^2} < \epsilon$ and $\epsilon' = \min\{\epsilon, \frac{\epsilon}{\sum_{l=1}^{L} |\tau_l|}\} > 0$. Next, we apply Proposition 2.2.1 with $A = H_0,\ \{f_1, f_2, \ldots, f_L\}$ and ϵ' in place of ϵ. Hence we get a projection P of finite rank in $\mathcal{H}$ such that

$$\|(I-P)f_l\| < \epsilon' < \epsilon \ \text{ for } \ 1 \leq l \leq L, \ \ \|(I-P)H_0P\|_2 < \epsilon' < \epsilon \ \text{ and } \ \|(I-P)\mathrm{e}^{\mathrm{i}tH_0}P\|_2 < \epsilon' < \epsilon,$$

uniformly for t with $|t| \leq T$. For *(ii)* we note that

$$\begin{aligned}
&\|(I-P)V\|_2 = \|(I-P)(V-V_L) + (I-P)V_L\|_2 \leq \|(I-P)(V-V_L)\|_2 + \|(I-P)V_L\|_2 \\
&\leq \|V - V_L\|_2 + \|(I-P)V_L\|_2 < \epsilon + \left\| \sum_{l=1}^{L} \tau_l |(I-P)f_l\rangle\langle f_l| \right\|_2 < \epsilon + \epsilon' \left(\sum_{l=1}^{L} |\tau_l| \right) < 2\epsilon
\end{aligned}$$

and therefore

$$\|(I-P)HP\|_2 \leq \|(I-P)H_0P\|_2 + \|(I-P)VP\|_2 < 3\epsilon.$$

□

Remark 2.2.3. We can reformulate the statement of Lemma 2.2.2 by saying that there exists a sequence $\{P_n\}$ of finite rank projections in $\mathcal{H}$ such that

$$\|(I-P_n)H_0P_n\|_2, \ \|(I-P_n)\mathrm{e}^{\mathrm{i}tH_0}P_n\|_2, \ \|(I-P_n)V\|_2, \ \|(I-P_n)HP_n\|_2 \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

It may also be noted that $\{P_n\}$ does not necessarily converge strongly to I.

The next two theorems show how Lemma 2.2.2 can be used to reduce the relevant problem into a finite dimensional one, in the cases when the self-adjoint pair (H_0, H) are bounded and unbounded respectively.

Theorem 2.2.4. *Let H and H_0 be two bounded self-adjoint operators in a separable infinite dimensional Hilbert space $\mathcal{H}$ such that $H - H_0 \equiv V \in \mathcal{B}_2(\mathcal{H})$. Then there exists a sequence $\{P_n\}$ of finite rank projections in $\mathcal{H}$ such that*

$$\begin{aligned}&\mathrm{Tr}\{p(H) - p(H_0) - D^{(1)}p(H_0) \bullet V\}\\ &= \lim_{n\to\infty} \mathrm{Tr}\{P_n\left[p(P_nHP_n) - p(P_nH_0P_n) - D^{(1)}p(P_nH_0P_n) \bullet P_nVP_n\right]P_n\},\end{aligned} \tag{2.2.1}$$

where $p(.)$ is a polynomial.

Proof. It will be sufficient to prove the theorem for $p(\lambda) = \lambda^r$. Note that for $r = 0$ or 1, both sides of (2.2.1) are identically zero. Using the sequence $\{P_n\}$ of finite rank projections as obtained in Lemma 2.2.2 and using an expression similar to (2.1.5) in $\mathcal{B}(\mathcal{H})$, we have that

$$\begin{aligned}
&\mathrm{Tr}\{\left[p(H) - p(H_0) - D^{(1)}p(H_0) \bullet V\right]\\
&\qquad - P_n\left[p(P_nHP_n) - p(P_nH_0P_n) - D^{(1)}p(P_nH_0P_n) \bullet P_nVP_n\right]P_n\}\\
&= \mathrm{Tr}\{\left[H^r - H_0^r - D^{(1)}(H_0^r) \bullet V\right] - P_n\left[(P_nHP_n)^r - (P_nH_0P_n)^r - D^{(1)}((P_nH_0P_n)^r) \bullet P_nVP_n\right]P_n\}\\
&= \mathrm{Tr}\{\left[\sum_{j=0}^{r-1} H^{r-j-1}VH_0^j - \sum_{j=0}^{r-1} H_0^{r-j-1}VH_0^j\right]\\
&\qquad - P_n\left[\sum_{j=0}^{r-1}(P_nHP_n)^{r-j-1}(P_nVP_n)(P_nH_0P_n)^j - \sum_{j=0}^{r-1}(P_nH_0P_n)^{r-j-1}(P_nVP_n)(P_nH_0P_n)^j\right]P_n\}\\
&= \mathrm{Tr}\{\left[\sum_{j=0}^{r-1}\left(H^{r-j-1} - H_0^{r-j-1}\right)VH_0^j\right]\\
&\qquad - P_n\left[\sum_{j=0}^{r-1}[(P_nHP_n)^{r-j-1} - (P_nH_0P_n)^{r-j-1}](P_nVP_n)(P_nH_0P_n)^j\right]P_n\}\\
&= \mathrm{Tr}\{\sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2} H^{r-j-k-2}VH_0^kVH_0^j
\end{aligned}$$

$$-\sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2} P_n(P_nHP_n)^{r-j-k-2}(P_nVP_n)(P_nH_0P_n)^k(P_nVP_n)(P_nH_0P_n)^jP_n\}$$

$$=\sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2} \text{Tr}\{H^{r-j-k-2}VH_0^kVH_0^j$$

$$-P_n(P_nHP_n)^{r-j-k-2}(P_nVP_n)(P_nH_0P_n)^k(P_nVP_n)(P_nH_0P_n)^jP_n\}$$

$$=\sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2} \text{Tr}\{\left[H^{r-j-k-2}P_n-(P_nHP_n)^{r-j-k-2}\right]P_nVH_0^kVH_0^j$$

$$+H^{r-j-k-2}P_n^{\perp}VH_0^kVH_0^j+(P_nHP_n)^{r-j-k-2}P_nVP_n^{\perp}H_0^kVH_0^j$$

$$+(P_nHP_n)^{r-j-k-2}(P_nVP_n)\left[P_nH_0^k-(P_nH_0P_n)^k\right]VH_0^j$$

$$+(P_nHP_n)^{r-j-k-2}(P_nVP_n)(P_nH_0P_n)^kP_nVP_n^{\perp}H_0^j$$

$$+(P_nHP_n)^{r-j-k-2}(P_nVP_n)(P_nH_0P_n)^k(P_nVP_n)\left[P_nH_0^j-(P_nH_0P_n)^j\right]\}. \qquad (2.2.2)$$

Using the results of Lemma 2.2.2, the first term of the expression (2.2.2) leads to

$$\left\|\left[H^{r-j-k-2}-(P_nHP_n)^{r-j-k-2}\right]P_n\right\|_2=\left\|\sum_{l=0}^{r-j-k-3}H^{r-j-k-l-3}(P_n^{\perp}HP_n)(P_nHP_n)^l\right\|_2$$
$$\leq(r-j-k-2)\|H\|^{r-j-k-3}\left\|P_n^{\perp}HP_n\right\|_2\leq r(1+\|H\|)^r\left\|P_n^{\perp}HP_n\right\|_2,$$

which converges to 0 as $n\longrightarrow\infty$ and hence

$$\left\|\left[H^{r-j-k-2}P_n-(P_nHP_n)^{r-j-k-2}\right]P_nVH_0^kVH_0^j\right\|_1$$
$$\leq\left\|\left[H^{r-j-k-2}-(P_nHP_n)^{r-j-k-2}\right]P_n\right\|_2\|VH_0^kVH_0^j\|_2$$
$$\leq r(1+\|H\|)^r\left\|P_n^{\perp}HP_n\right\|_2\|V\|_2^2\|H_0\|^{k+j},$$

which converges to 0 as $n\longrightarrow\infty$. Similarly for the fourth term in (2.2.2), we note that,

$$\left\|P_n\left[H_0^k-(P_nH_0P_n)^k\right]\right\|_2=\left\|\sum_{l=0}^{k-1}P_nH_0^{k-l-1}(P_n^{\perp}H_0P_n)(P_nH_0P_n)^l\right\|_2$$
$$\leq k\|H_0\|^{k-1}\left\|P_n^{\perp}H_0P_n\right\|_2\leq k(1+\|H_0\|)^k\left\|P_n^{\perp}H_0P_n\right\|_2,$$

which converges to 0 as $n \longrightarrow \infty$ by Lemma 2.2.2 and hence

$$\begin{aligned}
&\left\| \, (P_nHP_n)^{r-j-k-2}(P_nVP_n)\left[P_nH_0^k-(P_nH_0P_n)^k\right]VH_0^j\right\|_1 \\
&\qquad \leq \| \, (P_nHP_n)^{r-j-k-2}(P_nVP_n)\|_2 \left\|P_n\left[H_0^k-(P_nH_0P_n)^k\right]VH_0^j\right\|_2 \\
&\qquad \leq k(1+\|H_0\|)^k \left\|P_n^{\perp}H_0P_n\right\|_2 \|V\|_2^2\|H\|^{r-j-k-2}\|H_0\|^j,
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ and for the sixth term we have

$$\begin{aligned}
&\left\| \, (P_nHP_n)^{r-j-k-2}(P_nVP_n)(P_nH_0P_n)^k(P_nVP_n)\left[P_nH_0^j-(P_nH_0P_n)^j\right]\right\|_1 \\
&\quad \leq \| \, (P_nHP_n)^{r-j-k-2}(P_nVP_n)(P_nH_0P_n)^k(P_nVP_n)\|_2 \left\|P_n\left[H_0^j-(P_nH_0P_n)^j\right]\right\|_2 \\
&\quad \leq j(1+\|H_0\|)^j \left\|P_n^{\perp}H_0P_n\right\|_2 \|V\|_2^2\|H\|^{r-j-k-2}\|H_0\|^k,
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ by Lemma 2.2.2 . For the second term in (2.2.2) we have

$$\left\| \, H^{r-j-k-2}P_n^{\perp}VH_0^kVH_0^j\right\|_1 \leq \| \, H^{r-j-k-2}P_n^{\perp}V\|_2\|H_0^kVH_0^j\|_2 \leq \|P_n^{\perp}V\|_2\|H\|^{r-j-k-2}\|H_0\|^{j+k}\|V\|_2,$$

which converges to 0 as $n \longrightarrow \infty$ since by Lemma 2.2.2, $\left\|P_n^{\perp}V\right\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$. Similarly for the third term in (2.2.2) we have

$$\begin{aligned}
\|(P_nHP_n)^{r-j-k-2}P_nVP_n^{\perp}H_0^kVH_0^j\|_1 &\leq \|(P_nHP_n)^{r-j-k-2}P_nVP_n^{\perp}\|_2\|H_0^kVH_0^j\|_2 \\
&\leq \|P_nVP_n^{\perp}\|_2\|H\|^{r-j-k-2}\|V\|_2\|H_0\|^{k+j},
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ by Lemma 2.2.2 and for the fifth term in (2.2.2) we have the following estimate

$$\begin{aligned}
&\|(P_nHP_n)^{r-j-k-2}(P_nVP_n)(P_nH_0P_n)^kP_nVP_n^{\perp}H_0^j\|_1 \\
&\qquad\qquad \leq \|\|(P_nHP_n)^{r-j-k-2}(P_nVP_n)(P_nH_0P_n)^k\|_2\|P_nVP_n^{\perp}H_0^j\|_2 \\
&\qquad\qquad \leq \|P_nVP_n^{\perp}\|_2\|H\|^{r-j-k-2}\|V\|_2\|H_0\|^{k+j},
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ by Lemma 2.2.2. Therefore the right hand side of (2.2.2) converges to 0 as $n \longrightarrow \infty$ and hence the result follows.

□

Theorem 2.2.5. *Let H and H_0 be two self-adjoint operators (not necessarily bounded) in a separable infinite dimensional Hilbert space $\mathcal{H}$ such that $H - H_0 \equiv V \in \mathcal{B}_2(\mathcal{H})$. Then there exists a sequence $\{P_n\}$ of finite rank projections in $\mathcal{H}$ such that for any $T > 0$*

$$\begin{aligned}
&\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH}-\mathrm{e}^{\mathrm{i}tH_0}-D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0})\bullet V\} \\
&\qquad = \lim_{n\to\infty}\mathrm{Tr}\{P_n\left[\mathrm{e}^{\mathrm{i}tP_nHP_n}-\mathrm{e}^{\mathrm{i}tP_nH_0P_n}-D^{(1)}(\mathrm{e}^{\mathrm{i}tP_nH_0P_n})\bullet P_nVP_n\right]P_n\},
\end{aligned}$$

uniformly for all t with $|t| \leq T$.

Proof. As in the case of a finite dimensional Hilbert space, for $X \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned} \mathrm{e}^{\mathrm{i}t(H_0+X)} - \mathrm{e}^{\mathrm{i}tH_0} &= \int_0^1 d\alpha \, \frac{d}{d\alpha} \left[\mathrm{e}^{\mathrm{i}t\alpha(H_0+X)} . \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} \right] = \mathrm{i}t \int_0^1 d\alpha \, \mathrm{e}^{\mathrm{i}t\alpha(H_0+X)} X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} \\ &= \mathrm{i}t \int_0^1 d\alpha \, \mathrm{e}^{\mathrm{i}t\alpha H_0} X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} + \mathrm{i}t \int_0^1 d\alpha \, \left[\mathrm{e}^{\mathrm{i}t\alpha(H_0+X)} - \mathrm{e}^{\mathrm{i}t\alpha H_0} \right] X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} \\ &= \mathrm{i}t \int_0^1 d\alpha \, \mathrm{e}^{\mathrm{i}t\alpha H_0} X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} + (\mathrm{i}t)^2 \int_0^1 \alpha d\alpha \int_0^1 d\beta \, \mathrm{e}^{\mathrm{i}t\alpha\beta(H_0+X)} X \mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0} X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} \end{aligned} \tag{2.2.3}$$

on $\mathrm{Dom}(H_0)$ and hence

$$\left\| \mathrm{e}^{\mathrm{i}t(H_0+X)} - \mathrm{e}^{\mathrm{i}tH_0} - \mathrm{i}t \int_0^1 d\alpha \, \mathrm{e}^{\mathrm{i}t\alpha H_0} X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} \right\| \leq \frac{1}{2} t^2 \|X\|^2,$$

proving that $D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet X = \mathrm{i}t \int_0^1 d\alpha \, \mathrm{e}^{\mathrm{i}t\alpha H_0} X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}$. Since $\mathbb{R} \ni s \longrightarrow \mathrm{e}^{\mathrm{i}sH}, \mathrm{e}^{\mathrm{i}sH_0}$ are strongly continuous and since $V \in \mathcal{B}_2(\mathcal{H})$, it follows that $\alpha \longrightarrow \mathrm{e}^{\mathrm{i}t\alpha H_0} V \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}$ is $\mathcal{B}_2$-continuous, then we have

$$D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V = \mathrm{i}t \int_0^1 \mathrm{e}^{\mathrm{i}t\alpha H_0} V \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} d\alpha \in \mathcal{B}_2(\mathcal{H}).$$

Again from the above calculations, we have

$$\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0} - D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V = (\mathrm{i}t)^2 \int_0^1 \alpha d\alpha \int_0^1 d\beta \, \mathrm{e}^{\mathrm{i}t\alpha\beta H} V \mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0} V \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} \tag{2.2.4}$$

and since $\beta \longrightarrow \mathrm{e}^{\mathrm{i}t\alpha\beta(H_0+V)} V \mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0} V \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}$ is $\mathcal{B}_1$-continuous, the integral in right hand side of (2.2.4) exits in $\mathcal{B}_1(\mathcal{H})$ and hence $\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0} - D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V \in \mathcal{B}_1(\mathcal{H})$ and therefore by Fubini's theorem,

$$\begin{aligned} &\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0} - D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V\} \\ &\qquad = (\mathrm{i}t)^2 \int_0^1 \alpha d\alpha \int_0^1 d\beta \, \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}t\alpha\beta H} V \mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0} V \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\}. \end{aligned}$$

Thus,

$$\begin{aligned}
&\mathrm{Tr}\{e^{itH} - e^{itH_0} - D^{(1)}(e^{itH_0}) \bullet V\} \\
&\qquad - \mathrm{Tr}\{P_n \left[e^{itP_nHP_n} - e^{itP_nH_0P_n} - D^{(1)}(e^{itP_nH_0P_n}) \bullet P_nVP_n\right] P_n\} \\
&= (it)^2 \int_0^1 \alpha d\alpha \int_0^1 d\beta \; \mathrm{Tr}\{e^{it\alpha\beta H} V e^{it\alpha(1-\beta)H_0} V e^{it(1-\alpha)H_0} \\
&\qquad - (it)^2 \int_0^1 \alpha d\alpha \int_0^1 d\beta \; \mathrm{Tr}\{P_n e^{it\alpha\beta P_nHP_n} P_nVP_n e^{it\alpha(1-\beta)P_nH_0P_n} P_nVP_n e^{it(1-\alpha)P_nH_0P_n} P_n\} \\
&= (it)^2 \int_0^1 \alpha d\alpha \int_0^1 d\beta \; \mathrm{Tr}\{e^{it\alpha\beta H} V e^{it\alpha(1-\beta)H_0} V e^{it(1-\alpha)H_0} \\
&\qquad\qquad - P_n e^{it\alpha\beta P_nHP_n} P_nVP_n e^{it\alpha(1-\beta)P_nH_0P_n} P_nVP_n e^{it(1-\alpha)P_nH_0P_n} P_n\} \\
&= (it)^2 \int_0^1 \alpha d\alpha \int_0^1 d\beta \; \mathrm{Tr}\{\left[e^{it\alpha\beta H} - e^{it\alpha\beta P_nHP_n}\right] P_nV e^{it\alpha(1-\beta)H_0} V e^{it(1-\alpha)H_0} \\
&\qquad\qquad + \; e^{it\alpha\beta H} P_n^{\perp} V e^{it\alpha(1-\beta)H_0} V e^{it(1-\alpha)H_0} \\
&\qquad\qquad + P_n e^{it\alpha\beta P_nHP_n} P_nVP_n^{\perp} e^{it\alpha(1-\beta)H_0} V e^{it(1-\alpha)H_0} \\
&\qquad\qquad + \; P_n e^{it\alpha\beta P_nHP_n} P_nVP_n \left[e^{it\alpha(1-\beta)H_0} - e^{it\alpha(1-\beta)P_nH_0P_n}\right] P_nV e^{it(1-\alpha)H_0} \\
&\qquad\qquad + \; P_n e^{it\alpha\beta P_nHP_n} P_nVP_n e^{it\alpha(1-\beta)H_0} P_n^{\perp} V e^{it(1-\alpha)H_0} \\
&\qquad\qquad + \; P_n e^{it\alpha\beta P_nHP_n} P_nVP_n e^{it\alpha(1-\beta)P_nH_0P_n} P_nVP_n^{\perp} e^{it(1-\alpha)H_0} \\
&\qquad\qquad + \; P_n e^{it\alpha\beta P_nHP_n} P_nVP_n e^{it\alpha(1-\beta)P_nH_0P_n} P_nVP_n \left[e^{it(1-\alpha)H_0} - e^{it(1-\alpha)P_nH_0P_n}\right]\},
\end{aligned} \tag{2.2.5}$$

where $\{P_n\}$ is a sequence of finite rank projections as obtained in Lemma 2.2.2.

In the first term of the expression (2.2.5) :

$$\begin{aligned}
&\left\|\left[\mathrm{e}^{\mathrm{i}t\alpha\beta H}-\mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}\right]P_n\right\|_2\\
&\qquad=\left\|\left[\int_0^1 d\gamma\,\frac{d}{d\gamma}\left(\mathrm{e}^{\mathrm{i}t\alpha\beta\gamma H}.\mathrm{e}^{\mathrm{i}t\alpha\beta(1-\gamma)P_nHP_n}\right)\right]P_n\right\|_2\\
&\qquad\leq\left\|\mathrm{i}t\alpha\beta\int_0^1 d\gamma\mathrm{e}^{\mathrm{i}t\alpha\beta\gamma H}P_n^{\perp}HP_n\mathrm{e}^{\mathrm{i}t\alpha\beta(1-\gamma)P_nHP_n}P_n\right\|_2\leq\ T\left\|P_n^{\perp}HP_n\right\|_2,
\end{aligned}$$

which converges to 0 as $n\longrightarrow\infty$, uniformly for $|t|\leq T$ by Remark 2.2.3 and hence

$$\begin{aligned}
&\left\|\left[\mathrm{e}^{\mathrm{i}t\alpha\beta H}-\mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}\right]P_nV\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1\\
&\qquad\leq\left\|\left[\mathrm{e}^{\mathrm{i}t\alpha\beta H}-\mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}\right]P_n\right\|_2\|V\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\|_2\\
&\qquad\leq\ T\left\|P_n^{\perp}HP_n\right\|_2\|V\|_2^2,
\end{aligned}$$

which converges to 0 as $n\longrightarrow\infty$, uniformly for $|t|\leq T$. Similarly for the fourth term in (2.2.5), we note that,

$$\begin{aligned}
&\left\|\left[\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}-\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)P_nH_0P_n}\right]P_n\right\|_2\\
&\qquad=\left\|\left[\int_0^1 d\gamma\,\frac{d}{d\gamma}\left(\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)\gamma H_0}.\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)(1-\gamma)P_nH_0P_n}\right)\right]P_n\right\|_2\\
&\qquad\leq\left\|\mathrm{i}t\alpha(1-\beta)\int_0^1 d\gamma\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)\gamma H_0}P_n^{\perp}H_0P_n\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)(1-\gamma)P_nH_0P_n}P_n\right\|_2\leq\ T\left\|P_n^{\perp}H_0P_n\right\|_2,
\end{aligned}$$

which converges to 0 as $n\longrightarrow\infty$, uniformly for $|t|\leq T$ by Remark 2.2.3 and hence

$$\begin{aligned}
&\left\|P_n\mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}P_nVP_n\left[\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}-\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)P_nH_0P_n}\right]P_nV\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1\\
&\qquad\leq\left\|P_n\mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}P_nVP_n\right\|_2\|\left[\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}-\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)P_nH_0P_n}\right]P_nV\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\|_2\\
&\qquad\leq\ T\left\|P_n^{\perp}H_0P_n\right\|_2\|V\|_2^2,
\end{aligned}$$

which converges to 0 as $n\longrightarrow\infty$, uniformly for $|t|\leq T$ and for the seventh term in (2.2.5) we have

$$\begin{aligned}
&\left\|P_n\mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}P_nVP_n\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)P_nH_0P_n}P_nVP_n\left[\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}-\mathrm{e}^{\mathrm{i}t(1-\alpha)P_nH_0P_n}\right]\right\|_1\\
&\quad\leq\left\|P_n\mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}P_nVP_n\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)P_nH_0P_n}\right\|_2\|P_nVP_n\left[\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}-\mathrm{e}^{\mathrm{i}t(1-\alpha)P_nH_0P_n}\right]\|_2\\
&\quad\leq\ T\left\|P_n^{\perp}H_0P_n\right\|_2\|V\|_2^2,
\end{aligned}$$

which converges to 0 as $n\longrightarrow\infty$, uniformly for $|t|\leq T$ by Remark 2.2.3. For the second term in (2.2.5) we have

$$\begin{aligned}
&\left\|\mathrm{e}^{\mathrm{i}t\alpha\beta H}P_n^{\perp}V\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1\\
&\qquad\leq\left\|\mathrm{e}^{\mathrm{i}t\alpha\beta H}P_n^{\perp}V\right\|_2\left\|\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_2\leq\ \left\|P_n^{\perp}V\right\|_2\|V\|_2,
\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ by Remark 2.2.3. Similarly for the third term in (2.2.5) we have

$$\begin{aligned}&\left\|P_n \mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}P_nVP_n^{\perp}\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1\\ &\qquad\leq \left\|P_n \mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}P_nVP_n^{\perp}\right\|_2\left\|\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_2 \leq \left\|P_n^{\perp}VP_n\right\|_2\|V\|_2,\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ by Remark 2.2.3 and for the fifth term in (2.2.5) we have

$$\begin{aligned}&\left\|P_n \mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}P_nVP_n\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}P_n^{\perp}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1\\ &\qquad\leq \left\|P_n \mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}P_nVP_n\right\|_2\left\|\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}P_n^{\perp}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_2 \leq \left\|P_n^{\perp}V\right\|_2\|V\|_2,\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ by Remark 2.2.3. Finally for the sixth term in (2.2.5) we have the following estimate

$$\begin{aligned}&\left\|P_n \mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}P_nVP_n\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)P_nH_0P_n}P_nVP_n^{\perp}\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_1\\ &\qquad\leq \left\|P_n \mathrm{e}^{\mathrm{i}t\alpha\beta P_nHP_n}P_nVP_n\right\|_2\left\|\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)P_nH_0P_n}P_nVP_n^{\perp}\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\right\|_2 \leq \left\|P_n^{\perp}VP_n\right\|_2\|V\|_2,\end{aligned}$$

which converges to 0 as $n \longrightarrow \infty$ by Remark 2.2.3. Therefore the right hand side of (2.2.5) converges to 0 as $n \longrightarrow \infty$ and hence the result follows.

□

2.3 Koplienko formula for both bounded and unbounded cases

In this section, we derive the trace formulas for both bounded and unbounded self-adjoint pairs (H_0, H).

Theorem 2.3.1. *Let H and H_0 be two bounded self-adjoint operators in an infinite dimensional separable Hilbert space $\mathcal{H}$ such that $H - H_0 \equiv V \in \mathcal{B}_2(\mathcal{H})$. Then for any polynomial $p(.)$, $p(H) - p(H_0) - D^{(1)}p(H_0) \bullet V \in \mathcal{B}_1(\mathcal{H})$ and there exists a unique non-negative $L^1(\mathbb{R})$-function η supported on $[a, b]$ such that*

$$Tr\{p(H) - p(H_0) - D^{(1)}p(H_0) \bullet V\} = \int_a^b p''(\lambda)\eta(\lambda)d\lambda,$$

where, $a = \inf \sigma(H_0) - \|V\|,\ b = \sup \sigma(H_0) + \|V\|$. Furthermore $\int_a^b |\eta(\lambda)|d\lambda = \frac{1}{2}\|V\|_2^2$.

Proof. By Theorem 2.2.4 and Theorem 2.1.1, we have that $p(H)-p(H_0)-D^{(1)}p(H_0)\bullet V \in \mathcal{B}_1(\mathcal{H})$ and

$$\begin{aligned} &Tr\{p(H)-p(H_0)-D^{(1)}p(H_0)\bullet V\} \\ &\quad = \lim_{n\to\infty} Tr\{P_n\left[p(P_nHP_n)-p(P_nH_0P_n)-D^{(1)}p(P_nH_0P_n)\bullet P_nVP_n\right]P_n\} \\ &\quad = \lim_{n\to\infty}\int_a^b p''(\lambda)\eta_n(\lambda)d\lambda, \end{aligned}$$

with $\eta_n(\lambda)$ given by (2.1.2), and $\|\eta_n\|_1 = \frac{1}{2}\|P_n(H-H_0)P_n\|_2^2$, which clearly converges to $\frac{1}{2}\|V\|_2^2$ as $n \longrightarrow \infty$, since $|\|P_nVP_n\|_2 - \|V\|_2| \leq \|P_nVP_n - V\|_2 \leq \left\|P_nVP_n^{\perp}\right\|_2 + \left\|P_n^{\perp}V\right\|_2$,

which converges to 0 as $n \longrightarrow \infty$. Set $V_n \equiv P_nVP_n;\ H_n \equiv P_nHP_n;\ H_{0,n} \equiv P_nH_0P_n$ and $E_{H_{0,n}}(.)$, $E_{H_{s,n}}(.)$ are the spectral families of $H_{0,n}$ and $H_{s,n} \equiv P_nH_sP_n$ respectively.

Following the idea contained in the paper of Gestezy et.al ([3]) and using the expression (2.1.2), we have for $f \in L^\infty([a,b])$ and $g(\lambda) = \int_a^\lambda f(\mu)d\mu$ that

$$\begin{aligned} &\int_a^b f(\lambda)\left[\eta_n(\lambda)-\eta_m(\lambda)\right]d\lambda = \int_a^b g'(\lambda)\left[\eta_n(\lambda)-\eta_m(\lambda)\right]\,d\lambda \\ &= \int_a^b g'(\lambda)d\lambda\int_0^1 \mathrm{Tr}\{V_n\left[E_{H_{0,n}}(\lambda)-E_{H_{s,n}}(\lambda)\right]\}\,ds \\ &\qquad\qquad - \int_a^b g'(\lambda)d\lambda\int_0^1 \mathrm{Tr}\{V_m\left[E_{H_{0,m}}(\lambda)-E_{H_{s,m}}(\lambda)\right]\}\,ds. \end{aligned} \tag{2.3.1}$$

Again by using Fubini's theorem to interchange the orders of integration, the right hand side of (4.3.24) is equal to

$$\begin{aligned} &\int_0^1 ds\int_a^b g'(\lambda)\,\mathrm{Tr}\{V_n\left[E_{H_{0,n}}(\lambda)-E_{H_{s,n}}(\lambda)\right]\}\,d\lambda \\ &\qquad\qquad - \int_0^1 ds\int_a^b g'(\lambda)\mathrm{Tr}\{V_m\left[E_{H_{0,m}}(\lambda)-E_{H_{s,m}}(\lambda)\right]\}\,d\lambda \\ &\quad = \int_0^1 ds\ \int_a^b g'(\lambda)\,\mathrm{Tr}\{V_n\left[E_{H_{0,n}}(\lambda)-E_{H_{s,n}}(\lambda)\right] \\ &\qquad\qquad\qquad - V_m\left[E_{H_{0,m}}(\lambda)-E_{H_{s,m}}(\lambda)\right]\}\,d\lambda. \end{aligned} \tag{2.3.2}$$

Next by doing integration by-parts, the right hand side of (4.3.25) becomes

$$\int_0^1 ds\, \{g(\lambda)\,\mathrm{Tr}(V_n\left[E_{H_{0,n}}(\lambda)-E_{H_{s,n}}(\lambda)\right] - V_m\left[E_{H_{0,m}}(\lambda)-E_{H_{s,m}}(\lambda)\right])\}|_{\lambda=a}^{b}$$

$$-\int_0^1 ds\int_a^b g(\lambda)\,\mathrm{Tr}\{V_n\left[E_{H_{0,n}}(d\lambda)-E_{H_{s,n}}(d\lambda)\right] - V_m\left[E_{H_{0,m}}(d\lambda)-E_{H_{s,m}}(d\lambda)\right]\}$$

$$=\int_0^1 ds\int_a^b g(\lambda)\,\mathrm{Tr}\{V_n\left[E_{H_{s,n}}(d\lambda)-E_{H_{0,n}}(d\lambda)\right] - V_m\left[E_{H_{s,m}}(d\lambda)-E_{H_{n,m}}(d\lambda)\right]\}$$

$$=\int_0^1 ds\,\mathrm{Tr}\{V_n\left[g(H_{s,n})-g(H_{0,n})\right]-V_m\left[g(H_{s,m})-g(H_{0,m})\right]\}, \tag{2.3.3}$$

where we have noted that all the boundary terms vanishes. Next we note that as in (2.1.6)

$$g(H_0)-g(H_s)=-s\int_a^b\int_a^b \frac{g(\alpha)-g(\beta)}{\alpha-\beta}\mathcal{G}(d\alpha\times d\beta)V,$$

where $\mathcal{G}(\Delta\times\delta)X = E_{H_0}(\Delta)XE_{H_s}(\delta)$ ($X\in\mathcal{B}_2(\mathcal{H})$ and $\Delta\times\delta\subseteq\mathbb{R}\times\mathbb{R}$) extends to a spectral measure on $\mathbb{R}^2$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$. Therefore $\|g(H_s)-g(H_0)\|_2\leq s\,\|f\|_\infty\,\|V\|_2,$ since $\sup\limits_{\alpha,\beta\in[a,b];\alpha\neq\beta}\left|\frac{g(\alpha)-g(\beta)}{\alpha-\beta}\right|\leq\|f\|_\infty.$ Again as in (2.1.6), we have for $0\leq s\leq 1$,

$$\begin{aligned}P_n\left[g(H_{s,n})-g(H_s)\right]P_n &= \int_a^b\int_a^b\frac{g(\alpha)-g(\beta)}{\alpha-\beta}P_nE_{H_{s,n}}(d\alpha)\left[H_{s,n}-H_s\right]E_{H_s}(d\beta)P_n\\ &= \int_a^b\int_a^b\frac{g(\alpha)-g(\beta)}{\alpha-\beta}P_nE_{H_{s,n}}(d\alpha)\left[P_n(H_0+sV)P_n-(H_0+sV)\right]E_{H_s}(d\beta)P_n\\ &= -\int_a^b\int_a^b\frac{g(\alpha)-g(\beta)}{\alpha-\beta}P_nE_{H_{s,n}}(d\alpha)\left[P_nH_0P_n^{\perp}+sP_nVP_n^{\perp}\right]E_{H_s}(d\beta)P_n\\ &= -P_n\{\int_a^b\int_a^b\frac{g(\alpha)-g(\beta)}{\alpha-\beta}\,\mathcal{G}_{(s,n)}(d\alpha\times d\beta)\left[P_nH_0P_n^{\perp}+sP_nVP_n^{\perp}\right]\}P_n,\end{aligned}$$

where $\mathcal{G}_{(s,n)}(\Delta\times\delta)X = E_{H_{s,n}}(\Delta)XE_{H_s}(\delta)$ ($X\in\mathcal{B}_2(\mathcal{H})$ and $\Delta\times\delta\subseteq\mathbb{R}\times\mathbb{R}$) extends to a spectral measure on $\mathbb{R}^2$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$ (see section 1.3 for details) and hence

$$\|P_n\left[g(H_{s,n})-g(H_s)\right]P_n\|_2\leq\|f\|_\infty\left(\left\|P_nH_0P_n^{\perp}\right\|_2+s\left\|P_nVP_n^{\perp}\right\|_2\right).$$

In particular for $s=0$, we have

$$\|P_n\left[g(H_{0,n})-g(H_0)\right]P_n\|_2\leq\|f\|_\infty\left\|P_nH_0P_n^{\perp}\right\|_2.$$

Therefore

$$\begin{aligned}
&\left|\int_a^b f(\lambda)\,[\eta_n(\lambda)-\eta_m(\lambda)]\,d\lambda\right| \\
&\quad = \left|\int_0^1 ds\ \mathrm{Tr}\{V_n\,[g(H_{s,n})-g(H_{0,n})] - V_m\,[g(H_{s,m})-g(H_{0,m})]\}\right| \\
&\quad = |\int_0^1 ds\ (\mathrm{Tr}(V_n\{[g(H_{s,n})-g(H_{0,n})]-[g(H_s)-g(H_0)]\}) \\
&\qquad\qquad - \mathrm{Tr}(V_m\{[g(H_{s,m})-g(H_{0,m})]-[g(H_s)-g(H_0)]\}) \\
&\qquad\qquad\qquad + \mathrm{Tr}\{(V_n-V_m)\,[g(H_s)-g(H_0)]\})| \\
&\quad = |\int_0^1 ds\ (\mathrm{Tr}(V_n\{[g(H_{s,n})-g(H_s)]-[g(H_{0,n})-g(H_0)]\}) \\
&\qquad\qquad - \mathrm{Tr}(V_m\{[g(H_{s,m})-g(H_s)]-[g(H_{0,m})-g(H_0)]\}) \\
&\qquad\qquad\qquad + \mathrm{Tr}\{(V_n-V_m)\,[g(H_s)-g(H_0)]\})| \\
&\quad \le \int_0^1 ds\ \{\|V_n\|_2(\|P_n\,[g(H_{s,n})-g(H_s)]\,\|_2 + \|P_n\,[g(H_{0,n})-g(H_0)]\,\|_2) \\
&\qquad\qquad - \|V_m\|_2(\|P_m\,[g(H_{s,m})-g(H_s)]\,\|_2 + \|P_m\,[g(H_{0,m})-g(H_0)]\,\|_2) \\
&\qquad\qquad\qquad + \|(V_n-V_m)\|_2\|\,[g(H_s)-g(H_0)]\,\|_2\} \\
&\quad \le \|f\|_\infty\|V\|_2\ (\int_0^1 ds\{2\left(\left\|P_nH_0P_n^\perp\right\|_2+\left\|P_mH_0P_m^\perp\right\|_2\right) \\
&\qquad\qquad + s\left(\left\|P_nVP_n^\perp\right\|_2+\left\|P_mVP_m^\perp\right\|_2\right) + s\|V_n-V_m\|_2\}).
\end{aligned}$$

Hence

$$\begin{aligned}
\sup_{f\in L^\infty([a,b])}\frac{\left|\int_a^b f(\lambda)\,[\eta_n(\lambda)-\eta_m(\lambda)]\,d\lambda\right|}{\|f\|_\infty} &\le \|f\|_\infty\|V\|_2\ (\int_0^1 ds\{2\left(\left\|P_nH_0P_n^\perp\right\|_2+\left\|P_mH_0P_m^\perp\right\|_2\right) \\
&\qquad + s\left(\left\|P_nVP_n^\perp\right\|_2+\left\|P_mVP_m^\perp\right\|_2\right)+s\|V_n-V_m\|_2\}) \\
i.e.\quad \|\eta_n-\eta_m\|_{L^1([a,b])} &\le \|f\|_\infty\|V\|_2\ (\int_0^1 ds\{2\left(\left\|P_nH_0P_n^\perp\right\|_2+\left\|P_mH_0P_m^\perp\right\|_2\right) \\
&\qquad + s\left(\left\|P_nVP_n^\perp\right\|_2+\left\|P_mVP_m^\perp\right\|_2\right)+s\|V_n-V_m\|_2\}),
\end{aligned}$$

which converges to zero as $m, n \longrightarrow \infty$ and therefore $\{\eta_n\}$ is a Cauchy sequence of non-negative functions in $L^1([a,b])$ and hence there exists a non-negative $L^1([a,b])$- function η such that $\{\eta_n\}$ converges to η in L^1-norm. Thus

$$\mathrm{Tr}\{p(H)-p(H_0)-D^{(1)}p(H_0)\bullet V\} = \lim_{n\to\infty}\int_a^b p''(\lambda)\eta_n(\lambda)d\lambda = \int_a^b p''(\lambda)\eta(\lambda)d\lambda.$$

For uniqueness, let us assume that there exists $\eta_1,\ \eta_2 \in L^1([a,b])$ such that

$$\text{Tr}\left[p(H) - p(H_0) - D^{(1)}p(H_0) \bullet V\right] = \int_a^b p''(\lambda)\eta_j(\lambda)d\lambda,$$

where $p(.)$ is a polynomial and $j = 1, 2$. Therefore

$$\int_a^b p''(\lambda)\,\eta(\lambda)\,d\lambda = 0\ \forall \quad \text{polynomials} \quad p(.) \quad \text{and} \quad \eta \equiv \eta_1 - \eta_2 \in L^1([a,b]),$$

which together with the fact that $\int_a^b \eta_1(\lambda)\,d\lambda = \int_a^b \eta_2(\lambda)\,d\lambda = \frac{1}{2}\text{Tr}(V^2)$ (which one can easily arrive at by setting $p(\lambda) = \lambda^2$ in the above formula), implies that

$\displaystyle\int_a^b \lambda^r \eta(\lambda)d\lambda = 0\ \forall\ r \geq 0.$ Hence by an application of Fubini's theorem, we get that

$$\int_{-\infty}^{\infty} e^{-\text{i}t\lambda}\,\eta(\lambda)\,d\lambda \;=\; \sum_{n=0}^{\infty} \frac{1}{n!}\int_{-\infty}^{\infty} (-\text{i}t\lambda)^n\,\eta(\lambda)\,d\lambda = 0.$$

Hence

$$\int_{-\infty}^{\infty} \text{e}^{-\text{i}t\lambda}\,\eta(\lambda)\,d\lambda = 0\,\forall\,t \in \mathbb{R}.$$

Therefore η is an $L^1([a,b])$- function whose Fourier transform $\hat{\eta}(t)$ vanishes identically, implying that $\eta = 0$ or $\eta_1 = \eta_2$ a.e.

□

Lemma 2.3.2. *Let H and H_0 be two self-adjoint operators in an infinite dimensional separable Hilbert space $\mathcal{H}$ such that $H - H_0 \equiv V \in \mathcal{B}_2(\mathcal{H})$. Then $\text{e}^{\text{i}tH} - \text{e}^{\text{i}tH_0} - D^{(1)}(\text{e}^{\text{i}tH_0}) \bullet V \in \mathcal{B}_1(\mathcal{H})$ and there exists a unique non-negative $L^1(\mathbb{R})$-function η such that*

$$\text{Tr}\{\text{e}^{\text{i}tH} - \text{e}^{\text{i}tH_0} - D^{(1)}(\text{e}^{\text{i}tH_0}) \bullet V\} = (\text{i}t)^2 \int_{\mathbb{R}} \text{e}^{it\lambda}\eta(\lambda)d\lambda$$

and $\|\eta\|_1 = \frac{1}{2}\|V\|_2^2$.

Proof. By Theorem 2.2.5 we conclude that, there exists a sequence $\{P_n\}$ of finite rank projections

such that

$$\text{Tr}\{\text{e}^{\text{i}tH} - \text{e}^{\text{i}tH_0} - D^{(1)}(\text{e}^{\text{i}tH_0}) \bullet V\} = \lim_{n\to\infty} \text{Tr}\{P_n\left[\text{e}^{itH_n} - \text{e}^{itH_{0,n}} - D^{(1)}(\text{e}^{\text{i}tH_{0,n}}) \bullet V_n\right]P_n\}, \quad (2.3.4)$$

where $H_n \equiv P_nHP_n,\ H_{0,n} \equiv P_nH_0P_n$ and $V_n \equiv P_nVP_n$, and the convergence is uniform in t for $|t| \leq T$. Note that by construction $P_n\mathcal{H} \subseteq Dom(H_0) = Dom(H)$ (see proof of Proposition 2.2.1) and hence both H_n and $H_{0,n}$ are self-adjoint operators in the finite dimensional space $P_n\mathcal{H}$. By (2.1.4), there exists a unique non-negative $\eta_n \in L^1(\mathbb{R})$ such that

$$\mathrm{Tr}\{P_n\left[\mathrm{e}^{\mathrm{i}tH_n} - \mathrm{e}^{\mathrm{i}tH_{0,n}} - D^{(1)}(\mathrm{e}^{\mathrm{i}tH_{0,n}}) \bullet V_n\right]P_n\} = (\mathrm{i}t)^2 \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda}\eta_n(\lambda)d\lambda, \tag{2.3.5}$$

with $\eta_n(\lambda)$ given by (2.1.2), and hence

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0} - D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V\} = (\mathrm{i}t)^2 \lim_{n\to\infty} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\lambda}\eta_n(\lambda)d\lambda, \tag{2.3.6}$$

the convergence being uniform in t for $|t| \leq T$. Furthermore, $\|\eta_n\|_1 = \frac{1}{2}\|P_n(H - H_0)P_n\|_2^2$,

which clearly converges to $\frac{1}{2}\|V\|_2^2$ as $n \to \infty$. In order to prove the $L^1(\mathbb{R})$-convergence of $\{\eta_n\}$, we essentially repeat the procedure as in the proof of Theorem 2.3.1, except that one needs to take into account the possibility that the indefinite integral g of a $L^\infty(\mathbb{R})$-function f may have a linear part, which will make $g(H_0)$ and $g(H)$ unbounded operators. Next we note that as in (2.1.6)

$$g(H_0) - g(H_s) = -s \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{g(\alpha) - g(\beta)}{\alpha - \beta}\mathcal{G}(d\alpha \times d\beta)V,$$

where $\mathcal{G}(\Delta \times \delta)X = E_{H_0}(\Delta)XE_{H_s}(\delta)$ ($X \in \mathcal{B}_2(\mathcal{H})$ and $\Delta \times \delta \subseteq \mathbb{R} \times \mathbb{R}$) extends to a spectral measure on $\mathbb{R}^2$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$. Therefore $\|g(H_s) - g(H_0)\|_2 \leq s\, \|f\|_\infty\, \|V\|_2$, since $\sup\limits_{\alpha,\beta\in\mathbb{R};\alpha\neq\beta} \left|\frac{g(\alpha) - g(\beta)}{\alpha - \beta}\right| \leq \|f\|_\infty$. Again as in (2.1.6), we have for $0 \leq s \leq 1$,

$$P_n\left[g(H_{s,n})-g(H_s)\right]P_n=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{g(\alpha)-g(\beta)}{\alpha-\beta}P_nE_{H_{s,n}}(d\alpha)\left[H_{s,n}-H_s\right]E_{H_s}(d\beta)P_n$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{g(\alpha)-g(\beta)}{\alpha-\beta}P_nE_{H_{s,n}}(d\alpha)\left[P_n(H_0+sV)P_n-(H_0+sV)\right]E_{H_s}(d\beta)P_n$$
$$=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{g(\alpha)-g(\beta)}{\alpha-\beta}P_nE_{H_{s,n}}(d\alpha)\left[P_nH_0P_n^{\perp}+sP_nVP_n^{\perp}\right]E_{H_s}(d\beta)P_n$$
$$=-P_n\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{g(\alpha)-g(\beta)}{\alpha-\beta}\,\mathcal{G}_{(s,n)}(d\alpha\times d\beta)\left[P_nH_0P_n^{\perp}+sP_nVP_n^{\perp}\right]\}P_n,$$

where $\mathcal{G}_{(s,n)}(\Delta\times\delta)X=E_{H_{s,n}}(\Delta)XE_{H_s}(\delta)$ ($X\in\mathcal{B}_2(\mathcal{H})$ and $\Delta\times\delta\subseteq\mathbb{R}\times\mathbb{R}$) extends to a spectral measure on $\mathbb{R}^2$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$ and hence

$$\left\|P_n\left[g(H_{s,n})-g(H_s)\right]P_n\right\|_2\leq\|f\|_\infty\left(\left\|P_nH_0P_n^{\perp}\right\|_2+s\left\|P_nVP_n^{\perp}\right\|_2\right).$$

In particular for $s=0$, we have

$$\left\|P_n\left[g(H_{0,n})-g(H_0)\right]P_n\right\|_2\leq\|f\|_\infty\left\|P_nH_0P_n^{\perp}\right\|_2.$$

Since for fixed finite m and n, the support of the spectral measures involved are compact, then using the expression (2.1.2) and using the indefinite integral g of $f\in L^\infty(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty}f(\lambda)\left[\eta_n(\lambda)-\eta_m(\lambda)\right]d\lambda=\int_{-\infty}^{\infty}g'(\lambda)\left[\eta_n(\lambda)-\eta_m(\lambda)\right]\,d\lambda$$
$$=\int_{-\infty}^{\infty}g'(\lambda)d\lambda\int_0^1\text{Tr}\{V_n\left[E_{H_{0,n}}(\lambda)-E_{H_{s,n}}(\lambda)\right]\}\,ds \tag{2.3.7}$$
$$-\int_{-\infty}^{\infty}g'(\lambda)d\lambda\int_0^1\text{Tr}\{V_m\left[E_{H_{0,m}}(\lambda)-E_{H_{s,m}}(\lambda)\right]\}\,ds$$

Again by using Fubini's theorem to interchange the orders of integration, the right hand side of (2.3.7) is equal to

$$\int_0^1 ds\int_{-\infty}^{\infty}g'(\lambda)\,\text{Tr}\{V_n\left[E_{H_{0,n}}(\lambda)-E_{H_{s,n}}(\lambda)\right]\}\,d\lambda$$
$$-\int_0^1 ds\int_{-\infty}^{\infty}g'(\lambda)\text{Tr}\{V_m\left[E_{H_{0,m}}(\lambda)-E_{H_{s,m}}(\lambda)\right]\}\,d\lambda$$

$$=\int_0^1 ds\,\int_{-\infty}^{\infty}g'(\lambda)\,\text{Tr}\{V_n\left[E_{H_{0,n}}(\lambda)-E_{H_{s,n}}(\lambda)\right]-V_m\left[E_{H_{0,m}}(\lambda)-E_{H_{s,m}}(\lambda)\right]\}\,d\lambda \tag{2.3.8}$$

Next by doing integration by-parts, the right hand side of (2.3.8) becomes

$$\int_0^1 ds\ \{g(\lambda)\ \mathrm{Tr}(V_n\left[E_{H_{0,n}}(\lambda) - E_{H_{s,n}}(\lambda)\right] \\ - V_m\left[E_{H_{0,m}}(\lambda) - E_{H_{s,m}}(\lambda)\right])\}|_{\lambda=-\infty}^{\infty} \\ - \int_0^1 ds \int_{-\infty}^{\infty} g(\lambda)\ \mathrm{Tr}\{V_n\left[E_{H_{0,n}}(d\lambda) - E_{H_{s,n}}(d\lambda)\right] \\ - V_m\left[E_{H_{0,m}}(d\lambda) - E_{H_{s,m}}(d\lambda)\right]\}$$

$$= \int_0^1 ds \int_{-\infty}^{\infty} g(\lambda)\ \mathrm{Tr}\{V_n\left[E_{H_{s,n}}(d\lambda) - E_{H_{0,n}}(d\lambda)\right] \\ - V_m\left[E_{H_{s,m}}(d\lambda) - E_{H_{n,m}}(d\lambda)\right]\}$$

$$= \int_0^1 ds\ \mathrm{Tr}\{V_n\left[g(H_{s,n}) - g(H_{0,n})\right] - V_m\left[g(H_{s,m}) - g(H_{0,m})\right]\}, \tag{2.3.9}$$

where we have noted that all the boundary terms vanishes. Therefore

$$\left|\int_{-\infty}^{\infty} f(\lambda)\left[\eta_n(\lambda) - \eta_m(\lambda)\right] d\lambda\right| \\ = \left|\int_0^1 ds\ \mathrm{Tr}\{V_n\left[g(H_{s,n}) - g(H_{0,n})\right] - V_m\left[g(H_{s,m}) - g(H_{0,m})\right]\}\right| \\ = |\int_0^1 ds\ (\mathrm{Tr}(V_n\{[g(H_{s,n}) - g(H_{0,n})] - [g(H_s) - g(H_0)]\}) \\ - \mathrm{Tr}(V_m\{[g(H_{s,m}) - g(H_{0,m})] - [g(H_s) - g(H_0)]\}) \\ + \mathrm{Tr}\{(V_n - V_m)\left[g(H_s) - g(H_0)\right]\})| \\ = |\int_0^1 ds\ (\mathrm{Tr}(V_n\{[g(H_{s,n}) - g(H_s)] - [g(H_{0,n}) - g(H_0)]\}) \\ - \mathrm{Tr}(V_m\{[g(H_{s,m}) - g(H_s)] - [g(H_{0,m}) - g(H_0)]\}) \\ + \mathrm{Tr}\{(V_n - V_m)\left[g(H_s) - g(H_0)\right]\})| \\ \leq \int_0^1 ds\ \{\|V_n\|_2(\|P_n\left[g(H_{s,n}) - g(H_s)\right]\|_2 + \|P_n\left[g(H_{0,n}) - g(H_0)\right]\|_2) \\ - \|V_m\|_2(\|P_m\left[g(H_{s,m}) - g(H_s)\right]\|_2 + \|P_m\left[g(H_{0,m}) - g(H_0)\right]\|_2) \\ + \|(V_n - V_m)\|_2\|\left[g(H_s) - g(H_0)\right]\|_2\} \\ \leq \|f\|_\infty \|V\|_2\ (\int_0^1 ds\{2\left(\left\|P_n H_0 P_n^\perp\right\|_2 + \left\|P_m H_0 P_m^\perp\right\|_2\right) \\ + s\left(\left\|P_n V P_n^\perp\right\|_2 + \left\|P_m V P_m^\perp\right\|_2\right) + s\|V_n - V_m\|_2\}).$$

Therefore, by Remark 2.2.3 and the Hahn-Banach theorem, $\{\eta_n\}$ is a Cauchy sequence of non-negative functions in $L^1(\mathbb{R})$ and hence there exists a non-negative $L^1(\mathbb{R})$- function η such that $\{\eta_n\}$ converges to η in L^1-norm. Thus

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0} - D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V\} = (\mathrm{i}t)^2 \lim_{n\to\infty} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\eta_n(\lambda)d\lambda = (\mathrm{i}t)^2 \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda}\eta(\lambda)d\lambda. \quad (2.3.10)$$

For uniqueness, let us assume that there exists $\eta_1, \eta_2 \in L^1(\mathbb{R})$ such that

$$\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0} D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V\} = (\mathrm{i}t)^2 \int_{\mathbb{R}} \mathrm{e}^{it\lambda}\eta_j(\lambda)d\lambda,$$

for $j = 1, 2$ and hence

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda} \left[\eta_1(\lambda) - \eta_2(\lambda)\right] d\lambda = 0 \ \forall \ t \in \mathbb{R},$$

since $\int_{\mathbb{R}} \eta_1(\lambda)d\lambda = \int_{\mathbb{R}} \eta_2(\lambda)d\lambda = \frac{1}{2}\|V\|_2^2$ and $\eta_1 - \eta_2 \in L^1(\mathbb{R})$. Then by Fourier Inversion Theorem we conclude that $\eta_1 = \eta_2$ a.e.

□

Theorem 2.3.3. *Let H and H_0 be two self-adjoint operators in an infinite dimensional separable Hilbert space $\mathcal{H}$ such that $H - H_0 \equiv V \in \mathcal{B}_2(\mathcal{H})$ and $f \in \mathcal{S}(\mathbb{R})$(the Schwartz class of smooth functions of rapid decrease). Then $f(H) - f(H_0) - D^{(1)}f(H_0) \bullet V \in \mathcal{B}_1(\mathcal{H})$ and*

$$\mathrm{Tr}\{f(H) - f(H_0) - D^{(1)}f(H_0) \bullet V\} = \int_{\mathbb{R}} f''(\lambda)\eta(\lambda)d\lambda,$$

where η is a unique non-negative $L^1(\mathbb{R})$ -function with $\|\eta\|_1 = \frac{1}{2}\|V\|_2^2$.

Proof. By the spectral theorem and Fourier Inversion theorem, we have for $g, h \in \mathcal{H}$

$$\langle f(H)g, h\rangle = \int_{\mathbb{R}} f(\lambda)\langle E_H(d\lambda)g, h\rangle = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \hat{f}(t)\mathrm{e}^{\mathrm{i}t\lambda}\, dt\right) \langle E_H(d\lambda)g, h\rangle. \quad (2.3.11)$$

Again by an application of Fubini's theorem the right hand side of (4.3.27) is equal to

$$\int_{\mathbb{R}} \hat{f}(t)\, dt \left(\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda} \, \langle E_H(d\lambda)g, h\rangle\right) = \int_{\mathbb{R}} \hat{f}(t)\, dt\langle \mathrm{e}^{\mathrm{i}tH}g, h\rangle = \left\langle \int_{\mathbb{R}} \hat{f}(t)\mathrm{e}^{\mathrm{i}tH}\, dt\; g, h \right\rangle,$$

proving that

$$f(H) = \int_{\mathbb{R}} \hat{f}(t)\mathrm{e}^{\mathrm{i}tH}dt \quad \text{and similarly} \quad f(H_0) = \int_{\mathbb{R}} \hat{f}(t)\mathrm{e}^{\mathrm{i}tH_0}dt.$$

Again for $X \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned}
f(H_0+X) - f(H_0) &= \int_{\mathbb{R}} \hat{f}(t)\, dt \left[\mathrm{e}^{\mathrm{i}t(H_0+X)} - \mathrm{e}^{\mathrm{i}tH}\right] = \mathrm{i}t \int_{\mathbb{R}} \hat{f}(t)\, dt \int_0^1 d\alpha\, \mathrm{e}^{\mathrm{i}t\alpha(H_0+X)} X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\\
&= \mathrm{i}t \int_{\mathbb{R}} \hat{f}(t)\, dt \int_0^1 d\alpha\, \mathrm{e}^{\mathrm{i}t\alpha H_0} X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} + \mathrm{i}t \int_{\mathbb{R}} \hat{f}(t)\, dt \int_0^1 d\alpha\, \left[\mathrm{e}^{\mathrm{i}t\alpha(H_0+X)} - \mathrm{e}^{\mathrm{i}t\alpha H_0}\right] X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}\\
&= \int_{\mathbb{R}} \hat{f}(t)\, dt\, \left[D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet X\right] + \int_{\mathbb{R}} \hat{f}(t)\, dt\, (\mathrm{i}t)^2 \int_0^1 \alpha d\alpha \int_0^1 d\beta\, \mathrm{e}^{\mathrm{i}t\alpha\beta(H_0+X)} X \mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0} X \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0},
\end{aligned}$$

on $\mathrm{Dom}(H_0)$ and hence

$$\left\| f(H_0+X) - f(H_0) - \int_{\mathbb{R}} \hat{f}(t)\, dt\, \left[D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet X\right]\right\| \leq \|t^2 \hat{f}\|_{L^1}\, \frac{1}{2} \|X\|^2,$$

proving that

$$D^{(1)} f(H_0) \bullet X = \int_{\mathbb{R}} \hat{f}(t) \left[D^{(1)}\left(\mathrm{e}^{\mathrm{i}tH_0}\right) \bullet X\right] dt.$$

Since $\mathbb{R} \ni s \longrightarrow \mathrm{e}^{\mathrm{i}s(H_0+V)},\ \mathrm{e}^{\mathrm{i}sH_0}$ are strongly continuous and since $V \in \mathcal{B}_2(\mathcal{H})$, it follows that $\beta \longrightarrow \mathrm{e}^{\mathrm{i}t\alpha\beta(H_0+V)} V \mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0} V \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0}$ is $\mathcal{B}_1$-continuous and using the fact that $\hat{f} \in \mathcal{S}(\mathbb{R})$, from the above calculations we conclude that

$$\begin{aligned}
f(H) - f(H_0) - D^{(1)} f(H_0) \bullet V &= \int_{\mathbb{R}} \hat{f}(t)\, dt \left[\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0} - D^{(1)}\left(\mathrm{e}^{\mathrm{i}tH_0}\right) \bullet V\right]\\
&= \int_{-\infty}^{\infty} \hat{f}(t)\, dt\, (\mathrm{i}t)^2 \int_0^1 \alpha d\alpha \int_0^1 d\beta\, \mathrm{e}^{\mathrm{i}t\alpha\beta H} V \mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0} V \mathrm{e}^{\mathrm{i}t(1-\alpha)H_0} \in \mathcal{B}_1(\mathcal{H}).
\end{aligned}$$

Finally by an application of Fubini's theorem and using Lemma 2.3.2, we have

$$\begin{aligned}
&\mathrm{Tr}\{f(H) - f(H_0) - D^{(1)} f(H_0) \bullet V\} = \int_{\mathbb{R}} \hat{f}(t)\, \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH} - \mathrm{e}^{\mathrm{i}tH_0} - D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0}) \bullet V\}\, dt\\
&= \int_{\mathbb{R}} \hat{f}(t) \left((\mathrm{i}t)^2 \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda} \eta(\lambda) d\lambda\right) dt = \int_{\mathbb{R}} \eta(\lambda)\, d\lambda \left(\int_{\mathbb{R}} (\mathrm{i}t)^2 \hat{f}(t) \mathrm{e}^{\mathrm{i}t\lambda}\, dt\right) = \int_{\mathbb{R}} f''(\lambda) \eta(\lambda) d\lambda.
\end{aligned}$$

□

Remark 2.3.4. Let f be a function on $\mathbb{R}$ such that

$$f(\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}t\lambda} - 1 - \mathrm{i}t\lambda}{(\mathrm{i}t)^2} \nu(dt) + \widetilde{C_1}\lambda + \widetilde{C_2},$$

where $\widetilde{C_1}, \widetilde{C_2}$ are some constants and ν is a complex measure on $\mathbb{R}$. It is worth observing that $f(H)$ and $f(H_0)$ are normal and not necessarily bounded operators. Moreover, $\mathrm{Dom} f(H)$ and $\mathrm{Dom} f(H_0)$ contains $\mathrm{Dom}(H^2)$ and $\mathrm{Dom}(H_0^2)$ respectively. Indeed,

$$\mathrm{e}^{\mathrm{i}t\lambda} - 1 = \mathrm{i}t \int_0^{\lambda} dx\, \mathrm{e}^{\mathrm{i}tx} = \mathrm{i}t \int_0^{\lambda} dx \left(\mathrm{i}t \int_0^{x} dy\, \mathrm{e}^{\mathrm{i}ty} + 1\right) = (\mathrm{i}t)^2 \int_0^{\lambda} dx \int_0^{x} dy\, \mathrm{e}^{\mathrm{i}ty} + \mathrm{i}t\, \lambda$$

$$i.e.\ \frac{e^{it\lambda}-1-it\lambda}{(it)^2}=\int_0^\lambda dx\int_0^x dy\, e^{ity}\quad\text{and hence}\quad\left|\frac{e^{it\lambda}-1-it\lambda}{(it)^2}\right|\leq\frac{1}{2}\lambda^2.$$

Therefore

$$|f(\lambda)|\leq\int_{-\infty}^{\infty}\left|\frac{e^{it\lambda}-1-it\lambda}{(it)^2}\right||\nu|(dt)+\left|\widetilde{C_1}\right||\lambda|+\left|\widetilde{C_2}\right|\leq\frac{|\nu|(\mathbb{R})}{2}|\lambda|^2+\left|\widetilde{C_1}\right||\lambda|+\left|\widetilde{C_2}\right|,$$

where $|\nu|(\mathbb{R})(<\infty)$ is the total variation of the measure ν. Hence for $h\in \text{Dom}\,(H^2)\subseteq \text{Dom}\,(H)$, we have

$$\begin{aligned}\int_{-\infty}^{\infty}|f(\lambda)|^2\ \|E_H(d\lambda)h\|^2 &\leq\int_{-\infty}^{\infty}\left(\frac{|\nu|(\mathbb{R})}{2}|\lambda|^2+\left|\widetilde{C_1}\right||\lambda|+\left|\widetilde{C_2}\right|\right)^2\ \|E_H(d\lambda)h\|^2\\ &\leq 4\left(\frac{|\nu|(\mathbb{R})}{2}\right)^2\int_{-\infty}^{\infty}\lambda^4\ \|E_H(d\lambda)h\|^2+4\left(\left|\widetilde{C_1}\right|\right)^2\int_{-\infty}^{\infty}\lambda^2\ \|E_H(d\lambda)h\|^2\\ &\qquad+2\left(\left|\widetilde{C_2}\right|\right)^2\int_{-\infty}^{\infty}\|E_H(d\lambda)h\|^2\\ &<\infty,\end{aligned}$$

proving that $\text{Dom}\,(H^2)\subseteq\text{Dom}f(H)$. Similarly by the same above argument we conclude that $\text{Dom}\,(H_0^2)\subseteq\text{Dom}f(H_0)$. Moreover, f is twice continuously differentiable and

$$f'(\lambda)=\int_{-\infty}^{\infty}\frac{e^{it\lambda}-1}{(it)}\,\nu(dt)+\widetilde{C_1}\quad\text{and}\quad f''(\lambda)=\int_{-\infty}^{\infty}e^{it\lambda}\,\nu(dt).$$

For $X\in\mathcal{B}(\mathcal{H})$, consider the following expression on $\text{Dom}(H_0)$

$$\int_{-\infty}^{\infty}\left[\frac{e^{it(H_0+X)}-1-it(H_0+X)}{(it)^2}-\frac{e^{itH_0}-1-itH_0}{(it)^2}\right]\nu(dt)+\widetilde{C_1}\left[(H_0+X)-H_0\right].\quad(2.3.12)$$

But the right hand side of (2.3.12) is equal to

$$\begin{aligned}&\int_{-\infty}^{\infty}D^{(1)}\left(\frac{e^{itH_0}-1-itH_0}{(it)^2}\right)\bullet X\,\nu(dt)+\widetilde{C_1}D^{(1)}(H_0)\bullet X\\ &+\int_{-\infty}^{\infty}\left[\frac{e^{it(H_0+X)}-1-it(H_0+X)}{(it)^2}-\frac{e^{itH_0}-1-itH_0}{(it)^2}-D^{(1)}\left(\frac{e^{itH_0}-1-itH_0}{(it)^2}\right)\bullet X\right]\nu(dt)\\ &\qquad+\widetilde{C_1}\left[(H_0+X)-H_0-D^{(1)}(H_0)\bullet X\right]\\ &=\int_{-\infty}^{\infty}D^{(1)}\left(\frac{e^{itH_0}-1-itH_0}{(it)^2}\right)\bullet X\,\nu(dt)+\widetilde{C_1}D^{(1)}(H_0)\bullet X\\ &\qquad+\int_{-\infty}^{\infty}\frac{1}{(it)^2}\left[e^{it(H_0+X)}-e^{itH_0}-D^{(1)}(e^{itH_0})\bullet X\right]\nu(dt)\\ &=\int_{-\infty}^{\infty}D^{(1)}\left(\frac{e^{itH_0}-1-itH_0}{(it)^2}\right)\bullet X\,\nu(dt)+\widetilde{C_1}D^{(1)}(H_0)\bullet X\\ &\qquad+\int_{-\infty}^{\infty}\nu(dt)\int_0^1\alpha d\alpha\int_0^1 d\beta\ e^{it\alpha\beta(H_0+X)}Xe^{it\alpha(1-\beta)H_0}Xe^{it(1-\alpha)H_0},\end{aligned}$$

on Dom(H_0) and using the expression (2.2.4) and hence

$$\left\|f(H_0+X)-f(H_0)-\left[\int_{-\infty}^{\infty} D^{(1)}\left(\frac{\mathrm{e}^{\mathrm{i}tH_0}-1-\mathrm{i}tH_0}{(\mathrm{i}t)^2}\right)\bullet X\ \nu(dt)+\widetilde{C_1}D^{(1)}\left(H_0\right)\bullet X\right]\right\| \leq \frac{1}{2}\|X\|^2|\nu|(\mathbb{R}),$$

proving that

$$D^{(1)}f(H_0)\bullet X=\int_{-\infty}^{\infty} D^{(1)}\left(\frac{\mathrm{e}^{\mathrm{i}tH_0}-1-\mathrm{i}tH_0}{(\mathrm{i}t)^2}\right)\bullet X\ \nu(dt)+\widetilde{C_1}D^{(1)}\left(H_0\right)\bullet X.$$

Next consider the expression

$$\begin{aligned}\int_{-\infty}^{\infty}\left[\frac{\mathrm{e}^{\mathrm{i}tH}-1-\mathrm{i}tH}{(\mathrm{i}t)^2}-\frac{\mathrm{e}^{\mathrm{i}tH_0}-1-\mathrm{i}tH_0}{(\mathrm{i}t)^2}-D^{(1)}\left(\frac{\mathrm{e}^{\mathrm{i}tH_0}-1-\mathrm{i}tH_0}{(\mathrm{i}t)^2}\right)\bullet V\right]\ \nu(dt)\\ +\widetilde{C_1}[H-H_0-D^{(1)}\left(H_0\right)\bullet V]\end{aligned} \tag{2.3.13}$$

on Dom(H_0). But the right hand side of (2.3.13) is equal to

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{1}{(\mathrm{i}t)^2}\left[\mathrm{e}^{\mathrm{i}tH}-\mathrm{e}^{\mathrm{i}tH_0}-D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0})\bullet V\right]\nu(dt)\\ &=\int_{-\infty}^{\infty}\nu(dt)\int_0^1\alpha d\alpha\int_0^1 d\beta\ \mathrm{e}^{\mathrm{i}t\alpha\beta H}V\mathrm{e}^{\mathrm{i}t\alpha(1-\beta)H_0}V\mathrm{e}^{\mathrm{i}t(1-\alpha)H_0},\end{aligned} \tag{2.3.14}$$

on Dom(H_0) and using the expression (2.2.4). On the other hand the right hand side of (2.3.14) is a well-defined $\mathcal{B}_1(\mathcal{H})$ operator and hence if $\{f(H)-f(H_0)-D^{(1)}f(H_0)\bullet V\}$ is densely defined, then it can be extended to whole of $\mathcal{H}$ as a trace class operator and

$$\begin{aligned}&\mathrm{Tr}\{f(H)-f(H_0)-D^{(1)}f(H_0)\bullet V\}\\ &\qquad=\int_{-\infty}^{\infty}\frac{1}{(\mathrm{i}t)^2}\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}tH}-\mathrm{e}^{\mathrm{i}tH_0}-D^{(1)}(\mathrm{e}^{\mathrm{i}tH_0})\bullet V\}\nu(dt)\\ &\qquad=\int_{-\infty}^{\infty}\frac{1}{(\mathrm{i}t)^2}\left((\mathrm{i}t)^2\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}t\lambda}\eta(\lambda)d\lambda\right)\nu(dt)=\int_{-\infty}^{\infty}\left(\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}t\lambda}\nu(dt)\right)\eta(\lambda)d\lambda\\ &\qquad=\int_{-\infty}^{\infty}f''(\lambda)\eta(\lambda)d\lambda,\end{aligned}$$

by using the expression (2.3.10) and using Fubini's theorem.

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Chapter 3

Third Order Trace Formula

Restating the result as in Section 1.6 of Chapter 1 in the particular case where $p = 3$, we have the following [4]. Let A (possibly unbounded) be a self-adjoint operator in $\mathcal{H}$ with $\sigma(A)$ as the spectra and $E_A(\lambda)$ the spectral family and V be a self-adjoint operator in $\mathcal{H}$ such that $V \in \mathcal{B}_2(\mathcal{H})$, then

(i) there is a unique finite real-valued measure ν_3 on $\mathbb{R}$ such that the trace formula

$$\mathrm{Tr}\{\phi(A+V) - \phi(A) - D^{(1)}\phi(A) \bullet V - \frac{1}{2}D^{(2)}\phi(A) \bullet (V,V)\} = \int_{-\infty}^{\infty} \phi'''(\lambda) d\nu_3(\lambda), \quad (3.0.1)$$

holds for suitable functions ϕ, where $D^{(2)}\phi(A)$ is the second order Frechet derivative of ϕ at A [1]. *Moreover, the total variation of ν_3 is bounded by $\frac{1}{3!}\|V\|_2^3$.*

(ii) If, in addition, A is bounded, then ν_3 is absolutely continuous.

In this chapter we give a new proof of formula (3.0.1) in the form (ii) for both bounded and unbounded (but bounded below) self-adjoint cases [8] .

3.1 Bounded Case

The next three lemmas are preparatory for the proof of the main theorem of this section, Theorem 3.1.5.

Lemma 3.1.1. *Let, for a given* $n \in \mathbb{N}$, $\{a_k\}_{k=0}^{n-1}$ *be a sequence of complex numbers such that* $a_{n-k-1} = a_k$. *Then*

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} a_k + \sum_{j=1}^{n} \sum_{k=0}^{j-1} a_k = (n+1) \sum_{k=0}^{n-1} a_k.$$

Proof. By changing the indices of summation and using the fact $a_{n-k-1} = a_k$, we get that

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-j-1} a_k + \sum_{j=1}^{n} \sum_{k=0}^{j-1} a_k = \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} a_{n-k-1} + \sum_{j=0}^{n-1} \sum_{k=0}^{j} a_k = \sum_{j=0}^{n-1} \sum_{k=j}^{n-1} a_k + \sum_{j=0}^{n-1} \sum_{k=0}^{j} a_k$$
$$= \sum_{j=0}^{n-1} a_j + \left(\sum_{k=j+1}^{n-1} a_k + \sum_{k=0}^{j} a_k \right) = \sum_{j=0}^{n-1} a_j + \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} a_k = \sum_{j=0}^{n-1} a_j + n \sum_{k=0}^{n-1} a_k = (n+1) \sum_{k=0}^{n-1} a_k.$$

□

Lemma 3.1.2. *Let* A *and* V *be two bounded self-adjoint operators in an infinite dimensional Hilbert space* $\mathcal{H}$ *such that* $V \in \mathcal{B}_3(\mathcal{H})$. *Let* $p(\lambda) = \lambda^r$ $(r \geq 0)$.*Then*

$$\begin{aligned} &\mathrm{Tr}\left[(A+V)^r - A^r - D^{(1)}(A^r) \bullet V - \frac{1}{2} D^{(2)}(A^r) \bullet (V,V) \right] \\ &\qquad = r \sum_{k=0}^{r-2} \int_0^1 ds \int_0^s d\tau \ \mathrm{Tr}\left[V A_\tau^{r-k-2} V A_\tau^k - V A^{r-k-2} V A^k \right], \end{aligned} \tag{3.1.1}$$

where $A_\tau = A + \tau V$ *and* $0 \leq \tau \leq 1$.

Proof. We have already shown in (i) of Theorem 2.1.1 that

$$D^{(1)}(A^r) \bullet X = \sum_{j=0}^{r-1} A^{r-j-1} X A^j \quad \text{where} \quad X \in \mathcal{B}(\mathcal{H}).$$

Again for $X, Y \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned}
&D^{(1)}((A+X)^r)\bullet Y - D^{(1)}(A^r)\bullet Y \\
&\quad= \sum_{j=0}^{r-1}(A+X)^{r-j-1}Y(A+X)^j - \sum_{j=0}^{r-1}A^{r-j-1}YA^j \\
&\quad= \sum_{j=0}^{r-1}\left[(A+X)^{r-j-1} - A^{r-j-1}\right]Y(A+X)^j + \sum_{j=0}^{r-1}A^{r-j-1}Y\left[(A+X)^j - A^j\right] \\
&\quad= \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}(A+X)^{r-j-k-2}XA^kY(A+X)^j + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}A^{r-j-1}Y(A+X)^kXA^{j-k-1},
\end{aligned}$$

and hence

$$\begin{aligned}
&D^{(1)}((A+X)^r)\bullet Y - D^{(1)}(A^r)\bullet Y \\
&\qquad - \left(\sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}A^{r-j-k-2}XA^kYA^j + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}A^{r-j-1}YA^kXA^{j-k-1}\right) \\
&= \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}(A+X)^{r-j-k-2}XA^kY(A+X)^j + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}A^{r-j-1}Y(A+X)^kXA^{j-k-1} \\
&\qquad - \left(\sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}A^{r-j-k-2}XA^kYA^j + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}A^{r-j-1}YA^kXA^{j-k-1}\right) \\
&= \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\left[(A+X)^{r-j-k-2} - A^{r-j-k-2}\right]XA^kY(A+X)^j \\
&\qquad + \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}A^{r-j-k-2}XA^kY\left[(A+X)^j - A^j\right] \\
&\qquad + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}A^{r-j-1}Y\left[(A+X)^k - A^k\right]XA^{j-k-1} \\
&= \sum_{j=0}^{r-3}\sum_{k=0}^{r-j-3}\sum_{l=0}^{r-j-k-3}(A+X)^{r-j-k-l-3}XA^lXA^kY(A+X)^j \\
&\qquad + \sum_{j=1}^{r-2}\sum_{k=0}^{r-j-2}\sum_{l=0}^{j-1}A^{r-j-k-2}XA^kY(A+X)^{j-l-1}XA^l \\
&\qquad + \sum_{j=2}^{r-1}\sum_{k=0}^{j-2}\sum_{l=0}^{j-k-2}A^{r-j-1}Y(A+X)^{j-k-l-2}XA^lXA^k,
\end{aligned}$$

leading to the estimate

$$\begin{aligned}
&\|D^{(1)}((A+X)^r)\bullet Y - D^{(1)}(A^r)\bullet Y \\
&\qquad - \left(\sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2} A^{r-j-k-2}XA^kYA^j + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1} A^{r-j-1}YA^kXA^{j-k-1}\right)\| \\
\leq \quad & \sum_{j=0}^{r-3}\sum_{k=0}^{r-j-3}\sum_{l=0}^{r-j-k-3} \|A+X\|^{r-j-k-l-3}\|X\|\|A\|^l\|X\|\|A\|^k\|Y\|\|A+X\|^j \\
&\qquad + \sum_{j=1}^{r-2}\sum_{k=0}^{r-j-2}\sum_{l=0}^{j-1}\|A\|^{r-j-k-2}\|X\|\|A\|^k\|Y\|\|A+X\|^{j-l-1}\|X\|\|A\|^l \\
&\quad + \sum_{j=2}^{r-1}\sum_{k=0}^{j-2}\sum_{l=0}^{j-k-2} \|A\|^{r-j-1}\|Y\|\|A+X\|^{j-k-l-2}\|X\|\|A\|^l\|X\|\|A\|^k,
\end{aligned}$$

for $\|X\| \leq 1$, proving that

$$D^{(2)}(A^r)\bullet(X,Y) = \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2} A^{r-j-k-2}XA^kYA^j + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1} A^{r-j-1}YA^kXA^{j-k-1}. \tag{3.1.2}$$

Recall that $A_s = A + sV \in \mathcal{B}_{s.a.}(\mathcal{H})$ $(0 \leq s \leq 1)$, and a similar calculation as in the proof of (i) in Theorem 2.1.1 shows that the map $[0,1] \ni s \longmapsto A_s^r$ is continuously differentiable in norm-topology and

$$\frac{d}{ds}(A_s^r) = \sum_{j=0}^{r-1} A_s^{r-j-1}VA_s^j = \sum_{j=0}^{r-1} A_s^jVA_s^{r-j-1}.$$

Hence

$$\begin{aligned}
(A+V)^r - A^r - D^{(1)}(A^r)\bullet V &= \int_0^1 ds\, \frac{d}{ds}(A_s^r) - \int_0^1 ds\, \left(\sum_{j=0}^{r-1} A^{r-j-1}VA^j\right) \\
= \int_0^1 ds \sum_{j=0}^{r-1}\left(A_s^{r-j-1}VA_s^j - A^{r-j-1}VA^j\right) &= \int_0^1 ds \sum_{j=0}^{r-1}\int_0^s d\tau \frac{d}{d\tau}\left(A_\tau^{r-j-1}VA_\tau^j\right),
\end{aligned}$$

which by an application of Leibnitz's rule reduces to

$$\int_0^1 ds\int_0^s d\tau \left(\sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2} A_\tau^{r-j-k-2}VA_\tau^kVA_\tau^j + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1} A_\tau^{r-j-1}VA_\tau^kVA_\tau^{j-k-1}\right)$$

and using (3.1.2), we get

$$\begin{aligned}(A+V)^r &- A^r - D^{(1)}(A^r)\bullet V - \frac{1}{2}D^{(2)}(A^r)\bullet(V,V)\\
&= \int_0^1 ds\int_0^s d\tau\{\sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2} A_\tau^{r-j-k-2}VA_\tau^kVA_\tau^j + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}A_\tau^{r-j-1}VA_\tau^kVA_\tau^{j-k-1}\\
&\qquad -\sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}A^{r-j-k-2}VA^kVA^j - \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}A^{r-j-1}VA^kVA^{j-k-1}\ \}.\end{aligned} \tag{3.1.3}$$

Let us denote the sum of the first and third term inside the integral in (3.1.3) to be

$$\begin{aligned}\mathrm{I}_1 &\equiv \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\left[A_\tau^{r-j-k-2}VA_\tau^kVA_\tau^j - A^{r-j-k-2}VA^kVA^j\right]\\
&= \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\left[A_\tau^{r-j-k-2}-A^{r-j-k-2}\right]VA_\tau^kVA_\tau^j + \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}A^{r-j-k-2}V\left[A_\tau^k - A^k\right]VA_\tau^j\\
&\qquad + \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}A^{r-j-k-2}VA^kV\left[A_\tau^j - A^j\right] \in \mathcal{B}_1(\mathcal{H}),\end{aligned}$$

since $V\in\mathcal{B}_3(\mathcal{H})$ and $A_\tau^k - A^k \in \mathcal{B}_3(\mathcal{H})\ \ \forall\tau\in[0,1]$ and $k\in\{0,1,2,3,......\}$. Thus by the cyclicity of trace , we have that

$$\begin{aligned}\mathrm{Tr}(\mathrm{I}_1) &= \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\mathrm{Tr}\{\left[A_\tau^{r-j-k-2}-A^{r-j-k-2}\right]VA_\tau^kVA_\tau^j\} + \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\mathrm{Tr}\{A^{r-j-k-2}V\left[A_\tau^k-A^k\right]VA_\tau^j\}\\
&\qquad + \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\mathrm{Tr}\{A^{r-j-k-2}VA^kV\left[A_\tau^j-A^j\right]\}\\
&= \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\mathrm{Tr}\{A_\tau^j\left[A_\tau^{r-j-k-2}-A^{r-j-k-2}\right]VA_\tau^kV\} + \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\mathrm{Tr}\{A_\tau^jA^{r-j-k-2}V\left[A_\tau^k-A^k\right]V\}\\
&\qquad + \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\mathrm{Tr}\{\left[A_\tau^j-A^j\right]A^{r-j-k-2}VA^kV\}\\
&= \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\mathrm{Tr}\{A_\tau^j\left[A_\tau^{r-j-k-2}-A^{r-j-k-2}\right]VA_\tau^kV + A_\tau^jA^{r-j-k-2}V\left[A_\tau^k-A^k\right]V\\
&\qquad + \left[A_\tau^j-A^j\right]A^{r-j-k-2}VA^kV\}\\
&= \sum_{j=0}^{r-2}\sum_{k=0}^{r-j-2}\mathrm{Tr}\left[A_\tau^{r-k-2}VA_\tau^kV - A^{r-k-2}VA^kV\right].\end{aligned}$$

Again if we set the sum of the second and fourth term inside the integral in (3.1.3) to be

$$\begin{aligned}
\mathrm{I}_2 &\equiv \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}\left[A_\tau^{r-j-1}VA_\tau^kVA_\tau^{j-k-1} - A^{r-j-1}VA^kVA^{j-k-1}\right]\\
&= \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}\left[A_\tau^{r-j-1} - A^{r-j-1}\right]VA_\tau^kVA_\tau^{j-k-1} + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}A^{r-j-1}V\left[A_\tau^k - A^k\right]VA_\tau^{j-k-1}\\
&\qquad + \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}A^{r-j-1}VA^kV\left[A_\tau^{j-k-1} - A^{j-k-1}\right] \in \mathcal{B}_1(\mathcal{H}),
\end{aligned}$$

since $V \in \mathcal{B}_3(\mathcal{H})$; $A_\tau^k - A^k \in \mathcal{B}_3(\mathcal{H})$ $\forall \tau \in [0,1]$; $k \in \{0,1,2,3,......\}$ and a similar calculation as above (using the cyclicity of trace) shows that

$$\mathrm{Tr}(\mathrm{I}_2) = \sum_{j=1}^{r-1}\sum_{k=0}^{j-1}\mathrm{Tr}\left[A_\tau^{r-k-2}VA_\tau^kV - A^{r-k-2}VA^kV\right].$$

By applying Lemma 3.1.1 with $n = r-1$ and $a_k = \mathrm{Tr}\left[A_\tau^{r-k-2}VA_\tau^kV - A^{r-k-2}VA^kV\right]$ and using the cyclicity of trace, we conclude that

$$\begin{aligned}
\mathrm{Tr}(\mathrm{I}_1) + \mathrm{Tr}(\mathrm{I}_2) &= r\sum_{k=0}^{r-2}\mathrm{Tr}\left[A_\tau^{r-k-2}VA_\tau^kV - A^{r-k-2}VA^kV\right]\\
&= r\sum_{k=0}^{r-2}\mathrm{Tr}\left[VA_\tau^{r-k-2}VA_\tau^k - VA^{r-k-2}VA^k\right].
\end{aligned} \tag{3.1.4}$$

Hence combining (3.1.3) and (3.1.4), we get the required expression (3.1.1). □

Lemma 3.1.3. *Let B be a bounded operator in an infinite dimensional Hilbert space $\mathcal{H}$(i.e. $B \in \mathcal{B}(\mathcal{H})$). Define $\mathcal{M}_B : \mathcal{B}_2(\mathcal{H}) \longmapsto \mathcal{B}_2(\mathcal{H})$ (looking upon $\mathcal{B}_2(\mathcal{H}) \equiv \widetilde{\mathcal{H}}$ as a Hilbert space with inner product given by trace i.e.$\langle X, Y\rangle_2 = \mathrm{Tr}\{X^*Y\}$ for $X, Y \in \mathcal{B}_2(\mathcal{H})$) by $\mathcal{M}_B(X) = BX - XB$; $X \in \mathcal{B}_2(\mathcal{H})$. Then*

(i) $\mathcal{M}_B$ is a bounded operator on $\widetilde{\mathcal{H}}$ (i.e. $\mathcal{M}_B \in \mathcal{B}(\widetilde{\mathcal{H}})$) with $\mathcal{M}_B^* = \mathcal{M}_{B^*}$.

(ii) $\mathrm{Ker}\,(\mathcal{M}_B)$ and its orthogonal complement $\overline{\mathrm{Ran}\,(\mathcal{M}_{B^*})}$ in $\widetilde{\mathcal{H}}$ are left invariant by left and right multiplication by B^n and $(B^*)^n$ $(n = 1,2,3,...)$ respectively.

(iii) $\widetilde{\mathcal{H}} = \mathrm{Ker}\,(\mathcal{M}_B) \bigoplus \overline{\mathrm{Ran}\,(\mathcal{M}_{B^*})}$; $\mathcal{B}_2(\mathcal{H}) \ni X = X_1 \oplus X_2$, where $X_1 \in \mathrm{Ker}\,(\mathcal{M}_B)$ and $X_2 \in \overline{\mathrm{Ran}\,(\mathcal{M}_{B^*})}$.

(iv) If $\text{Ker}\,(\mathcal{M}_B) = \text{Ker}\,(\mathcal{M}_{B^*})$*, then* $\text{Ker}\,(\mathcal{M}_B)$ *and* $\overline{\text{Ran}\,(\mathcal{M}_B)}$ *are generated by their self-adjoint elements and for* $X \in \widetilde{\mathcal{H}}$*, we have* $(X^*)_1 = X_1^*$ *and* $(X^*)_2 = X_2^*$*, where* $X = X_1 \oplus X_2$ *and* $X^* = (X^*)_1 \oplus (X^*)_2$ *are the respective decompositions of* X *and* X^* *in* $\widetilde{\mathcal{H}}$*.*

(v) If $\text{Ker}\,(\mathcal{M}_B) = \text{Ker}\,(\mathcal{M}_{B^*})$*, then for* $X = X^* \in \widetilde{\mathcal{H}},\ X = X_1 \oplus X_2$ *with* X_1 *and* X_2 *both self-adjoint.*

(vi) (a) For $B = B^* \in \mathcal{B}(\mathcal{H})$, $\mathcal{M}_B$ *is self-adjoint in* $\widetilde{\mathcal{H}}$ *and for* $X = X^* \in \widetilde{\mathcal{H}}$*, we have* $X_1 = X_1^*$ *and* $X_2 = X_2^*$*.*

(b) For $B = (A + \mathrm{i})^{-1}$ *(where* A *is an unbounded self-adjoint operator in* $\mathcal{H}$*),* $\mathcal{M}_B$ *is bounded normal in* $\widetilde{\mathcal{H}}$ *and for* $X = X^* \in \widetilde{\mathcal{H}}$*, we have* $X_1 = X_1^*$ *and* $X_2 = X_2^*$*, where* $X = X_1 \oplus X_2$ *is the decomposition of* X *in* $\widetilde{\mathcal{H}}$*.*

(vii) (a) Let $[0,1] \ni \tau \longrightarrow A_\tau \in \mathcal{B}_{s.a}(\mathcal{H})$*(set of bounded self-adjoint operators in* $\mathcal{H}$*) be continuous in operator norm , and let* $\widetilde{\mathcal{H}} \ni X \equiv X_{1\tau} \oplus X_{2\tau}$ *be the self-adjoint decomposition with respect to* A_τ*. Then* $\tau \longrightarrow X_{1\tau},\ X_{2\tau} \in \widetilde{\mathcal{H}}$ *are continuous.*

(b) Let $\{A_\tau\}_{\tau\in[0,1]}$ *be a family of unbounded self-adjoint operators in* $\mathcal{H}$ *such that* $[0,1] \ni \tau \longrightarrow (A_\tau + \mathrm{i})^{-1}$ *is continuous in operator norm. Then the conclusions of* $(vii)(a)$ *is valid for the decomposition of* $\widetilde{\mathcal{H}}$ *with respect to* $B_\tau \equiv (A_\tau + \mathrm{i})^{-1}$*.*

Proof. (i) For $X \in \widetilde{\mathcal{H}}$,

$$\|\mathcal{M}_B(X)\|_2 = \|BX - XB\|_2 \leq 2\|B\|\|X\|_2,$$

proving that $\mathcal{M}_B$ is a bounded operator and $\|\mathcal{M}_B\| \leq 2\|B\|$. Next for $X, Y \in \widetilde{\mathcal{H}}$,

$$\begin{aligned}\langle X, \mathcal{M}_B^*(Y)\rangle_2 &= \langle \mathcal{M}_B(X), Y\rangle_2 = \langle BX - XB, Y\rangle_2 = \text{Tr}\{(BX - XB)^*Y\}\\ &= \text{Tr}\{X^*B^*Y - B^*X^*Y\} = \text{Tr}\{X^*B^*Y\} - \text{Tr}\{B^*X^*Y\}\\ &= \text{Tr}\{X^*B^*Y\} - \text{Tr}\{X^*YB^*\} = \text{Tr}\{X^*(B^*Y - YB^*)\} = \langle X, \mathcal{M}_{B^*}(Y)\rangle_2,\end{aligned}$$

proving that $\mathcal{M}_B^* = \mathcal{M}_{B^*}$.

(ii) Notice that for $X \in \widetilde{\mathcal{H}}$,

$$X \in \text{Ker}\,(\mathcal{M}_B) \quad \text{if and only if} \quad XB = BX. \quad \text{Hence}$$

$$X \in \text{Ker}\,(\mathcal{M}_B) \Longrightarrow XB = BX \Longrightarrow \quad XB^n = B^nX \quad \text{for} \quad n = 1, 2, 3, ...$$

proving that $\text{Ker}\,(\mathcal{M}_B)$ is invariant by left and right multiplication by B^n.

Again for $X \in \text{Ker}\,(\mathcal{M}_B)^{\perp} = \overline{\text{Ran}\,(\mathcal{M}_{B^*})}$ and $Y \in \text{Ker}\,(\mathcal{M}_B)$, we have

$$\langle (B^*)^n X, Y\rangle_2 = \text{Tr}\{[(B^*)^n X]^* Y\} = \text{Tr}\{X^* B^n Y\} = \langle X, B^n Y\rangle_2 = 0 \quad \forall \quad Y,$$

since $\text{Ker}\,(\mathcal{M}_B)$ is invariant by left multiplication by B^n and hence $(B^*)^n X \in \text{Ker}\,(\mathcal{M}_B)^{\perp} = \overline{\text{Ran}\,(\mathcal{M}_{B^*})}$, proving that $\overline{\text{Ran}\,(\mathcal{M}_{B^*})}$ is invariant by left multiplication by $(B^*)^n$. Similarly by the same above argument we conclude that $\overline{\text{Ran}\,(\mathcal{M}_{B^*})}$ is invariant by right multiplication by $(B^*)^n$.

(iii) This is a standard decomposition of the Hilbert space $\widetilde{\mathcal{H}}$.

(iv) We note that since $\text{Ker}\,(\mathcal{M}_B) = \text{Ker}\,(\mathcal{M}_{B^*})$, $X \in \text{Ker}\,(\mathcal{M}_B)$ if and only if $X^* \in \text{Ker}\,(\mathcal{M}_B)$ and hence for any $X \in \text{Ker}\,(\mathcal{M}_B)$ can be written as $X = \left(\frac{X+X^*}{2}\right) + \mathrm{i}\left(\frac{X-X^*}{2\mathrm{i}}\right)$, proving that $\text{Ker}\,(\mathcal{M}_B)$ is generated by its self-adjoint elements. Similarly,

$$\text{Ker}\,(\mathcal{M}_B) = \text{Ker}\,(\mathcal{M}_{B^*}) \Longrightarrow \text{Ker}\,(\mathcal{M}_B)^{\perp} = \text{Ker}\,(\mathcal{M}_{B^*})^{\perp}\; i.e.\;\; \overline{\text{Ran}\,(\mathcal{M}_B)} = \overline{\text{Ran}\,(\mathcal{M}_{B^*})}$$

and hence $Y \in \overline{\text{Ran}\,(\mathcal{M}_B)}$ if and only if $Y^* \in \overline{\text{Ran}\,(\mathcal{M}_{B^*})}$ and therefore for any $Y \in \overline{\text{Ran}\,(\mathcal{M}_B)}$ can be written as $Y = \left(\frac{Y+Y^*}{2}\right) + \mathrm{i}\left(\frac{Y-Y^*}{2\mathrm{i}}\right)$, proving that $\overline{\text{Ran}\,(\mathcal{M}_B)}$ is generated by its self-adjoint elements.

Let $X \in \widetilde{\mathcal{H}}$, and $X = X_1 \oplus X_2$ and $X^* = (X^*)_1 \oplus (X^*)_2$ be the corresponding decompositions of X and X^* in $\widetilde{\mathcal{H}}$. Then for any $Y_1 = Y_1^* \in \text{Ker}\,(\mathcal{M}_B)$,

$$\langle X, Y_1\rangle_2 = \langle X_1, Y_1\rangle_2 = \text{Tr}\{X_1^* Y_1\} = \text{Tr}\{Y_1 X_1^*\} = \langle Y_1, X_1^*\rangle_2 = \overline{\langle X_1^*, Y_1\rangle_2}.$$

But on the other hand,

$$\langle X, Y_1\rangle_2 = \text{Tr}\{X^* Y_1\} = \text{Tr}\{(Y_1 X)^*\} = \overline{\text{Tr}\{Y_1 X\}} = \overline{\langle X^*, Y_1\rangle_2} = \overline{\langle (X^*)_1, Y_1\rangle_2}$$

and hence $\langle (X^*)_1 - X_1^*, Y_1\rangle_2 = 0 \;\forall\; Y_1 = Y_1^* \in \text{Ker}\,(\mathcal{M}_B)$, which implies that

$\langle (X^*)_1 - X_1^*, Y\rangle_2 = 0 \;\; \forall \;\; Y \in \text{Ker}\,(\mathcal{M}_B)$, proving that $(X^*)_1 = X_1^*$. Similarly, for any $Y_2 = Y_2^* \in \text{Ker}\,(\mathcal{M}_B)^{\perp}$,

$$\langle X, Y_2\rangle_2 = \langle X_2, Y_2\rangle_2 = \text{Tr}\{X_2^* Y_2\} = \text{Tr}\{Y_2 X_2^*\} = \langle Y_2, X_2^*\rangle_2 = \overline{\langle X_2^*, Y_2\rangle_2}.$$

But on the other hand,

$$\langle X, Y_2\rangle_2 = \text{Tr}\{X^*Y_2\} = \text{Tr}\{(Y_2X)^*\} = \overline{\text{Tr}\{Y_2X\}} = \overline{\langle X^*, Y_2\rangle_2} = \overline{\langle (X^*)_2, Y_2\rangle_2}$$

and hence $\langle (X^*)_2 - X_2^*, Y_2\rangle_2 = 0 \,\forall\; Y_2 = Y_2^* \in \text{Ker}\,(\mathcal{M}_B)$, which implies that

$$\langle (X^*)_2 - X_2^*, Y\rangle_2 = 0 \;\;\forall\;\; Y \in \text{Ker}\,(\mathcal{M}_B)^{\perp}, \text{ proving that } (X^*)_2 = X_2^*.$$

(v) The result follows easily from (iv).

(vi(a)) Since $B = B^*$, then $\mathcal{M}_B = \mathcal{M}_B^*$ and hence $\text{Ker}\,(\mathcal{M}_B) = \text{Ker}\,(\mathcal{M}_B^*) = \text{Ker}\,(\mathcal{M}_{B^*})$ and therefore the result follows easily from (v).

(vi(b)) Since $(A+\mathrm{i})^{-1}$ is a bounded normal operator with $((A+\mathrm{i})^{-1})^* = (A-\mathrm{i})^{-1}$, then $\mathcal{M}_{(A+\mathrm{i})^{-1}}$ is a bounded operator and for any $X \in \widetilde{\mathcal{H}}$

$$\begin{aligned}
&\mathcal{M}_{(A+\mathrm{i})^{-1}}\mathcal{M}^*_{(A+\mathrm{i})^{-1}}(X) = \mathcal{M}_{(A+\mathrm{i})^{-1}}\mathcal{M}_{(A-\mathrm{i})^{-1}}(X) = \mathcal{M}_{(A+\mathrm{i})^{-1}}((A-\mathrm{i})^{-1}X - X(A-\mathrm{i})^{-1})\\
&\quad = (A+\mathrm{i})^{-1}\left[(A-\mathrm{i})^{-1}X - X(A-\mathrm{i})^{-1}\right] - \left[(A-\mathrm{i})^{-1}X - X(A-\mathrm{i})^{-1}\right](A+\mathrm{i})^{-1}\\
&\quad = (A+\mathrm{i})^{-1}(A-\mathrm{i})^{-1}X - (A+\mathrm{i})^{-1}X(A-\mathrm{i})^{-1} - (A-\mathrm{i})^{-1}X(A+\mathrm{i})^{-1} + X(A-\mathrm{i})^{-1}(A+\mathrm{i})^{-1}\\
&\quad = (A-\mathrm{i})^{-1}(A+\mathrm{i})^{-1}X - (A-\mathrm{i})^{-1}X(A+\mathrm{i})^{-1} - (A+\mathrm{i})^{-1}X(A-\mathrm{i})^{-1} + X(A+\mathrm{i})^{-1}(A-\mathrm{i})^{-1}\\
&\quad = (A-\mathrm{i})^{-1}\left[(A+\mathrm{i})^{-1}X - X(A+\mathrm{i})^{-1}\right] - \left[(A+\mathrm{i})^{-1}X - X(A+\mathrm{i})^{-1}\right](A-\mathrm{i})^{-1}\\
&\quad = \mathcal{M}_{(A-\mathrm{i})^{-1}}((A+\mathrm{i})^{-1}X - X(A+\mathrm{i})^{-1}) = \mathcal{M}_{(A-\mathrm{i})^{-1}}\mathcal{M}_{(A+\mathrm{i})^{-1}}(X) = \mathcal{M}^*_{(A+\mathrm{i})^{-1}}\mathcal{M}_{(A+\mathrm{i})^{-1}}(X),
\end{aligned}$$

proving that $\mathcal{M}_{(A+\mathrm{i})^{-1}}$ is normal. Moreover for any $X \in \mathcal{B}_2(\mathcal{H})$

$$\begin{aligned}
&X \in \text{Ker}\left(\mathcal{M}_{(A+\mathrm{i})^{-1}}\right) \Leftrightarrow X(A+\mathrm{i})^{-1} = (A+\mathrm{i})^{-1}X\\
&\Leftrightarrow XE_A(.) = E_A(.)X \Leftrightarrow X(A-\mathrm{i})^{-1} = (A-\mathrm{i})^{-1}X \Leftrightarrow X \in \text{Ker}\left(\mathcal{M}_{(A-\mathrm{i})^{-1}}\right),
\end{aligned}$$

where $E_A(.)$ is the spectral family of A, proving that $\text{Ker}\left(\mathcal{M}_{(A+\mathrm{i})^{-1}}\right) = \text{Ker}\left(\mathcal{M}^*_{(A+\mathrm{i})^{-1}}\right)$ and hence the rest part of the result follows easily from (v).

(vii(a)) Since the map $[0,1] \ni \tau \longrightarrow \mathcal{M}_{A_\tau}$ is holomorphic, then (using Theorem 1.8, page 370, [6]) we conclude that the map $[0,1] \ni \tau \longrightarrow P_0(\tau)$ (where $P_0(\tau)$ is the projection onto Ker($\mathcal{M}_{A_\tau}$)) is continuous and since $X_{1\tau} \equiv P_0(\tau)X$ we get that the map $[0,1] \ni \tau \longrightarrow X_{1\tau}$ is continuous. Similarly, since the map $[0,1] \ni \tau \longrightarrow I - P_0(\tau)$ is continuous and $X_{2\tau} = (I - P_0(\tau))X$ then we conclude that the map $[0,1] \ni \tau \longrightarrow X_{2\tau}$ is also continuous.

(vii(b)) Result follows immediately from $(vii(a))$ since the map $[0,1] \ni \tau \longrightarrow \mathcal{M}_{(A_\tau+\mathrm{i})^{-1}}$ is

holomorphic, and since $\mathcal{M}_{(A_\tau+\mathrm{i})^{-1}}$ is normal for each τ.

$\square$

Remark 3.1.4. Let A and V be two bounded self-adjoint operators in an infinite dimensional Hilbert space $\mathcal{H}$ such that $V \in \mathcal{B}_2(\mathcal{H})$ and $A_\tau = A + \tau V$ $(0 \leq \tau \leq 1)$. Apply Lemma 3.1.3 with $B = A$ and A_τ respectively to get $V = V_1 \oplus V_2 = V_{1\tau} \oplus V_{2\tau}$, with V_j and $V_{j\tau}$ $(j = 1, 2)$ self-adjoint and therefore $\|V\|_2^2 = \|V_1\|_2^2 + \|V_2\|_2^2 = \|V_{1\tau}\|_2^2 + \|V_{2\tau}\|_2^2$ $\forall\, 0 \leq \tau \leq 1$.

Theorem 3.1.5. *Let A and V be two bounded self-adjoint operators in an infinite dimensional Hilbert space $\mathcal{H}$ such that $V \in \mathcal{B}_2(\mathcal{H})$. Then there exist a unique real-valued function $\eta \in L^1([a, b])$ such that*

$$\mathrm{Tr}\left[p(A+V) - p(A) - D^{(1)}p(A) \bullet V - \frac{1}{2}D^{(2)}p(A) \bullet (V, V)\right] = \int_a^b p'''(\lambda)\eta(\lambda)d\lambda, \qquad (3.1.5)$$

where $p(.)$ is a polynomial in $[a, b]$, $a = [\inf \sigma(A)] - \|V\|$, $b = [\sup \sigma(A)] + \|V\|$ and $\int_a^b \eta(\lambda)d\lambda = \frac{1}{6}\mathrm{Tr}(V^3)$.

Proof. It will be sufficient to prove the theorem for $p(\lambda) = \lambda^r$ $(r \geq 0)$. Note that for $r = 0, 1$ or 2, both sides of (3.1.5) are identically zero. We set $A_\tau = A + \tau V$ and $0 \leq \tau \leq 1$. Then by Lemma 3.1.2, we have that

$$\begin{aligned}\mathrm{Tr}&\left[(A+V)^r - A^r - D^{(1)}(A^r) \bullet V - \frac{1}{2}D^{(2)}(A^r) \bullet (V, V)\right] \\ &= r\sum_{k=0}^{r-2} \int_0^1 ds \int_0^s d\tau \ \mathrm{Tr}\left[VA_\tau^{r-k-2}VA_\tau^k - VA^{r-k-2}VA^k\right]\end{aligned}$$

$$\begin{aligned}&= r\sum_{k=0}^{r-2} \int_0^1 ds \int_0^s d\tau \ \langle V, A_\tau^{r-k-2}VA_\tau^k\rangle_2 - \langle V, A^{r-k-2}VA^k\rangle_2 \\ &= r\sum_{k=0}^{r-2} \int_0^1 ds \int_0^s d\tau \ \langle (V_{1\tau} \oplus V_{2\tau}), A_\tau^{r-k-2}(V_{1\tau} \oplus V_{2\tau})A_\tau^k\rangle_2 - \langle (V_1 \oplus V_2), A^{r-k-2}(V_1 \oplus V_2)A^k\rangle_2,\end{aligned}$$

(3.1.6)

where we set $V = V_1 \oplus V_2 = V_{1\tau} \oplus V_{2\tau} \in \mathcal{B}_2(\mathcal{H})$ as in Remark 3.1.4. Again by using the invariance, orthogonality and continuity properties in Lemma 3.1.3 $(ii) - (vii)$, the right hand side of (3.1.6) is equal to

$$\begin{aligned}
& r\sum_{k=0}^{r-2}\int_0^1 ds\int_0^s d\tau\ \{\left[\langle V_{1\tau}, A_\tau^{r-k-2}V_{1\tau}A_\tau^k\rangle_2 - \langle V_1, A^{r-k-2}V_1A^k\rangle_2\right] \\
&\qquad\qquad + \left[\langle V_{2\tau}, A_\tau^{r-k-2}V_{2\tau}A_\tau^k\rangle_2 - \langle V_2, A^{r-k-2}V_2A^k\rangle_2\right]\} \\
& = r(r-1)\int_0^1 ds\int_0^s d\tau\ \mathrm{Tr}\left[V_{1\tau}^2A_\tau^{r-2} - V_1^2A^{r-2}\right] \\
&\qquad + r\sum_{k=0}^{r-2}\int_0^1 ds\int_0^s d\tau\ \mathrm{Tr}\left[V_{2\tau}A_\tau^{r-k-2}V_{2\tau}A_\tau^k - V_2A^{r-k-2}V_2A^k\right].
\end{aligned} \tag{3.1.7}$$

Next by using the spectral families $E_{A_\tau}(.)$ and $E_A(.)$ of the self-adjoint operators A_τ and A respectively and integrating by-parts, the first term of the expression (3.1.7) is equal to

$$\begin{aligned}
& r(r-1)\int_0^1 ds\int_0^s d\tau\int_a^b \lambda^{r-2}\ \mathrm{Tr}\left[V_{1\tau}^2E_{A_\tau}(d\lambda) - V_1^2E_A(d\lambda)\right] \\
&\quad = r(r-1)\int_0^1 ds\int_0^s d\tau\{\lambda^{r-2}\ \mathrm{Tr}\left[V_{1\tau}^2E_{A_\tau}(\lambda) - V_1^2E_A(\lambda)\right]\ |_{\lambda=a}^b \\
&\qquad\qquad - \int_a^b (r-2)\lambda^{r-3}\ \mathrm{Tr}\left[V_{1\tau}^2E_{A_\tau}(\lambda) - V_1^2E_A(\lambda)\right]d\lambda\}
\end{aligned}$$

$$\begin{aligned}
& = r(r-1)b^{r-2}\int_0^1 ds\int_0^s d\tau\ \mathrm{Tr}\left[V_{1\tau}^2 - V_1^2\right] \\
&\qquad + r(r-1)(r-2)\int_0^1 ds\int_0^s d\tau\int_a^b \lambda^{r-3}\ \mathrm{Tr}\left[V_1^2E_A(\lambda) - V_{1\tau}^2E_{A_\tau}(\lambda)\right]d\lambda.
\end{aligned} \tag{3.1.8}$$

Since $V_2 \in \overline{\mathrm{Ran}(\mathcal{M}_A)}$, then there exists a sequence $\{V_2^{(n)}\} \subseteq \mathrm{Ran}(\mathcal{M}_A)$ such that

$\|V_2^{(n)} - V_2\|_2 \longrightarrow 0$ as $n \longrightarrow \infty$ and $V_2^{(n)} = AY_0^{(n)} - Y_0^{(n)}A$, for a sequence $\{Y_0^{(n)}\} \subseteq \mathcal{B}_2(\mathcal{H})$. Without loss of generality we can assume that $V_2^{(n)}$ is self-adjoint for each n , since if it is not self-adjoint, consider the sequence $\{\frac{V_2^{(n)}+(V_2^{(n)})^*}{2}\}$, which converges to V_2 in $\|.\|_2$ norm (since V_2 is self-adjoint) and $\frac{V_2^{(n)}+(V_2^{(n)})^*}{2}$ is self-adjoint for each n. Also without loss of generality $Y_0^{(n)}$ can be chosen to be skew-adjoint for each n, since if it is not skew-adjoint, consider $(Y_0^{(n)})' =$

$Y_0^{(n)} - \left(\frac{Y_0^{(n)}+(Y_0^{(n)})^*}{2}\right) = \mathrm{i}\frac{Y_0^{(n)}-(Y_0^{(n)})^*}{2\mathrm{i}}$, where $(Y_0^{(n)})'$ is skew-adjoint and

$$\begin{aligned}
A(Y_0^{(n)})' - (Y_0^{(n)})'A &= A\left[Y_0^{(n)} - \left(\frac{Y_0^{(n)} + (Y_0^{(n)})^*}{2}\right)\right] - \left[Y_0^{(n)} - \left(\frac{Y_0^{(n)} + (Y_0^{(n)})^*}{2}\right)\right]A \\
&= \left[AY_0^{(n)} - Y_0^{(n)}A\right] - \left[A\left(\frac{Y_0^{(n)} + (Y_0^{(n)})^*}{2}\right) - \left(\frac{Y_0^{(n)} + (Y_0^{(n)})^*}{2}\right)A\right] \\
&= \left[AY_0^{(n)} - Y_0^{(n)}A\right] - \left[\left(\frac{AY_0^{(n)} - Y_0^{(n)}A}{2}\right) - \left(\frac{(Y_0^{(n)})^*A - A(Y_0^{(n)})^*}{2}\right)\right] \\
&= \left[AY_0^{(n)} - Y_0^{(n)}A\right] - \left[\left(\frac{AY_0^{(n)} - Y_0^{(n)}A}{2}\right) - \left(\frac{\{AY_0^{(n)} - Y_0^{(n)}A\}^*}{2}\right)\right] \\
&= V_2^{(n)} - \left[\frac{V_2^{(n)}}{2} - \frac{(V_2^{(n)})^*}{2}\right] = V_2^{(n)},
\end{aligned}$$

since $V_2^{(n)}$ is self-adjoint. Similarly, for every $\tau \in (0,1]$, there exists a sequence $\{V_{2\tau}^{(n)}\} \subseteq$ Ran$(\mathcal{M}_{A_\tau})$ such that $\|V_{2\tau}^{(n)} - V_{2\tau}\|_2 \longrightarrow 0$ point-wise as $n \longrightarrow \infty$ and $V_{2\tau}^{(n)}$ is self-adjont for each n and $V_{2\tau}^{(n)} = A_\tau Y^{(n)} - Y^{(n)}A_\tau$, for some sequence $\{Y^{(n)}\} \subseteq \mathcal{B}_2(\mathcal{H})$, where $Y^{(n)}$ can be chosen to be skew-adjoint for each n. Furthermore, by Lemma 3.1.3 $(vii)(a)$, the map $[0,1] \ni \tau \longrightarrow V_{1\tau}, V_{2\tau}$ are continuous.

Hence the second term of the expression (3.1.7) is equal to

$$\begin{aligned}
& r\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \sum_{k=0}^{r-2} \mathrm{Tr}\{V_{2\tau}A_\tau^{r-k-2}V_{2\tau}^{(n)}A_\tau^k - V_2A^{r-k-2}V_2^{(n)}A^k\} \\
&= r\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \sum_{k=0}^{r-2} \int_a^b \int_a^b \lambda^{r-k-2}\mu^k \,\mathrm{Tr}\{V_{2\tau}E_{A_\tau}(d\lambda)V_{2\tau}^{(n)}E_{A_\tau}(d\mu) - V_2E_A(d\lambda)V_2^{(n)}E_A(d\mu)\} \\
&= r\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \int_a^b \sum_{k=0}^{r-2} \lambda^{r-k-2}\mu^k \,\mathrm{Tr}\{V_{2\tau}E_{A_\tau}(d\lambda)V_{2\tau}^{(n)}E_{A_\tau}(d\mu) - V_2E_A(d\lambda)V_2^{(n)}E_A(d\mu)\} \\
&= r\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \int_a^b \phi(\lambda,\mu) \,\mathrm{Tr}\{V_{2\tau}E_{A_\tau}(d\lambda)V_{2\tau}^{(n)}E_{A_\tau}(d\mu) - V_2E_A(d\lambda)V_2^{(n)}E_A(d\mu)\},
\end{aligned}$$

where $\phi(\lambda,\mu) = \frac{\lambda^{r-1}-\mu^{r-1}}{\lambda-\mu}$ if $\lambda \neq \mu$; $= (r-1)\lambda^{r-2}$ if $\lambda = \mu$, and where the interchange of the limit and the integration is justified by an application of the bounded convergence theorem.

Furthermore using the representation of $V_{2\tau}^{(n)} \in \text{Ran}(\mathcal{M}_{A_\tau})$, the above reduces to

$$
\begin{aligned}
& r\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \int_a^b \phi(\lambda,\mu) \, \text{Tr}\{V_{2\tau} E_{A_\tau}(d\lambda) \left[A_\tau Y^{(n)} - Y^{(n)} A_\tau\right] E_{A_\tau}(d\mu) \\
& \qquad\qquad\qquad\qquad\qquad\qquad - V_2 E_A(d\lambda) \left[A Y_0^{(n)} - Y_0^{(n)} A\right] E_A(d\mu)\} \\
&= r\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \int_a^b \phi(\lambda,\mu)(\lambda-\mu) \text{Tr}\{V_{2\tau} E_{A_\tau}(d\lambda) Y^{(n)} E_{A_\tau}(d\mu) - V_2 E_A(d\lambda) Y_0^{(n)} E_A(d\mu)\} \\
&= r\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \int_a^b \left(\lambda^{r-1} - \mu^{r-1}\right) \text{Tr}\{V_{2\tau} E_{A_\tau}(d\lambda) Y^{(n)} E_{A_\tau}(d\mu) - V_2 E_A(d\lambda) Y_0^{(n)} E_A(d\mu)\} \\
&= r\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \{\int_a^b \lambda^{r-1} \, \text{Tr}[V_{2\tau} E_{A_\tau}(d\lambda) Y^{(n)} - V_2 E_A(d\lambda) Y_0^{(n)}] \\
& \qquad\qquad\qquad\qquad - \int_a^b \mu^{r-1} \, \text{Tr}[V_{2\tau} Y^{(n)} E_{A_\tau}(d\mu) - V_2 Y_0^{(n)} E_A(d\mu)]\}
\end{aligned}
$$

$$
= r\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \lambda^{r-1} \, \text{Tr}\{V_{2\tau} \left[E_{A_\tau}(d\lambda), Y^{(n)}\right] - V_2 \left[E_A(d\lambda), Y_0^{(n)}\right]\}. \tag{3.1.9}
$$

Again by twice integrating by-parts, the expression in (3.1.9) is equal to

$$
\begin{aligned}
& r\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \{\lambda^{r-1} \, \text{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\lambda), Y^{(n)}\right] - V_2\left[E_A(\lambda), Y_0^{(n)}\right]\right) |_{\lambda=a}^{b} \\
& \qquad\qquad - \int_a^b (r-1)\lambda^{r-2} \, \text{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\lambda), Y^{(n)}\right] - V_2\left[E_A(\lambda), Y_0^{(n)}\right]\right) d\lambda\} \\
&= -r(r-1)\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \lambda^{r-2} \, \text{Tr}\{V_{2\tau}\left[E_{A_\tau}(\lambda), Y^{(n)}\right] - V_2\left[E_A(\lambda), Y_0^{(n)}\right]\} \, d\lambda
\end{aligned}
$$

$$
\begin{aligned}
&= -r(r-1)\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \{\lambda^{r-2}\left(\int_a^\lambda \text{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right) d\mu\right)\}|_{\lambda=a}^{b} \\
&+ r(r-1)\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b (r-2)\lambda^{r-3}\left(\int_a^\lambda \text{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right) d\mu\right) d\lambda \\
&= -r(r-1)b^{r-2}\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \text{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right) d\mu \\
&+ r(r-1)(r-2)\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \lambda^{r-3}\left(\int_a^\lambda \text{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right) d\mu\right) d\lambda.
\end{aligned} \tag{3.1.10}
$$

Next we note that by an integration by-parts,

$$\begin{aligned}
&\mathrm{Tr}\left(V_{2\tau}^2 - V_2^2\right) = \lim_{n\to\infty} \mathrm{Tr}\left(V_{2\tau}V_{2\tau}^{(n)} - V_2 V_2^{(n)}\right) = \lim_{n\to\infty} \mathrm{Tr}\left(V_{2\tau}\left[A_\tau, Y^{(n)}\right] - V_2\left[A, Y_0^{(n)}\right]\right)\\
&= \lim_{n\to\infty} \int_a^b \mu \mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(d\mu), Y^{(n)}\right] - V_2\left[E_A(d\mu), Y_0^{(n)}\right]\right)\\
&= \lim_{n\to\infty}\left[\mu \mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right)|_{\mu=a}^{b} - \int_a^b \mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right) d\mu\right]\\
&= -\lim_{n\to\infty}\int_a^b \mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right) d\mu.
\end{aligned} \tag{3.1.11}$$

The boundary term above vanishes and substituting the above in the first expression in (3.1.10), we get that the right hand side of (3.1.10)

$$= r(r-1)b^{r-2}\int_0^1 ds \int_0^s d\tau\, \mathrm{Tr}\left(V_{2\tau}^2 - V_2^2\right) + r(r-1)(r-2)\int_0^1 ds\int_0^s d\tau \lim_{n\to\infty}\int_a^b \lambda^{r-3}\eta_{2\tau}^{(n)}(\lambda)d\lambda,$$

where $\eta_{2\tau}^{(n)}(\lambda) = \int_a^\lambda \mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right) d\mu$ and $\eta_{2\tau}^{(n)}(\lambda)$ is real-valued for each n, because

$$\begin{aligned}
\overline{\eta_{2\tau}^{(n)}(\lambda)} &= \int_a^\lambda \overline{\mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right)} d\mu\\
&= \int_a^\lambda \{\mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right]\right)^* - \mathrm{Tr}\left(V_2\left[E_A(\mu), Y_0^{(n)}\right]\right)^*\} d\mu\\
&= \int_a^\lambda \{\mathrm{Tr}\left(\left[E_{A_\tau}(\mu), Y^{(n)}\right]^* V_{2\tau}^*\right) - \mathrm{Tr}\left(\left[E_A(\mu), Y_0^{(n)}\right]^* V_2^*\right)\} d\mu\\
&= \int_a^\lambda \{\mathrm{Tr}\left(\left[E_{A_\tau}(\mu), Y^{(n)}\right] V_{2\tau}\right) - \mathrm{Tr}\left(\left[E_A(\mu), Y_0^{(n)}\right] V_2\right)\} d\mu\\
&= \int_a^\lambda \mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right) d\mu\\
&= \eta_{2\tau}^{(n)}(\lambda),
\end{aligned}$$

since $V_{2\tau}$ and V_2 are self-adjoint and $Y^{(n)}$ and $Y_0^{(n)}$ are skew-adjoint.

Hence

$$\begin{aligned}
&r\sum_{k=0}^{r-2}\int_0^1 ds\int_0^s d\tau\, \mathrm{Tr}\left[V_{2\tau}A_\tau^{r-k-2}V_{2\tau}A_\tau^k - V_2 A^{r-k-2} V_2 A^k\right]\\
&= r(r-1)b^{r-2}\int_0^1 ds \int_0^s d\tau\, \mathrm{Tr}\left(V_{2\tau}^2 - V_2^2\right) + r(r-1)(r-2)\int_0^1 ds\int_0^s d\tau \lim_{n\to\infty}\int_a^b \lambda^{r-3}\eta_{2\tau}^{(n)}(\lambda)d\lambda.
\end{aligned} \tag{3.1.12}$$

Combining (3.1.8) and (3.1.12) and since $\|V\|_2^2 = \text{Tr}\,(V_{1\tau}^2 + V_{2\tau}^2) = \text{Tr}\,(V_1^2 + V_2^2)$, we conclude that

$$\begin{aligned}
&\text{Tr}\left[(A+V)^r - A^r - D^{(1)}(A^r) \bullet V - \frac{1}{2} D^{(2)}(A^r) \bullet (V,V)\right] \\
&\quad = r(r-1)b^{r-2} \int_0^1 ds \int_0^s d\tau \ \text{Tr}\left(\left[V_{1\tau}^2 - V_1^2\right] + \left[V_{2\tau}^2 - V_2^2\right]\right) \\
&\qquad + r(r-1)(r-2) \int_0^1 ds \int_0^s d\tau \int_a^b \lambda^{r-3} \ \text{Tr}\left[V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\right] d\lambda \\
&\qquad + r(r-1)(r-2) \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \lambda^{r-3} \ \eta_{2\tau}^{(n)}(\lambda) d\lambda \\
&\quad = r(r-1)(r-2) \int_0^1 ds \int_0^s d\tau \int_a^b \lambda^{r-3} \ \text{Tr}\left[V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\right] d\lambda \\
&\qquad + r(r-1)(r-2) \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \lambda^{r-3} \ \eta_{2\tau}^{(n)}(\lambda) d\lambda \\
&\quad = r(r-1)(r-2) \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_a^b \lambda^{r-3} \ \eta_{\tau}^{(n)}(\lambda) d\lambda \\
&\quad = \lim_{n\to\infty} \int_a^b (\lambda^r)''' \ \eta^{(n)}(\lambda) d\lambda\,, \quad \text{where}
\end{aligned}$$

$\eta^{(n)}(\lambda) \equiv \int_0^1 ds \int_0^s d\tau \ \eta_\tau^{(n)}(\lambda)$ and $\eta_\tau^{(n)}(\lambda) = \left[\ \text{Tr}\{V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\} + \eta_{2\tau}^{(n)}(\lambda)\right]$, the interchange of limit and the τ- and s- integral is justified by an easy application of bounded convergence theorem. Note that $\eta_\tau^{(n)}$ is a real-valued function $\forall\ n$, because

$$\begin{aligned}
\overline{\eta_\tau^{(n)}} = \overline{\text{Tr}\{V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\} + \eta_{2\tau}^{(n)}(\lambda)} &= \text{Tr}\{\left(V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\right)^*\} + \overline{\eta_{2\tau}^{(n)}(\lambda)} \\
&= \text{Tr}\{E_A(\lambda) V_1^2 - E_{A_\tau}(\lambda) V_{1\tau}^2\} + \eta_{2\tau}^{(n)}(\lambda) \\
&= \text{Tr}\{V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\} + \eta_{2\tau}^{(n)}(\lambda) \\
&= \eta_\tau^{(n)},
\end{aligned}$$

since $V_{1\tau}$ and V_1 are self-adjoint and hence $\eta^{(n)}$ is real-valued for each n.

Next we want to show that $\{\eta^{(n)}\}$ is cauchy in $L^1([a,b])$ and we proceed by a method similar to that used in the proof of Theorem 2.3.1. Let $f \in L^\infty([a,b])$ and define $g(\lambda) = \int_a^\lambda f(t)dt$,

$h(\lambda) = \int_a^\lambda g(\mu)d\mu$ so that $g'(\lambda) = f(\lambda)$ a.e. and $h'(\lambda) = g(\lambda)$. Now consider the expression

$$\begin{aligned}
&\int_a^b f(\lambda)\left[\eta_\tau^{(n)}(\lambda) - \eta_\tau^{(m)}(\lambda)\right] d\lambda = \int_a^b f(\lambda)\left[\eta_{2\tau}^{(n)}(\lambda) - \eta_{2\tau}^{(m)}(\lambda)\right] d\lambda \\
&\qquad = \int_a^b h''(\lambda)d\lambda \left(\int_a^\lambda \operatorname{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)} - Y^{(m)}\right] - V_2\left[E_A(\mu), Y_0^{(n)} - Y_0^{(m)}\right]\right) d\mu\right),
\end{aligned}$$

which on integration by-parts twice and on observing that the boundary term for $\lambda = a$ vanishes, leads to

$$\begin{aligned}
&h'(\lambda) \int_a^\lambda \operatorname{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)} - Y^{(m)}\right] - V_2\left[E_A(\mu), Y_0^{(n)} - Y_0^{(m)}\right]\right) d\mu\Big|_{\lambda=a}^{b} \\
&\qquad - \int_a^b h'(\lambda) \operatorname{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\lambda), Y^{(n)} - Y^{(m)}\right] - V_2\left[E_A(\lambda), Y_0^{(n)} - Y_0^{(m)}\right]\right) d\lambda \\
&= h'(b)\int_a^b \operatorname{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)} - Y^{(m)}\right] - V_2\left[E_A(\mu), Y_0^{(n)} - Y_0^{(m)}\right]\right) d\mu \\
&\qquad - \{h(\lambda)\operatorname{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\lambda), Y^{(n)} - Y^{(m)}\right] - V_2\left[E_A(\lambda), Y_0^{(n)} - Y_0^{(m)}\right]\right)\Big|_{\lambda=a}^{b} \\
&\qquad\qquad - \int_a^b h(\lambda)\operatorname{Tr}\left(V_{2\tau}\left[E_{A_\tau}(d\lambda), Y^{(n)} - Y^{(m)}\right] - V_2\left[E_A(d\lambda), Y_0^{(n)} - Y_0^{(m)}\right]\right)\} \\
&= h'(b)\int_a^b \operatorname{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)} - Y^{(m)}\right] - V_2\left[E_A(\mu), Y_0^{(n)} - Y_0^{(m)}\right]\right) d\mu \\
&\qquad + \int_a^b h(\lambda)\operatorname{Tr}\left(V_{2\tau}\left[E_{A_\tau}(d\lambda), Y^{(n)} - Y^{(m)}\right] - V_2\left[E_A(d\lambda), Y_0^{(n)} - Y_0^{(m)}\right]\right) \\
&= h'(b)\int_a^b \operatorname{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)} - Y^{(m)}\right] - V_2\left[E_A(\mu), Y_0^{(n)} - Y_0^{(m)}\right]\right) d\mu \\
&\qquad + \operatorname{Tr}\left(V_{2\tau}\left[h(A_\tau), Y^{(n)} - Y^{(m)}\right] - V_2\left[h(A), Y_0^{(n)} - Y_0^{(m)}\right]\right).
\end{aligned} \tag{3.1.13}$$

Next we use the identity (follows from (3.1.11))

$$\operatorname{Tr}\left(V_{2\tau}V_{2\tau}^{(n)} - V_2V_2^{(n)}\right) = -\int_a^b \operatorname{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), Y^{(n)}\right] - V_2\left[E_A(\mu), Y_0^{(n)}\right]\right) d\mu$$

to reduce the the above expression in (3.1.13) to

$$g(b)\operatorname{Tr}\left(V_2\left[V_2^{(n)} - V_2^{(m)}\right] - V_{2\tau}\left[V_{2\tau}^{(n)} - V_{2\tau}^{(m)}\right]\right) + \operatorname{Tr}\left(V_{2\tau}\left[h(A_\tau), Y^{(n)} - Y^{(m)}\right] - V_2\left[h(A), Y_0^{(n)} - Y_0^{(m)}\right]\right). \tag{3.1.14}$$

But on the other hand,

$$\left[h(A), Y_0^{(n)}\right] = h(A)Y_0^{(n)} - Y_0^{(n)}h(A) = \int_a^b h(\lambda)E_A(d\lambda)Y_0^{(n)} - \int_a^b h(\mu)Y_0^{(n)}E_A(d\mu)$$
$$= \int_a^b \int_a^b [h(\lambda) - h(\mu)]E_A(d\lambda)Y_0^{(n)}E_A(d\mu) = \int_a^b \int_a^b \frac{h(\lambda) - h(\mu)}{\lambda - \mu} (\lambda - \mu)E_A(d\lambda)Y_0^{(n)}E_A(d\mu)$$
$$= \int_a^b \int_a^b \frac{h(\lambda) - h(\mu)}{\lambda - \mu} E_A(d\lambda)[AY_0^{(n)} - Y_0^{(n)}A]E_A(d\mu) = \int_a^b \int_a^b \frac{h(\lambda) - h(\mu)}{\lambda - \mu} E_A(d\lambda)V_2^{(n)}E_A(d\mu),$$

and

$$\text{Tr}\left(V_2\left[h(A), Y_0^{(n)} - Y_0^{(m)}\right]\right) = \int_a^b \int_a^b \frac{h(\lambda) - h(\mu)}{\lambda - \mu} \text{Tr}\left(V_2E_A(d\lambda)\left[V_2^{(n)} - V_2^{(m)}\right]E_A(d\mu)\right) \tag{3.1.15}$$

and hence as in Birman-Solomyak ([2],[3]) and in [7] ,

$$\left|\text{Tr}\left(V_2\left[h(A), Y_0^{(n)} - Y_0^{(m)}\right]\right)\right| \leq \|h\|_{\text{Lip}}\|V_2\|_2 \left\|\left[V_2^{(n)} - V_2^{(m)}\right]\right\|_2 \leq (b-a)\|f\|_\infty\|V\|_2 \left\|\left[V_2^{(n)} - V_2^{(m)}\right]\right\|_2$$

and hence

$$\sup_{f\in L^\infty([a,b])} \frac{\left|\int_a^b f(\lambda)\left[\eta_\tau^{(n)}(\lambda) - \eta_\tau^{(m)}(\lambda)\right]d\lambda\right|}{\|f\|_\infty} \leq 2(b-a)\|V\|_2\left(\left\|\left[V_2^{(n)} - V_2^{(m)}\right]\right\|_2 + \left\|\left[V_{2\tau}^{(n)} - V_{2\tau}^{(m)}\right]\right\|_2\right)$$

$$i.e. \quad \|\eta_\tau^{(n)} - \eta_\tau^{(m)}\|_{L^1} \leq 2(b-a)\|V\|_2\left(\left\|\left[V_2^{(n)} - V_2^{(m)}\right]\right\|_2 + \left\|\left[V_{2\tau}^{(n)} - V_{2\tau}^{(m)}\right]\right\|_2\right),$$

which converges to 0 as $n, m \to \infty$ and $\forall\ \tau \in [0, 1]$. Therefore

$$\|\eta^{(n)} - \eta^{(m)}\|_{L^1} \leq \int_0^1 ds \int_0^s d\tau\ \|\eta_\tau^{(n)} - \eta_\tau^{(m)}\|_{L^1},$$

which converges to 0 as $m, n \longrightarrow \infty$, by the bounded convergence theorem and hence $\{\eta^{(n)}\}$ is a Cauchy sequence in $L^1([a, b])$ and thus there exists a function $\eta \in L^1([a, b])$ such that $\|\eta^{(n)} - \eta\|_{L^1} \to 0$ as $n \longrightarrow \infty$. Therefore,

$$\lim_{n\longrightarrow\infty} \int_a^b \lambda^{r-3}\eta^{(n)}(\lambda)d\lambda = \int_a^b \lambda^{r-3}\eta(\lambda)d\lambda \quad \text{and hence}$$

$$\text{Tr}\left[(A + V)^r - A^r - D^{(1)}(A^r) \bullet V - \frac{1}{2}D^{(2)}(A^r) \bullet (V, V)\right] = r(r-1)(r-2)\int_a^b \lambda^{r-3}\eta(\lambda)d\lambda. \tag{3.1.16}$$

In particular if we set $p(\lambda) = \lambda^3$ in the formula (3.1.5), we get

$$\begin{aligned} 6\int_a^b \eta(\lambda)d\lambda &= \mathrm{Tr}\left[(A+V)^3 - A^3 - D^{(1)}(A^3)\bullet V - \frac{1}{2}D^{(2)}(A^3)\bullet (V,V)\right] \\ &= \mathrm{Tr}\left[(A+V)^3 - A^3 - \{A^2V + AVA + VA^2\} - \frac{1}{2}\{2(V^2A + VAV + AV^2)\}\right] \\ &= \mathrm{Tr}\{V^3\}. \end{aligned}$$

For uniqueness, let us assume that there exists $\eta_1,\ \eta_2 \in L^1([a,b])$ such that

$$\mathrm{Tr}\left[p(A+V) - p(A) - D^{(1)}p(A)\bullet V - \frac{1}{2}D^{(2)}p(A)\bullet (V,V)\right] = \int_a^b p'''(\lambda)\eta_j(\lambda)d\lambda,$$

where $p(.)$ is a polynomial and $j = 1,2$. Therefore

$$\int_a^b p'''(\lambda)\ \eta(\lambda)\ d\lambda = 0\ \ \forall \quad \text{polynomials} \quad p(.) \quad \text{and} \quad \eta \equiv \eta_1 - \eta_2 \in L^1([a,b]),$$

which together with the fact that $\int_a^b \eta_1(\lambda)\ d\lambda = \int_a^b \eta_2(\lambda)\ d\lambda = \frac{1}{6}\mathrm{Tr}(V^3)$ (which one can easily arrive at by setting $p(\lambda) = \lambda^3$ in the above formula), implies that

$$\int_a^b \lambda^r \eta(\lambda)d\lambda = 0\ \ \forall\ \ r \geq 0.$$ Hence by an application of Fubini's theorem, we get that

$$\int_{-\infty}^{\infty} e^{-\mathrm{i}t\lambda}\ \eta(\lambda)\ d\lambda\ =\ \sum_{n=0}^{\infty} \frac{1}{n!}\int_{-\infty}^{\infty} (-\mathrm{i}t\lambda)^n\ \eta(\lambda)\ d\lambda = 0.$$

Hence

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}t\lambda}\ \eta(\lambda)\ d\lambda = 0\ \forall\ t \in \mathbb{R}.$$

Therefore η is an $L^1([a,b])$- function whose Fourier transform $\hat{\eta}(t)$ vanishes identically, implying that $\eta = 0$ or $\eta_1 = \eta_2$ a.e.

□

Corollary 3.1.6. *Let A and V be two bounded self-adjoint operators in an infinite dimensional Hilbert space $\mathcal{H}$ such that $V \in \mathcal{B}_2(\mathcal{H})$. Then the function $\eta \in L^1([a,b])$ obtained as in Theorem 3.1.5 satisfies the following equation*

$$\int_a^b f(\lambda)\eta(\lambda)d\lambda = \int_0^1 ds \int_0^s d\tau \int_a^b \int_a^b \frac{h(\lambda)-h(\mu)}{\lambda - \mu}\ \mathrm{Tr}\left[VE_{A_\tau}(d\lambda)VE_{A_\tau}(d\mu) - VE_A(d\lambda)VE_A(d\mu)\right],$$

where $f(\lambda)$, $g(\lambda)$ and $h(\lambda)$ are as in the proof of the Theorem 3.1.5. Moreover

$$\|\eta\|_{L^1} \leq (b-a)\|V\|_2^2.$$

Proof. By Fubini's theorem we have that

$$\int_a^b f(\lambda)\eta^{(n)}(\lambda)d\lambda = \int_0^1 ds \int_0^s d\tau \int_a^b f(\lambda) \Big[\text{Tr}\{V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\} + \eta_{2\tau}^{(n)}(\lambda)\Big] d\lambda.$$

But

$$\int_a^b f(\lambda)\ \text{Tr}\{V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\}d\lambda = \int_a^b g'(\lambda)\text{Tr}\left[V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\right] d\lambda,$$

which by integrating by-parts and using Lemma 3.1.3 (ii)leads to

$$\begin{aligned}
&g(\lambda)\ \text{Tr}\left[V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\right] |_{\lambda=a}^b \\
&\qquad\qquad - \int_a^b g(\lambda)\text{Tr}\left[V_1^2 E_A(d\lambda) - V_{1\tau}^2 E_{A_\tau}(d\lambda)\right] \\
&= g(b)\ \text{Tr}\left[V_1^2 - V_{1\tau}^2\right] + \int_a^b g(\lambda)\text{Tr}\left[V_{1\tau}^2 E_{A_\tau}(d\lambda) - V_1^2 E_A(d\lambda)\right] \\
&= g(b)\ \text{Tr}\left[V_1^2 - V_{1\tau}^2\right] + \text{Tr}\left[V_{1\tau}^2 h'(A_\tau) - V_1^2 h'(A)\right] \\
&= g(b)\ \text{Tr}\left[V_1^2 - V_{1\tau}^2\right] + \int_a^b\int_a^b \frac{h(\lambda)-h(\mu)}{\lambda-\mu}\ \text{Tr}\left[V_{1\tau}E_{A_\tau}(d\lambda)V_{1\tau}E_{A_\tau}(d\mu) - V_1 E_A(d\lambda)V_1 E_A(d\mu)\right].
\end{aligned} \tag{3.1.17}$$

Again by repeating the same above calculations to get (3.1.14) and (3.1.15) as in the proof of the Theorem 3.1.5, we conclude that

$$\begin{aligned}
\int_a^b f(\lambda)\eta_{2\tau}^{(n)}(\lambda)d\lambda &= g(b)\ \text{Tr}\left[V_2 V_2^{(n)} - V_{2\tau}V_{2\tau}^{(n)}\right] \\
&\quad + \int_a^b\int_a^b \frac{h(\lambda)-h(\mu)}{\lambda-\mu}\ \text{Tr}\left[V_{2\tau}E_{A_\tau}(d\lambda)V_{2\tau}^{(n)}E_{A_\tau}(d\mu) - V_2 E_A(d\lambda)V_2^{(n)}E_A(d\mu)\right].
\end{aligned} \tag{3.1.18}$$

Combining (3.1.17) and (3.1.18) we have,

$$\begin{aligned}
&\int_a^b f(\lambda)\eta^{(n)}(\lambda)d\lambda \\
&= \int_0^1 ds\int_0^s d\tau\ g(b)\ \text{Tr}\left[\left(V_1^2 + V_2V_2^{(n)}\right) - \left(V_{2\tau}^2 + V_{2\tau}V_{2\tau}^{(n)}\right)\right] + \\
&\int_0^1 ds\int_0^s d\tau \int_a^b\int_a^b \frac{h(\lambda)-h(\mu)}{\lambda-\mu}\ \text{Tr}\left[VE_{A_\tau}(d\lambda)\left(V_{1\tau}\oplus V_{2\tau}^{(n)}\right)E_{A_\tau}(d\mu) - VE_A(d\lambda)\left(V_1\oplus V_2^{(n)}\right)E_A(d\mu)\right].
\end{aligned} \tag{3.1.19}$$

But by definition $V_2^{(n)}, V_{2\tau}^{(n)}$ converges to $V_2, V_{2\tau}$ respectively in $\|.\|_2$ and we have already proved that $\eta^{(n)}$ converges to η in $L^1([a,b])$. Hence by taking limit on both sides of (3.1.19) we get that

$$\int_a^b f(\lambda)\eta(\lambda)d\lambda = \int_0^1 ds \int_0^s d\tau \int_a^b \int_a^b \frac{h(\lambda)-h(\mu)}{\lambda-\mu} \text{Tr}\left[VE_{A_\tau}(d\lambda)VE_{A_\tau}(d\mu) - VE_A(d\lambda)VE_A(d\mu)\right]. \tag{3.1.20}$$

In the right hand side of (3.1.20) we have used the fact that

$\text{Var}\left(\mathcal{G}_2^{(n)} - \mathcal{G}_2\right) \leq \|V\|_2 \left(\|V_{2\tau}^{(n)} - V_{2\tau}\|_2 + \|V_2 - V_2^{(n)}\|\right) \longrightarrow 0$ as $n \longrightarrow \infty$, where

$\mathcal{G}_2^{(n)}(\Delta \times \delta) = \text{Tr}\left[VE_{A_\tau}(\Delta)\left(V_{1\tau} \oplus V_{2\tau}^{(n)}\right)E_{A_\tau}(\delta) - VE_A(\Delta)\left(V_1 \oplus V_2^{(n)}\right)E_A(\delta)\right]$ and

$\mathcal{G}_2(\Delta \times \delta) = \text{Tr}\left[VE_{A_\tau}(\Delta)VE_{A_\tau}(\delta) - VE_A(\Delta)VE_A(\delta)\right]$ are complex measures on $\mathbb{R}^2$ and

$\text{Var}\left(\mathcal{G}_2^{(n)} - \mathcal{G}_2\right)$ is the variation of $\left(\mathcal{G}_2^{(n)} - \mathcal{G}_2\right)$ and also noted that $\|h\|_{\text{Lip}} \leq (b-a)\|f\|_\infty$.

Again from (3.1.20) we have the following estimate (as in Birman-Solomyak ([2],[3]) and in [7])

$$\left|\int_a^b f(\lambda)\eta(\lambda)d\lambda\right| \leq \int_0^1 ds \int_0^s d\tau\ 2\ \|h\|_{\text{Lip}}\ \|V\|_2^2 \leq (b-a)\ \|f\|_\infty\ \|V\|_2^2$$

and hence

$$\sup_{f\in L^\infty([a,b])} \frac{\left|\int_a^b f(\lambda)\eta(\lambda)d\lambda\right|}{\|f\|_\infty} \leq (b-a)\ \|V\|_2^2 \quad \text{i.e.} \quad \|\eta\|_{L^1} \leq (b-a)\ \|V\|_2^2.$$

□

3.2 Unbounded Case

Theorem 3.2.1. *Let A be an unbounded self-adjoint operator in a Hilbert space $\mathcal{H}$ and let $\phi : \mathbb{R} \longrightarrow \mathbb{C}$ be such that $\int_{-\infty}^{\infty} |\hat{\phi}(t)|\ (1+|t|)^3\ dt < \infty$, where $\hat{\phi}$ is the Fourier transform of ϕ. Then $\phi(A),\ D^{(1)}\phi(A),\ D^{(2)}\phi(A)$ exist and*

$$D^{(1)}\phi(A) \bullet X = \text{i}\int_{-\infty}^{\infty} \hat{\phi}(t)dt \int_0^t d\beta\ \text{e}^{\text{i}\beta A}X\text{e}^{\text{i}(t-\beta)A} \quad \textit{and}$$

$$D^{(2)}\phi(A) \bullet (X,Y) = \text{i}^2 \int_{-\infty}^{\infty} \hat{\phi}(t)dt \int_0^t d\beta\ \{\int_0^\beta d\nu\ \text{e}^{\text{i}\nu A}X\text{e}^{\text{i}(\beta-\nu)A}Y\text{e}^{\text{i}(t-\beta)A} + \int_0^{t-\beta} d\nu\ \text{e}^{\text{i}\beta A}Y\text{e}^{\text{i}\nu A}X\text{e}^{\text{i}(t-\beta-\nu)A}\},$$

where $X, Y \in \mathcal{B}(\mathcal{H})$.

Proof. That $\phi(A)$ and the expressions on the right hand side above exist in $\mathcal{B}(\mathcal{H})$ are consequences of the functional calculus and the assumption on $\hat{\phi}$. Next for $X \in \mathcal{B}(\mathcal{H})$,

$$\phi(A+X)-\phi(A)=\int_{-\infty}^{\infty}\hat{\phi}(t)\left[\mathrm{e}^{\mathrm{i}t(A+X)}-\mathrm{e}^{\mathrm{i}tA}\right]dt=\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta(A+X)}\mathrm{i}X\mathrm{e}^{\mathrm{i}(t-\beta)A}.$$

Therefore

$$\begin{aligned}\phi(A+X)-\phi(A)-\mathrm{i}\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ e^{\mathrm{i}\beta A}Xe^{\mathrm{i}(t-\beta)A}&\\ =\mathrm{i}\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \left[\mathrm{e}^{\mathrm{i}\beta(A+X)}X\mathrm{e}^{\mathrm{i}(t-\beta)A}-\mathrm{e}^{\mathrm{i}\beta A}X\mathrm{e}^{\mathrm{i}(t-\beta)A}\right]&\\ =\mathrm{i}\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \{\left[\mathrm{e}^{\mathrm{i}\beta(A+X)}-\mathrm{e}^{\mathrm{i}\beta A}\right]X\mathrm{e}^{\mathrm{i}(t-\beta)A}\}.&\end{aligned}$$

In one hand we have, $\left\|\mathrm{e}^{\mathrm{i}\beta(A+X)}-\mathrm{e}^{\mathrm{i}\beta A}\right\|\leq 2$ but on the other hand

$$\left\|\mathrm{e}^{\mathrm{i}\beta(A+X)}-\mathrm{e}^{\mathrm{i}\beta A}\right\|=\left\|\int_0^{\beta}d\nu\ \mathrm{e}^{\mathrm{i}\nu(A+X)}\mathrm{i}X\mathrm{e}^{\mathrm{i}(\beta-\nu)A}\right\|\leq|\beta|\|X\|$$

and hence using the interpolation inequality

$$\left\|\mathrm{e}^{\mathrm{i}\beta(A+X)}-\mathrm{e}^{\mathrm{i}\beta A}\right\|\leq 2^{(1-\epsilon)}\left(|\beta|\|X\|\right)^{\epsilon}\ \ (0\leq\epsilon\leq 1),\quad \text{we get that}$$

$$\begin{aligned}&\left\|\phi(A+X)-\phi(A)-\mathrm{i}\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A}X\mathrm{e}^{\mathrm{i}(t-\beta)A}\right\|\\ &\leq 2^{(1-\epsilon)}\|X\|^{\epsilon+1}\int_{-\infty}^{\infty}|\hat{\phi}(t)|\ dt\int_0^{|t|}\beta^{\epsilon}\ d\beta\leq\left(\frac{2^{(1-\epsilon)}}{\epsilon+1}\right)\|X\|^{\epsilon+1}\int_{-\infty}^{\infty}|\hat{\phi}(t)|\ (1+|t|)^{\epsilon+2}\ dt,\end{aligned}$$

which by virtue of the assumption on $\hat{\phi}$ implies that $D^{(1)}\phi(A)$ exist and that

$$D^{(1)}\phi(A)\bullet X=\mathrm{i}\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A}X\mathrm{e}^{\mathrm{i}(t-\beta)A}.\quad \text{Similarly for}\quad X,Y\in\mathcal{B}(\mathcal{H}),$$

$$
\begin{aligned}
&D^{(1)}\phi(A+X)\bullet Y - D^{(1)}\phi(A)\bullet Y \\
&\quad = \mathrm{i}\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta(A+X)}Y\mathrm{e}^{\mathrm{i}(t-\beta)(A+X)} - \mathrm{i}\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A}Y\mathrm{e}^{\mathrm{i}(t-\beta)A} \\
&\quad = \mathrm{i}\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \{\left[\mathrm{e}^{\mathrm{i}\beta(A+X)} - \mathrm{e}^{\mathrm{i}\beta A}\right]Y\mathrm{e}^{\mathrm{i}(t-\beta)(A+X)} \\
&\qquad\qquad + \mathrm{e}^{\mathrm{i}\beta A}Y\left[\mathrm{e}^{\mathrm{i}(t-\beta)(A+X)} - \mathrm{e}^{\mathrm{i}(t-\beta)A}\right]\} \\
&\quad = \mathrm{i}^2\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \{\int_0^{\beta} d\nu\ \mathrm{e}^{\mathrm{i}\nu(A+X)}X\mathrm{e}^{\mathrm{i}(\beta-\nu)A}Y\mathrm{e}^{\mathrm{i}(t-\beta)(A+X)} \\
&\qquad\qquad + \int_0^{t-\beta} d\nu\ \mathrm{e}^{\mathrm{i}\beta A}Y\mathrm{e}^{\mathrm{i}\nu(A+X)}X\mathrm{e}^{\mathrm{i}(t-\beta-\nu)A}\}
\end{aligned}
$$

and one can verify as before that

$$
\begin{aligned}
&\|D^{(1)}\phi(A+X)\bullet Y - D^{(1)}\phi(A)\bullet Y \\
&\qquad - \mathrm{i}^2\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \{\int_0^{\beta} d\nu\ \mathrm{e}^{\mathrm{i}\nu A}X\mathrm{e}^{\mathrm{i}(\beta-\nu)A}Y\mathrm{e}^{\mathrm{i}(t-\beta)A} \\
&\qquad\qquad + \int_0^{t-\beta} d\nu\ \mathrm{e}^{\mathrm{i}\beta A}Y\mathrm{e}^{\mathrm{i}\nu A}X\mathrm{e}^{\mathrm{i}(t-\beta-\nu)A}\}\| \\
&= \|\mathrm{i}^2\int_{-\infty}^{\infty}\hat{\phi}(t)dt\int_0^t d\beta\ \{\int_0^{\beta} d\nu\left[\mathrm{e}^{\mathrm{i}\nu(A+X)} - \mathrm{e}^{\mathrm{i}\nu A}\right]X\mathrm{e}^{\mathrm{i}(\beta-\nu)A}Y\mathrm{e}^{\mathrm{i}(t-\beta)(A+X)} \\
&\qquad + \int_0^{\beta} d\nu\ \mathrm{e}^{\mathrm{i}\nu A}X\mathrm{e}^{\mathrm{i}(\beta-\nu)A}Y\left[\mathrm{e}^{\mathrm{i}(t-\beta)(A+X)} - \mathrm{e}^{\mathrm{i}(t-\beta)A}\right] \\
&\qquad + \int_0^{t-\beta} d\nu\ \mathrm{e}^{\mathrm{i}\beta A}Y\left[\mathrm{e}^{\mathrm{i}\nu(A+X)} - \mathrm{e}^{\mathrm{i}\nu A}\right]X\mathrm{e}^{\mathrm{i}(t-\beta-\nu)A}\}\| \\
&\le\ K\|X\|^{\epsilon+1}\|Y\|\int_{-\infty}^{\infty}|\hat{\phi}(t)|\,(1+|t|)^{\epsilon+2}\,dt \quad \text{(for some } \epsilon>0 \text{ and some constant} \quad K\equiv K(\epsilon)),
\end{aligned}
$$

proving the expression for $D^{(2)}\phi(A)\bullet(X,Y)$. □

Theorem 3.2.2. *Let A be an unbounded self-adjoint operator in a Hilbert space $\mathcal{H}$, V be a self-adjoint operator such that $V\in\mathcal{B}_3(\mathcal{H})$ and furthermore let $\phi\in\mathcal{S}(\mathbb{R})$ (the Schwartz class of smooth functions of rapid decrease). Then*

$$
\phi(A+V)-\phi(A)-D^{(1)}\phi(A)\bullet V-\frac{1}{2}D^{(2)}\phi(A)\bullet(V,V)\in\mathcal{B}_1(\mathcal{H}) \quad \textit{and}
$$

$$
\begin{aligned}
&\mathrm{Tr}\left[\phi(A+V)-\phi(A)-D^{(1)}\phi(A)\bullet V-\frac{1}{2}D^{(2)}\phi(A)\bullet(V,V)\right] \\
&\quad = \int_{-\infty}^{\infty}\mathrm{i}^2\ t\hat{\phi}(t)\ dt\int_0^t d\nu\int_0^1 ds\int_0^s d\tau\ \mathrm{Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right],
\end{aligned}
$$

where $A_\tau = A+\tau V$ and $0\le\tau\le 1$.

Proof. Now

$$\begin{aligned}\frac{\mathrm{e}^{\mathrm{i}tA_{s+h}}-\mathrm{e}^{\mathrm{i}tA_s}}{h} &= \frac{1}{h}\int_0^t d\beta\ \frac{d}{d\beta}\left(\mathrm{e}^{\mathrm{i}\beta A_{s+h}}.\mathrm{e}^{\mathrm{i}(t-\beta)A_s}\right) = \mathrm{i}\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A_{s+h}}V\mathrm{e}^{\mathrm{i}(t-\beta)A_s}\\
&= \mathrm{i}\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A_s}V\mathrm{e}^{\mathrm{i}(t-\beta)A_s} + \mathrm{i}\int_0^t d\beta\left[\mathrm{e}^{\mathrm{i}\beta A_{s+h}}-\mathrm{e}^{\mathrm{i}\beta A_s}\right]V\mathrm{e}^{\mathrm{i}(t-\beta)A_s}\\
&= \mathrm{i}\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A_s}V\mathrm{e}^{\mathrm{i}(t-\beta)A_s} + (\mathrm{i})^2 h\int_0^t d\beta\int_0^\beta d\nu\ \mathrm{e}^{\mathrm{i}\nu A_{s+h}}V\mathrm{e}^{\mathrm{i}(\beta-\nu)A_s}V\mathrm{e}^{\mathrm{i}(t-\beta)A_s},\end{aligned}$$

and hence

$$\left\|\frac{\mathrm{e}^{\mathrm{i}tA_{s+h}}-\mathrm{e}^{\mathrm{i}tA_s}}{h} - \mathrm{i}\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A_s}V\mathrm{e}^{\mathrm{i}(t-\beta)A_s}\right\|_{\frac{3}{2}} \leq \frac{1}{2}\,|t|^2\,|h|\,\|V\|_3^2,$$

which converges to 0 as $h \longrightarrow 0$, proving that the map $[0,1] \ni s \longrightarrow \mathrm{e}^{\mathrm{i}tA_s}$ is $\mathcal{B}_{\frac{3}{2}}(\mathcal{H})$(and hence $\mathcal{B}_3(\mathcal{H})$)- continuously differentiable, uniformly in t and

$$\frac{d}{ds}\left(\mathrm{e}^{\mathrm{i}tA_s}\right) = \mathrm{i}\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A_s}V\mathrm{e}^{\mathrm{i}(t-\beta)A_s}.$$

Then

$$\begin{aligned}\phi(A+V)-\phi(A)-D^{(1)}\phi(A)\bullet V &= \int_{-\infty}^{\infty}\hat{\phi}(t)\ dt\left[\mathrm{e}^{\mathrm{i}t(A+V)}-\mathrm{e}^{\mathrm{i}tA}\right] - D^{(1)}\phi(A)\bullet V\\
&= \int_{-\infty}^{\infty}\hat{\phi}(t)\ dt\int_0^1 ds\frac{d}{ds}(\mathrm{e}^{\mathrm{i}tA_s}) - \mathrm{i}\int_{-\infty}^{\infty}\hat{\phi}(t)\ dt\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A}V\mathrm{e}^{\mathrm{i}(t-\beta)A}\\
&= \int_{-\infty}^{\infty}\hat{\phi}(t)\ dt\left[\mathrm{i}\int_0^1 ds\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A_s}V\mathrm{e}^{\mathrm{i}(t-\beta)A_s} - \mathrm{i}\int_0^1 ds\int_0^t d\beta\ \mathrm{e}^{\mathrm{i}\beta A}V\mathrm{e}^{\mathrm{i}(t-\beta)A}\right]\end{aligned}$$

$$\begin{aligned}&= \int_{-\infty}^{\infty}\mathrm{i}\hat{\phi}(t)\ dt\int_0^1 ds\int_0^t d\beta\left[\mathrm{e}^{\mathrm{i}\beta A_s}V\mathrm{e}^{\mathrm{i}(t-\beta)A_s}-\mathrm{e}^{\mathrm{i}\beta A}V\mathrm{e}^{\mathrm{i}(t-\beta)A}\right]\\
&= \int_{-\infty}^{\infty}\mathrm{i}\hat{\phi}(t)\ Q(t)\ dt, \quad \text{where} \quad Q(t) \equiv \int_0^1 ds\int_0^t d\beta\left[\mathrm{e}^{\mathrm{i}\beta A_s}V\mathrm{e}^{\mathrm{i}(t-\beta)A_s}-\mathrm{e}^{\mathrm{i}\beta A}V\mathrm{e}^{\mathrm{i}(t-\beta)A}\right].\end{aligned} \tag{3.2.1}$$

Again

$$\frac{e^{i\beta A_{\tau+h}}Ve^{i(t-\beta)A_{\tau+h}} - e^{i\beta A_\tau}Ve^{i(t-\beta)A_\tau}}{h}$$
$$= \frac{1}{h}\left[e^{i\beta A_{\tau+h}} - e^{i\beta A_\tau}\right]Ve^{i(t-\beta)A_{\tau+h}} + \frac{1}{h}e^{i\beta A_\tau}V\left[e^{i(t-\beta)A_{\tau+h}} - e^{i(t-\beta)A_\tau}\right]$$
$$= i\int_0^\beta d\nu\, e^{i\nu A_{\tau+h}}Ve^{i(\beta-\nu)A_\tau}Ve^{i(t-\beta)A_{\tau+h}} + i\int_0^{t-\beta} d\nu\, e^{i\beta A_\tau}Ve^{i\nu A_{\tau+h}}Ve^{i(t-\beta-\nu)A_\tau}$$
$$= i\int_0^\beta d\nu\, e^{i\nu A_\tau}Ve^{i(\beta-\nu)A_\tau}Ve^{i(t-\beta)A_\tau} + i\int_0^{t-\beta} d\nu\, e^{i\beta A_\tau}Ve^{i\nu A_\tau}Ve^{i(t-\beta-\nu)A_\tau}$$
$$+ i\int_0^\beta d\nu\, \left[e^{i\nu A_{\tau+h}} - e^{i\nu A_\tau}\right]Ve^{i(\beta-\nu)A_\tau}Ve^{i(t-\beta)A_\tau} + i\int_0^\beta d\nu\, e^{i\nu A_\tau}Ve^{i(\beta-\nu)A_\tau}V\left[e^{i(t-\beta)A_{\tau+h}} - e^{i(t-\beta)A_\tau}\right]$$
$$+ i\int_0^{t-\beta} d\nu\, e^{i\beta A_\tau}V\left[e^{i\nu A_{\tau+h}} - e^{i\nu A_\tau}\right]Ve^{i(t-\beta-\nu)A_\tau}$$

$$\begin{aligned}
&= i\int_0^\beta d\nu\, e^{i\nu A_\tau}Ve^{i(\beta-\nu)A_\tau}Ve^{i(t-\beta)A_\tau} + i\int_0^{t-\beta} d\nu\, e^{i\beta A_\tau}Ve^{i\nu A_\tau}Ve^{i(t-\beta-\nu)A_\tau} \\
&\quad + i^2 h\int_0^\beta d\nu\int_0^\nu d\eta\, e^{i\eta A_{\tau+h}}Ve^{i(\nu-\eta)A_\tau}Ve^{i(\beta-\nu)A_\tau}Ve^{i(t-\beta)A_\tau} \\
&\quad + i^2 h\int_0^\beta d\nu\int_0^\nu d\eta\, e^{i\nu A_\tau}Ve^{i(\beta-\nu)A_\tau}Ve^{i\eta A_{\tau+h}}Ve^{i(t-\beta-\eta)A_\tau} \\
&\quad + i^2 h\int_0^{t-\beta} d\nu\int_0^\nu d\eta\, e^{i\beta A_\tau}Ve^{i\eta A_{\tau+h}}Ve^{i(\nu-\eta)A_\tau}Ve^{i(t-\beta-\nu)A_\tau},
\end{aligned}$$

and hence

$$\begin{aligned}
&\Big\|\frac{e^{i\beta A_{\tau+h}}Ve^{i(t-\beta)A_{\tau+h}} - e^{i\beta A_\tau}Ve^{i(t-\beta)A_\tau}}{h} \\
&\quad - \left(i\int_0^\beta d\nu\, e^{i\nu A_\tau}Ve^{i(\beta-\nu)A_\tau}Ve^{i(t-\beta)A_\tau} + i\int_0^{t-\beta} d\nu\, e^{i\beta A_\tau}Ve^{i\nu A_\tau}Ve^{i(t-\beta-\nu)A_\tau}\right)\Big\|_1 \\
&\le \frac{1}{2}|h|\{2\beta^2 + (t-\beta)^2\}\|V\|_3^3,
\end{aligned}$$

proving that the map $\tau \in [0,1] \longrightarrow e^{i\beta A_\tau}Ve^{i(t-\beta)A_\tau} \in \mathcal{B}_3(\mathcal{H})$ is $\mathcal{B}_1(\mathcal{H})$ (and hence $\mathcal{B}_{\frac{3}{2}}(\mathcal{H})$) -continuously differentiable, uniformly with respect to β. Then by an application of Leibnitz's rule we get

$$Q(t) = \int_0^1 ds\int_0^t d\beta\int_0^s d\tau\frac{d}{d\tau}\left(e^{i\beta A_\tau}Ve^{i(t-\beta)A_\tau}\right)$$
$$= i\int_0^1 ds\int_0^s d\tau\int_0^t d\beta\,\{\int_0^\beta d\nu e^{i\nu A_\tau}Ve^{i(\beta-\nu)A_\tau}Ve^{i(t-\beta)A_\tau} + \int_0^{t-\beta} d\nu e^{i\beta A_\tau}Ve^{i\nu A_\tau}Ve^{i(t-\beta-\nu)A_\tau}\},$$

where Fubini's theorem has been used to interchange the order of integration. Hence

$$\begin{aligned}
&\phi(A+V)-\phi(A)-D^{(1)}\phi(A)\bullet V-\frac{1}{2}D^{(2)}\phi(A)\bullet(V,V)\\
=&\int_{-\infty}^{\infty}\mathrm{i}^2\hat{\phi}(t)dt\int_0^1 ds\int_0^s d\tau\int_0^t d\beta\{\int_0^\beta d\nu\left[\mathrm{e}^{\mathrm{i}\nu A_\tau}V\mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau}Ve^{\mathrm{i}(t-\beta)A_\tau}-\mathrm{e}^{\mathrm{i}\nu A}V\mathrm{e}^{\mathrm{i}(\beta-\nu)A}V\mathrm{e}^{\mathrm{i}(t-\beta)A}\right]\\
&+\int_0^{t-\beta}d\nu\left[\mathrm{e}^{\mathrm{i}\beta A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau}V\mathrm{e}^{\mathrm{i}(t-\beta-\nu)A_\tau}-\mathrm{e}^{\mathrm{i}\beta A}V\mathrm{e}^{\mathrm{i}\nu A}V\mathrm{e}^{\mathrm{i}(t-\beta-\nu)A}\right]\}.
\end{aligned} \tag{3.2.2}$$

Though each of the four individual terms in the integral in (3.2.2) belong to $\mathcal{B}_{\frac{3}{2}}(\mathcal{H})$, each of the differences in parenthesis $[.]$ belong to $\mathcal{B}_1(\mathcal{H})$, e.g.

$$\begin{aligned}
&\left[\mathrm{e}^{\mathrm{i}\nu A_\tau}V\mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}-\mathrm{e}^{\mathrm{i}\nu A}V\mathrm{e}^{\mathrm{i}(\beta-\nu)A}V\mathrm{e}^{\mathrm{i}(t-\beta)A}\right]\\
&=\left[\mathrm{e}^{\mathrm{i}\nu A_\tau}-\mathrm{e}^{\mathrm{i}\nu A}\right]Ve^{\mathrm{i}(\beta-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}+\mathrm{e}^{\mathrm{i}\nu A}V\left[\mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau}-\mathrm{e}^{\mathrm{i}(\beta-\nu)A}\right]V\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}\\
&\quad+\mathrm{e}^{\mathrm{i}\nu A}V\mathrm{e}^{\mathrm{i}(\beta-\nu)A}V\left[\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}-\mathrm{e}^{\mathrm{i}(t-\beta)A}\right]\\
&=\int_0^\nu d\eta\,\mathrm{e}^{\mathrm{i}\eta A_\tau}\mathrm{i}\tau V\mathrm{e}^{\mathrm{i}(\nu-\eta)A}Ve^{\mathrm{i}(\beta-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}\\
&\quad+\int_0^{(\beta-\nu)}d\eta\,\mathrm{e}^{\mathrm{i}\nu A}V\mathrm{e}^{\mathrm{i}\eta A_\tau}\mathrm{i}\tau V\mathrm{e}^{\mathrm{i}(\beta-\nu-\eta)A}V\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}\\
&\quad+\int_0^{(t-\beta)}d\eta\,\mathrm{e}^{\mathrm{i}\nu A}V\mathrm{e}^{\mathrm{i}(\beta-\nu)A}V\mathrm{e}^{\mathrm{i}\eta A_\tau}\mathrm{i}\tau V\mathrm{e}^{\mathrm{i}(t-\beta-\eta)A}\in\mathcal{B}_1(\mathcal{H})
\end{aligned}$$

and its norm $\|[.]\|_1 \;\leq\; |\tau|\;\|V\|_3^3\;\{|\nu|+|\beta-\nu|+|t-\beta|\}$, where we have used the estimates:

$$\left\|\int_0^\nu d\eta\,\mathrm{e}^{\mathrm{i}\eta A_\tau}\mathrm{i}\tau V\mathrm{e}^{\mathrm{i}(\nu-\eta)A}Ve^{\mathrm{i}(\beta-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}\right\|_1\leq|\nu||\tau|\|V\|_3^3\;;$$

$$\left\|\int_0^{(\beta-\nu)}d\eta\,\mathrm{e}^{\mathrm{i}\nu A}V\mathrm{e}^{\mathrm{i}\eta A_\tau}\mathrm{i}\tau V\mathrm{e}^{\mathrm{i}(\beta-\nu-\eta)A}V\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}\right\|_1\leq|\beta-\nu||\tau|\|V\|_3^3\quad\text{and}$$

$$\left\|\int_0^{(t-\beta)}d\eta\,\mathrm{e}^{\mathrm{i}\nu A}V\mathrm{e}^{\mathrm{i}(\beta-\nu)A}V\mathrm{e}^{\mathrm{i}\eta A_\tau}\mathrm{i}\tau V\mathrm{e}^{\mathrm{i}(t-\beta-\eta)A}\right\|_1\leq|t-\beta||\tau|\|V\|_3^3.$$

Hence by the hypothesis on $\hat{\phi}$,

$$\phi(A+V)-\phi(A)-D^{(1)}\phi(A)\bullet V-\frac{1}{2}D^{(2)}\phi(A)\bullet(V,V)\in\mathcal{B}_1(\mathcal{H})\quad\text{and}$$

$$\mathcal{Z} \equiv \mathrm{Tr}\left[\phi(A+V) - \phi(A) - D^{(1)}\phi(A) \bullet V - \frac{1}{2}D^{(2)}\phi(A) \bullet (V,V)\right]$$
$$= \int_{-\infty}^{\infty} \mathrm{i}^2\hat{\phi}(t)dt \int_0^1 ds \int_0^s d\tau \int_0^t d\beta \, \{\int_0^\beta d\nu \mathrm{Tr}\left[\mathrm{e}^{\mathrm{i}\nu A_\tau} V \mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau} V \mathrm{e}^{\mathrm{i}(t-\beta)A_\tau} - \mathrm{e}^{\mathrm{i}\nu A} V \mathrm{e}^{\mathrm{i}(\beta-\nu)A} V \mathrm{e}^{\mathrm{i}(t-\beta)A}\right]$$
$$+ \int_0^{t-\beta} d\nu \mathrm{Tr}\left[\mathrm{e}^{\mathrm{i}\beta A_\tau} V \mathrm{e}^{\mathrm{i}\nu A_\tau} V \mathrm{e}^{\mathrm{i}(t-\beta-\nu)A_\tau} - \mathrm{e}^{\mathrm{i}\beta A} V \mathrm{e}^{\mathrm{i}\nu A} V \mathrm{e}^{\mathrm{i}(t-\beta-\nu)A}\right]\}. \tag{3.2.3}$$

Again by the cyclicity of trace and a change of variable, the first integral in $\{.\}$ in (3.2.3) is reduces to

$$\int_0^\beta d\nu \ \mathrm{Tr}\{\left[\mathrm{e}^{\mathrm{i}\nu A_\tau} - \mathrm{e}^{\mathrm{i}\nu A}\right] V \mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau} V \mathrm{e}^{\mathrm{i}(t-\beta)A_\tau} + \mathrm{e}^{\mathrm{i}\nu A} V \left[\mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau} - \mathrm{e}^{\mathrm{i}(\beta-\nu)A}\right] V \mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}$$
$$+ \mathrm{e}^{\mathrm{i}\nu A} V \mathrm{e}^{\mathrm{i}(\beta-\nu)A} V \left[\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau} - \mathrm{e}^{\mathrm{i}(t-\beta)A}\right]\}$$
$$= \int_0^\beta d\nu \ (\mathrm{Tr}\{\left[\mathrm{e}^{\mathrm{i}\nu A_\tau} - \mathrm{e}^{\mathrm{i}\nu A}\right] V \mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau} V \mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}\} + \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}\nu A} V \left[\mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau} - \mathrm{e}^{\mathrm{i}(\beta-\nu)A}\right] V \mathrm{e}^{\mathrm{i}(t-\beta)A_\tau}\}$$
$$+ \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}\nu A} V \mathrm{e}^{\mathrm{i}(\beta-\nu)A} V \left[\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau} - \mathrm{e}^{\mathrm{i}(t-\beta)A}\right]\})$$
$$= \int_0^\beta d\nu \ (\mathrm{Tr}\{\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau} \left[\mathrm{e}^{\mathrm{i}\nu A_\tau} - \mathrm{e}^{\mathrm{i}\nu A}\right] V \mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau} V\} + \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau} \mathrm{e}^{\mathrm{i}\nu A} V \left[\mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau} - \mathrm{e}^{\mathrm{i}(\beta-\nu)A}\right] V\}$$
$$+ \mathrm{Tr}\{\left[\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau} - \mathrm{e}^{\mathrm{i}(t-\beta)A}\right] \mathrm{e}^{\mathrm{i}\nu A} V \mathrm{e}^{\mathrm{i}(\beta-\nu)A} V\})$$
$$= \int_0^\beta d\nu \ \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau} \left[\mathrm{e}^{\mathrm{i}\nu A_\tau} - \mathrm{e}^{\mathrm{i}\nu A}\right] V \mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau} V + \mathrm{e}^{\mathrm{i}(t-\beta)A_\tau} \mathrm{e}^{\mathrm{i}\nu A} V \left[\mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau} - \mathrm{e}^{\mathrm{i}(\beta-\nu)A}\right] V$$
$$+ \left[\mathrm{e}^{\mathrm{i}(t-\beta)A_\tau} - \mathrm{e}^{\mathrm{i}(t-\beta)A}\right] \mathrm{e}^{\mathrm{i}\nu A} V \mathrm{e}^{\mathrm{i}(\beta-\nu)A} V\}$$
$$= \int_0^\beta d\nu \ \mathrm{Tr}\{\mathrm{e}^{\mathrm{i}(t-\beta+\nu)A_\tau} V \mathrm{e}^{\mathrm{i}(\beta-\nu)A_\tau} V - \mathrm{e}^{\mathrm{i}(t-\beta+\nu)A} V \mathrm{e}^{\mathrm{i}(\beta-\nu)A} V\}$$
$$= \int_0^\beta d\nu \ \mathrm{Tr}\left[V \mathrm{e}^{\mathrm{i}(t-\nu)A_\tau} V \mathrm{e}^{\mathrm{i}\nu A_\tau} - V \mathrm{e}^{\mathrm{i}(t-\nu)A} V \mathrm{e}^{\mathrm{i}\nu A}\right]. \tag{3.2.4}$$

Similarly, the second integral in $\{.\}$ in (3.2.3) is equal to

$$\int_0^{t-\beta} d\nu \ \mathrm{Tr}\left[V \mathrm{e}^{\mathrm{i}(t-\nu)A_\tau} V \mathrm{e}^{\mathrm{i}\nu A_\tau} - V \mathrm{e}^{\mathrm{i}(t-\nu)A} V \mathrm{e}^{\mathrm{i}\nu A}\right]. \tag{3.2.5}$$

Combining (3.2.4) and (3.2.5), we conclude that

$$\mathcal{Z} = \int_{-\infty}^{\infty} \mathrm{i}^2\hat{\phi}(t)dt \int_0^1 ds \int_0^s d\tau \int_0^t d\beta \, \{\int_0^\beta d\nu \ \mathrm{Tr}\left[V \mathrm{e}^{\mathrm{i}(t-\nu)A_\tau} V \mathrm{e}^{\mathrm{i}\nu A_\tau} - V \mathrm{e}^{\mathrm{i}(t-\nu)A} V \mathrm{e}^{\mathrm{i}\nu A}\right]$$
$$+ \int_0^{t-\beta} d\nu \ \mathrm{Tr}\left[V \mathrm{e}^{\mathrm{i}(t-\nu)A_\tau} V \mathrm{e}^{\mathrm{i}\nu A_\tau} - V \mathrm{e}^{\mathrm{i}(t-\nu)A} V \mathrm{e}^{\mathrm{i}\nu A}\right]\}. \tag{3.2.6}$$

By a change of variable and the cyclicity of trace, we get that

$$
\begin{aligned}
& t\int_0^t d\nu \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right] \\
&= \int_0^t d\beta \,\{\int_0^\beta d\nu \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right] \\
&\qquad\qquad + \int_\beta^t d\nu \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right]\} \\
&= \int_0^t d\beta \,\{\int_0^\beta d\nu \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right] \\
&\qquad\qquad + \int_0^{t-\beta} d\nu \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right]\}, \\
&= \int_0^t d\beta \,\{\int_0^\beta d\nu \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right] \\
&\qquad\qquad + \int_0^{t-\beta} d\nu \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}\nu A_\tau}V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau} - V\mathrm{e}^{\mathrm{i}\nu A}V\mathrm{e}^{\mathrm{i}(t-\nu)A}\right]\} \\
&= \int_0^t d\beta \,\{\int_0^\beta d\nu \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right] \\
&\qquad\qquad + \int_0^{t-\beta} d\nu \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right]\},
\end{aligned}
$$

using which in (3.2.6) we are lead to the equation

$$
\mathcal{Z} = \int_{-\infty}^{\infty} \mathrm{i}^2\, t\hat{\phi}(t)dt \int_0^t d\nu \int_0^1 ds \int_0^s d\tau \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right], \tag{3.2.7}
$$

by an application of Fubini's theorem. ☐

Theorem 3.2.3. *Let A be an unbounded self-adjoint operator in a Hilbert space $\mathcal{H}$ with $\sigma(A) \subseteq [a,\infty)$ for some $a \in \mathbb{R}$ and V be a self-adjoint operator such that $V \in \mathcal{B}_2(\mathcal{H})$. Then there exist a unique real-valued function $\eta \in L^1\left(\mathbb{R}, \frac{d\lambda}{(1+\lambda^2)^{1+\epsilon}}\right)$ (for some $\epsilon > 0$) such that for every $\phi \in \mathcal{S}(\mathbb{R})$ (the Schwartz class of smooth functions of rapid decrease)*

$$
\text{Tr}\left[\phi(A+V) - \phi(A) - D^{(1)}\phi(A)\bullet V - \frac{1}{2}D^{(2)}\phi(A)\bullet(V,V)\right] = \int_{-\infty}^{\infty}\phi'''(\lambda)\eta(\lambda)d\lambda.
$$

Proof. Equation (3.2.7), after an application of Fubini's theorem, yields that

$$
\begin{aligned}
\mathcal{Z} &\equiv \text{Tr}\left[\phi(A+V) - \phi(A) - D^{(1)}\phi(A)\bullet V - \frac{1}{2}D^{(2)}\phi(A)\bullet(V,V)\right] \\
&= \int_0^1 ds \int_0^s d\tau \int_{-\infty}^{\infty} \mathrm{i}^2\, t\hat{\phi}(t)dt \int_0^t d\nu \text{ Tr}\left[V\mathrm{e}^{\mathrm{i}(t-\nu)A_\tau}V\mathrm{e}^{\mathrm{i}\nu A_\tau} - V\mathrm{e}^{\mathrm{i}(t-\nu)A}V\mathrm{e}^{\mathrm{i}\nu A}\right].
\end{aligned} \tag{3.2.8}
$$

Now

$$\int_0^t d\nu \ \mathrm{Tr}\left[V e^{i(t-\nu)A_\tau} V e^{i\nu A_\tau} - V e^{i(t-\nu)A} V e^{i\nu A}\right]$$
$$= \int_0^t d\nu \int_b^\infty \int_b^\infty e^{i(t-\nu)\lambda} e^{i\nu\mu} \ \mathrm{Tr}\left[V E_{A_\tau}(d\lambda) V E_{A_\tau}(d\mu) - V E_A(d\lambda) V E_A(d\mu)\right], \tag{3.2.9}$$

where $b = a - \|V\|$ and $E_{A_\tau}(.)$ and $E_A(.)$ are the spectral families of the operator A_τ and A respectively and the measure

$\mathcal{F} : \Delta \times \delta \subseteq Borel(\mathbb{R}^2) \longrightarrow \mathrm{Tr}\left[V E_{A_\tau}(\Delta) V E_{A_\tau}(\delta) - V E_A(\Delta) V E_A(\delta)\right]$ is a complex measure with total variation $\leq 2\|V\|_2^2$ and hence by Fubini's theorem the right hand side expression in (3.2.8) is equal to

$$\int_0^1 ds \int_0^s d\tau \int_{-\infty}^\infty i^2 \, t\hat{\phi}(t) dt \int_b^\infty \int_b^\infty \int_0^t d\nu \ e^{i(t-\nu)\lambda} e^{i\nu\mu} \ \mathrm{Tr}\left[V E_{A_\tau}(d\lambda) V E_{A_\tau}(d\mu) - V E_A(d\lambda) V E_A(d\mu)\right]$$
$$= \int_0^1 ds \int_0^s d\tau \int_{-\infty}^\infty i^2 \, t\hat{\phi}(t) dt \int_b^\infty \int_b^\infty \frac{e^{it\lambda} - e^{it\mu}}{i(\lambda - \mu)} \ \mathrm{Tr}\left[V E_{A_\tau}(d\lambda) V E_{A_\tau}(d\mu) - V E_A(d\lambda) V E_A(d\mu)\right]$$
$$= \int_0^1 ds \int_0^s d\tau \int_{-\infty}^\infty i^2 \, t\hat{\phi}(t) dt \int_b^\infty \int_b^\infty \frac{e^{it\lambda} - e^{it\mu}}{i(\lambda - \mu)} \ \left\{\langle V, E_{A_\tau}(d\lambda) V E_{A_\tau}(d\mu)\rangle_2 - \langle V, E_A(d\lambda) V E_A(d\mu)\rangle_2\right\}$$
$$= \int_0^1 ds \int_0^s d\tau \int_{-\infty}^\infty i^2 \, t\hat{\phi}(t) dt \int_b^\infty \int_b^\infty \frac{e^{it\lambda} - e^{it\mu}}{i(\lambda - \mu)} \ \{\langle (V_{1\tau} \oplus V_{2\tau}), E_{A_\tau}(d\lambda)(V_{1\tau} \oplus V_{2\tau}) E_{A_\tau}(d\mu)\rangle_2$$
$$- \langle (V_1 \oplus V_2), E_A(d\lambda)(V_1 \oplus V_2) E_A(d\mu)\rangle_2\}, \tag{3.2.10}$$

where we set $V = V_1 \oplus V_2 = V_{1\tau} \oplus V_{2\tau} \in \mathcal{B}_2(\mathcal{H})$ as in Lemma 3.1.3 $vi(b)$. Again by using the invariance and orthogonality properties in Lemma 3.1.3 $(ii) - (vii)$, the right hand side of (3.2.10) is equal to

$$\int_0^1 ds \int_0^s d\tau \int_{-\infty}^\infty i^2 \, t\hat{\phi}(t) dt \int_b^\infty \int_b^\infty \frac{e^{it\lambda} - e^{it\mu}}{i(\lambda - \mu)} \ [\{\langle (V_{1\tau}, E_{A_\tau}(d\lambda) V_{1\tau} E_{A_\tau}(d\mu)\rangle_2 - \langle (V_1, E_A(d\lambda) V_1 E_A(d\mu)\rangle_2\}$$
$$+ \{\langle (V_{2\tau}, E_{A_\tau}(d\lambda) V_{2\tau} E_{A_\tau}(d\mu)\rangle_2 - \langle (V_2, E_A(d\lambda) V_2 E_A(d\mu)\rangle_2\}]$$
$$= \int_0^1 ds \int_0^s d\tau \int_{-\infty}^\infty i^2 \, t\hat{\phi}(t) dt \int_b^\infty t e^{it\lambda} \ \mathrm{Tr}\left[V_{1\tau}^2 E_{A_\tau}(d\lambda) - V_1^2 E_A(d\lambda)\right]$$
$$+ \int_0^1 ds \int_0^s d\tau \int_{-\infty}^\infty i^2 \, t\hat{\phi}(t) dt \int_b^\infty \int_b^\infty \frac{e^{it\lambda} - e^{it\mu}}{i(\lambda - \mu)} \ \mathrm{Tr}\left[V_{2\tau} E_{A_\tau}(d\lambda) V_{2\tau} E_{A_\tau}(d\mu) - V_2 E_A(d\lambda) V_2 E_A(d\mu)\right]. \tag{3.2.11}$$

Applying Fubini's theorem in the first expression in right hand side of (3.2.11), we conclude that

$$\begin{aligned}
&\int_0^1 ds \int_0^s d\tau \int_{-\infty}^{\infty} \mathrm{i}^2\, t\hat{\phi}(t)dt \int_b^{\infty} t\mathrm{e}^{\mathrm{i}t\lambda}\mathrm{Tr}\left[V_{1\tau}^2 E_{A_\tau}(d\lambda) - V_1^2 E_A(d\lambda)\right]\\
&= \int_0^1 ds \int_0^s d\tau \int_b^{\infty} \phi''(\lambda)\ \mathrm{Tr}\left[V_{1\tau}^2 E_{A_\tau}(d\lambda) - V_1^2 E_A(d\lambda)\right]\\
&= \int_0^1 ds \int_0^s d\tau\ \{\phi''(\lambda)\ \mathrm{Tr}\left[V_{1\tau}^2 E_{A_\tau}(\lambda) - V_1^2 E_A(\lambda)\right] |_{\lambda=b}^{\infty}\\
&\qquad - \int_b^{\infty} \phi'''(\lambda)\ \mathrm{Tr}\left[V_{1\tau}^2 E_{A_\tau}(\lambda) - V_1^2 E_A(\lambda)\right] d\lambda\}\\
&= \int_0^1 ds \int_0^s d\tau \int_b^{\infty} \phi'''(\lambda)\ \mathrm{Tr}\left[V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\right] d\lambda,
\end{aligned}$$

where we have integrated by-parts and observed that the boundary term vanishes. Thus the first term in (3.2.11) is equal to

$$\int_0^1 ds \int_0^s d\tau \int_b^{\infty} \phi'''(\lambda)\ \eta_{1\tau}(\lambda)\ d\lambda, \quad \text{where} \quad \eta_{1\tau}(\lambda) = \mathrm{Tr}\left[V_1^2 E_A(\lambda) - V_{1\tau}^2 E_{A_\tau}(\lambda)\right]. \tag{3.2.12}$$

Now consider the second expression in the right hand side of (3.2.11) :

$$\int_0^1 ds \int_0^s d\tau \int_{-\infty}^{\infty} \mathrm{i}^2\, t\hat{\phi}(t)dt \int_b^{\infty}\int_b^{\infty} \frac{\mathrm{e}^{\mathrm{i}t\lambda} - \mathrm{e}^{\mathrm{i}t\mu}}{\mathrm{i}(\lambda-\mu)}\ \mathrm{Tr}\left[V_{2\tau}E_{A_\tau}(d\lambda)V_{2\tau}E_{A_\tau}(d\mu) - V_2 E_A(d\lambda)V_2 E_A(d\mu)\right] \tag{3.2.13}$$

Since $V_2 \in \left[\mathrm{Ker}\left(\mathcal{M}_{(A+\mathrm{i})^{-1}}\right)\right]^{\perp} = \overline{\mathrm{Ran}\left(\mathcal{M}^*_{(A+\mathrm{i})^{-1}}\right)} = \overline{\mathrm{Ran}\left(\mathcal{M}_{(A-\mathrm{i})^{-1}}\right)}$, then there exists a sequence $\{V_2^{(n)}\} \subseteq \mathrm{Ran}\left(\mathcal{M}_{(A-\mathrm{i})^{-1}}\right)$ such that $V_2^{(n)} \longrightarrow V_2$ in $\|.\|_2$ as $n \longrightarrow \infty$ and $V_2^{(n)} = (A-\mathrm{i})^{-1} Y_0^{(n)} - Y_0^{(n)} (A-\mathrm{i})^{-1}$ for some $Y_0^{(n)} \in \mathcal{B}_2(\mathcal{H})$. Without loss of generality we can assume that $V_2^{(n)}$ is self-adjoint for each n , since if it is not self-adjoint, consider the sequence $\{\frac{V_2^{(n)}+(V_2^{(n)})^*}{2}\}$, which converges to V_2 in $\|.\|_2$ (since V_2 is self-adjoint) and $\frac{V_2^{(n)}+(V_2^{(n)})^*}{2}$ is self-adjoint for each n. Similarly, for every $\tau \in (0,1]$, there exists a sequence $\{V_{2\tau}^{(n)}\} \subseteq \mathrm{Ran}\left(\mathcal{M}_{(A_\tau-\mathrm{i})^{-1}}\right)$ such that $\left\|V_{2\tau}^{(n)} - V_{2\tau}\right\|_2 \longrightarrow 0$ point-wise as $n \longrightarrow \infty$ and $V_{2\tau}^{(n)}$ is self-adjoint for each n and $V_{2\tau}^{(n)} = (A_\tau-\mathrm{i})^{-1} Y^{(n)} - Y^{(n)} (A_\tau-\mathrm{i})^{-1}$ for some sequence $\{Y^{(n)}\} \subseteq \mathcal{B}_2(\mathcal{H})$. Furthermore, by Lemma 3.1.3 $(vii)(b)$, the map $[0,1] \ni \tau \longrightarrow V_{1\tau},\ V_{2\tau}$ are continuous. Thus the expression in (3.2.13) is equal to

$$\begin{aligned}
&\int_0^1 ds \int_0^s d\tau \int_{-\infty}^{\infty} \mathrm{i}^2\, t\hat{\phi}(t)dt \int_b^{\infty}\int_b^{\infty} \frac{\mathrm{e}^{\mathrm{i}t\lambda} - \mathrm{e}^{\mathrm{i}t\mu}}{\mathrm{i}(\lambda-\mu)} \lim_{n\to\infty}\ \mathrm{Tr}\left[V_{2\tau}E_{A_\tau}(d\lambda)V_{2\tau}^{(n)}E_{A_\tau}(d\mu) - V_2 E_A(d\lambda)V_2^{(n)} E_A(d\mu)\right]\\
&= \int_0^1 ds \int_0^s d\tau \int_{-\infty}^{\infty} \mathrm{i}^2\, t\hat{\phi}(t)dt \lim_{n\to\infty} \int_b^{\infty}\int_b^{\infty} \frac{\mathrm{e}^{\mathrm{i}t\lambda} - \mathrm{e}^{\mathrm{i}t\mu}}{\mathrm{i}(\lambda-\mu)}\ \mathrm{Tr}\left[V_{2\tau}E_{A_\tau}(d\lambda)V_{2\tau}^{(n)}E_{A_\tau}(d\mu) - V_2 E_A(d\lambda)V_2^{(n)} E_A(d\mu)\right]
\end{aligned} \tag{3.2.14}$$

since Var $\left(\mathcal{F}_2^{(n)} - \mathcal{F}_2\right) \leq \|V_{2\tau}\|_2 \, \|V_{2\tau}^{(n)} - V_{2\tau}\|_2 + \|V\|_2 \, \|V_2 - V_2^{(n)}\| \longrightarrow 0$ as $n \longrightarrow \infty$, where $\mathcal{F}_2(\Delta \times \delta) = \text{Tr}\left[V_{2\tau}E_{A_\tau}(\Delta)V_{2\tau}E_{A_\tau}(\delta) - V_2E_A(\Delta)V_2E_A(\delta)\right]$ and $\mathcal{F}_2^{(n)}(\Delta \times \delta)$ is the same expression with second V_2 and $V_{2\tau}$ terms are replaced by $V_2^{(n)}$ and $V_{2\tau}^{(n)}$ respectively. These are complex measures on $\mathbb{R}^2$ and Var $\left(\mathcal{F}_2^{(n)} - \mathcal{F}_2\right)$ is the variation of $\left(\mathcal{F}_2^{(n)} - \mathcal{F}_2\right)$. Note that

$$\begin{aligned}
\text{Tr}\left(V_{2\tau}E_{A_\tau}(d\lambda)V_{2\tau}^{(n)}E_{A_\tau}(d\mu)\right) &= \text{Tr}\left(V_{2\tau}E_{A_\tau}(d\lambda)\left[(A_\tau - \mathrm{i})^{-1}Y^{(n)} - Y^{(n)}(A_\tau - \mathrm{i})^{-1}\right]E_{A_\tau}(d\mu)\right] \\
&= \left[(\lambda - \mathrm{i})^{-1} - (\mu - \mathrm{i})^{-1}\right] \text{Tr}\left(V_{2\tau}E_{A_\tau}(d\lambda)Y^{(n)}E_{A_\tau}(d\mu)\right) \\
&= \frac{-(\lambda-\mu)}{(\lambda - \mathrm{i})(\mu - \mathrm{i})} \text{Tr}\left(V_{2\tau}E_{A_\tau}(d\lambda)Y^{(n)}E_{A_\tau}(d\mu)\right)
\end{aligned}$$

and since $\int_{-\infty}^{\infty}|t\hat{\phi}(t)|dt < \infty$ and the other functions are bounded, the right hand side expression in (3.2.14) is equal to

$$\begin{aligned}
&\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_{-\infty}^{\infty} \mathrm{i}^2\, t\hat{\phi}(t)dt \int_b^{\infty}\int_b^{\infty} \frac{\mathrm{e}^{\mathrm{i}t\lambda} - \mathrm{e}^{\mathrm{i}t\mu}}{\mathrm{i}(\lambda-\mu)} \left[\frac{-(\lambda-\mu)}{(\lambda - \mathrm{i})(\mu - \mathrm{i})}\right] \text{Tr}\{V_{2\tau}E_{A_\tau}(d\lambda)Y^{(n)}E_{A_\tau}(d\mu) \\
&\qquad\qquad - V_2E_A(d\lambda)Y_0^{(n)}E_A(d\mu)\} \\
&= \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_{-\infty}^{\infty} -\mathrm{i}\, t\hat{\phi}(t)dt \int_b^{\infty}\int_b^{\infty} \frac{[\mathrm{e}^{\mathrm{i}t\lambda} - \mathrm{e}^{\mathrm{i}t\mu}]}{[(\lambda - \mathrm{i})(\mu - \mathrm{i})]} \text{Tr}\{V_{2\tau}E_{A_\tau}(d\lambda)Y^{(n)}E_{A_\tau}(d\mu) \\
&\qquad\qquad - V_2E_A(d\lambda)Y_0^{(n)}E_A(d\mu)\} \\
&= \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_{-\infty}^{\infty} -\mathrm{i}t\hat{\phi}(t)dt \int_b^{\infty}\int_b^{\infty} \left[\mathrm{e}^{\mathrm{i}t\lambda} - \mathrm{e}^{\mathrm{i}t\mu}\right] \text{Tr}\{V_{2\tau}E_{A_\tau}(d\lambda)(A_\tau - \mathrm{i})^{-1}Y^{(n)}(A_\tau - \mathrm{i})^{-1}E_{A_\tau}(d\mu) \\
&\qquad\qquad - V_2E_A(d\lambda)(A - \mathrm{i})^{-1}Y_0^{(n)}(A - \mathrm{i})^{-1}E_A(d\mu)\} \\
&= \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_{-\infty}^{\infty} -\mathrm{i}t\hat{\phi}(t)dt \, \{\int_b^{\infty} \mathrm{e}^{\mathrm{i}t\lambda} \, (\text{Tr}[V_{2\tau}E_{A_\tau}(d\lambda)(A_\tau - \mathrm{i})^{-1}Y^{(n)}(A_\tau - \mathrm{i})^{-1}] \\
&\qquad\qquad - \text{Tr}[V_2E_A(d\lambda)(A - \mathrm{i})^{-1}Y_0^{(n)}(A - \mathrm{i})^{-1}]) \\
&\qquad - \int_b^{\infty} \mathrm{e}^{\mathrm{i}t\mu} \, (\text{Tr}[V_{2\tau}(A_\tau - \mathrm{i})^{-1}Y^{(n)}(A_\tau - \mathrm{i})^{-1}E_{A_\tau}(d\mu)] \\
&\qquad\qquad - \text{Tr}[V_2(A - \mathrm{i})^{-1}Y_0^{(n)}(A - \mathrm{i})^{-1}E_A(d\mu)])\} \\
&= \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_{-\infty}^{\infty} -\mathrm{i}t\hat{\phi}(t)dt \int_b^{\infty} \mathrm{e}^{\mathrm{i}t\lambda} \, \text{Tr}\left(V_{2\tau}\left[E_{A_\tau}(d\lambda), \tilde{Y}^{(n)}\right] - V_2\left[E_A(d\lambda), \tilde{Y_0}^{(n)}\right]\right),
\end{aligned} \tag{3.2.15}$$

where $\tilde{Y}^{(n)} = (A_\tau - \mathrm{i})^{-1}Y^{(n)}(A_\tau - \mathrm{i})^{-1}$ and $\tilde{Y_0}^{(n)} = (A - \mathrm{i})^{-1}Y_0^{(n)}(A - \mathrm{i})^{-1}$. Again by applying Fubini's theorem the right hand side expression in (3.2.15) is equal to

$$\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_b^{\infty} -\phi'(\lambda)\, \text{Tr}\left(V_{2\tau}\left[E_{A_\tau}(d\lambda), \tilde{Y}^{(n)}\right] - V_2\left[E_A(d\lambda), \tilde{Y_0}^{(n)}\right]\right),$$

and by integrating by-parts twice and on observing that the boundary term vanishes, this reduces to

$$\begin{aligned}
&\int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \; - \{\phi'(\lambda)\, \mathrm{Tr}(V_{2\tau}[E_{A_\tau}(\lambda), \tilde{Y}^{(n)}] - V_2[E_A(\lambda), \tilde{Y_0}^{(n)}]) \mid_{\lambda=a}^{\infty} \\
&\qquad\qquad - \int_b^\infty -\phi''(\lambda)\, \mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\lambda), \tilde{Y}^{(n)}\right] - V_2\left[E_A(\lambda), \tilde{Y_0}^{(n)}\right]\right) d\lambda\} \\
&= \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_b^\infty \phi''(\lambda)\, \mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\lambda), \tilde{Y}^{(n)}\right] - V_2\left[E_A(\lambda), \tilde{Y_0}^{(n)}\right]\right) d\lambda \\
&= \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \{\phi''(\lambda) \int_b^\lambda \mathrm{Tr}(V_{2\tau}[E_{A_\tau}(\mu), \tilde{Y}^{(n)}] - V_2[E_A(\mu), \tilde{Y_0}^{(n)}])\, d\mu \mid_{\lambda=b}^{\infty} \\
&\qquad - \int_b^\infty \phi'''(\lambda) \left(\int_b^\lambda \mathrm{Tr}\left(V_{2\tau}\left[E_{A_\tau}(\mu), \tilde{Y}^{(n)}\right] - V_2\left[E_A(\mu), \tilde{Y_0}^{(n)}\right]\right) d\mu\right) d\lambda\} \\
&= \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_b^\infty \phi'''(\lambda) \left(\int_b^\lambda \mathrm{Tr}\left(V_2\left[E_A(\mu), \tilde{Y_0}^{(n)}\right] - V_{2\tau}\left[E_{A_\tau}(\mu), \tilde{Y}^{(n)}\right]\right) d\mu\right) d\lambda \\
&= \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_b^\infty \phi'''(\lambda)\, \eta_{2\tau}^{(n)}(\lambda)\, d\lambda, \qquad (3.2.16)
\end{aligned}$$

where $\eta_{2\tau}^{(n)}(\lambda) = \int_b^\lambda \mathrm{Tr}\left(V_2\left[E_A(\mu), \tilde{Y_0}^{(n)}\right] - V_{2\tau}\left[E_{A_\tau}(\mu), \tilde{Y}^{(n)}\right]\right) d\mu.$

Here it is worth observing that the hypothesis that A is bounded below is used for the first time here, only for performing the second integration-by-parts. Combining (3.2.12) and (3.2.16), we conclude that

$$\begin{aligned}
&\mathrm{Tr}\left[\phi(A+V) - \phi(A) - D^{(1)}\phi(A) \bullet V - \frac{1}{2} D^{(2)}\phi(A) \bullet (V,V)\right] \\
&\qquad = \int_0^1 ds \int_0^s d\tau \lim_{n\to\infty} \int_b^\infty \phi'''(\lambda)\, \eta_\tau^{(n)}(\lambda)\, d\lambda,
\end{aligned} \qquad (3.2.17)$$

where $\eta_\tau^{(n)}(\lambda) = \eta_{1\tau}(\lambda) + \eta_{2\tau}^{(n)}(\lambda)$. We claim that $\{\eta_\tau^{(n)}\}$ is a cauchy sequence in

$L^1\left(\mathbb{R}, \frac{d\lambda}{(1+\lambda^2)^{1+\epsilon}}\right)$ ($\epsilon > 0$) and we follow the idea from [5]. First note that

$L^\infty(\mathbb{R}, d\lambda) = L^\infty(\mathbb{R}, \psi(\lambda) d\lambda)$ [where $\psi(\lambda) = \frac{1}{(1+\lambda^2)^{1+\epsilon}}$] since the two measures are equivalent. Next, let $f \in L^\infty(\mathbb{R}, d\lambda)$ and define

$g(\lambda) = \int_\lambda^\infty f(t)\psi(t) dt$ for $\lambda \in \mathbb{R}$, then g is absolutely continuous with

$g'(\lambda) = -f(\lambda)\psi(\lambda)$ a.e. and that $|g(\lambda)| \leq$ Const. $\frac{1}{(1+\lambda^2)^{\frac{1}{2}+\epsilon'}}$ (for some $\epsilon' > 0$) for $\lambda \to \infty$ and bounded. Next consider the expression

$$\int_{-\infty}^{\infty} f(\lambda)\psi(\lambda)\left[\eta_\tau^{(n)}(\lambda) - \eta_\tau^{(m)}(\lambda)\right] d\lambda = \int_b^{\infty} f(\lambda)\psi(\lambda)\left[\eta_{2\tau}^{(n)}(\lambda) - \eta_{2\tau}^{(m)}(\lambda)\right] d\lambda$$
$$= \int_b^{\infty} -g'(\lambda)\, d\lambda \left(\int_b^{\lambda} \text{Tr}\left(V_2\left[E_A(\mu), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right] - V_{2\tau}\left[E_{A_\tau}(\mu), \tilde{Y}^{(n)} - \tilde{Y}^{(m)}\right]\right) d\mu\right),$$

which on integration by-parts and on observing that the boundary terms vanishes, leads to

$$-g(\lambda)\int_b^{\lambda} \text{Tr}\left(V_2\left[E_A(\mu), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right] - V_{2\tau}\left[E_{A_\tau}(\mu), \tilde{Y}^{(n)} - \tilde{Y}^{(m)}\right]\right) d\mu|_{\lambda=b}^{\infty}$$
$$+ \int_b^{\infty} g(\lambda)\text{Tr}\left(V_2\left[E_A(\lambda), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right] - V_{2\tau}\left[E_{A_\tau}(\lambda), \tilde{Y}^{(n)} - \tilde{Y}^{(m)}\right]\right) d\lambda$$
$$= \int_b^{\infty} g(\lambda)\text{Tr}\left(V_2\left[E_A(\lambda), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right] - V_{2\tau}\left[E_{A_\tau}(\lambda), \tilde{Y}^{(n)} - \tilde{Y}^{(m)}\right]\right) d\lambda. \tag{3.2.18}$$

Define

$$h(\lambda) = \int_b^{\lambda} g(t)dt \quad \text{for} \quad \lambda \in [a, \infty), \quad \text{then} \quad h \quad \text{is bounded, differentiable on} \quad [a, \infty)$$

with $h'(\lambda) = g(\lambda)\ \forall\ \lambda \in [b, \infty)$. Hence by integrating by-parts and observing that the boundary term vanishes, the right hand side expression in (3.2.18) is equal to

$$\int_b^{\infty} h'(\lambda)\text{Tr}\left(V_2\left[E_A(\lambda), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right] - V_{2\tau}\left[E_{A_\tau}(\lambda), \tilde{Y}^{(n)} - \tilde{Y}^{(m)}\right]\right) d\lambda$$
$$= h(\lambda)\ \text{Tr}\left(V_2\left[E_A(\lambda), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right] - V_{2\tau}\left[E_{A_\tau}(\lambda), \tilde{Y}^{(n)} - \tilde{Y}^{(m)}\right]\right)|_{\lambda=b}^{\infty}$$
$$- \int_b^{\infty} h(\lambda)\text{Tr}\left(V_2\left[E_A(d\lambda), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right] - V_{2\tau}\left[E_{A_\tau}(d\lambda), \tilde{Y}^{(n)} - \tilde{Y}^{(m)}\right]\right)$$
$$= \int_b^{\infty} h(\lambda)\ \text{Tr}\left(V_{2\tau}\left[E_{A_\tau}(d\lambda), \tilde{Y}^{(n)} - \tilde{Y}^{(m)}\right] - V_2\left[E_A(d\lambda), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right]\right)$$
$$= \text{Tr}\left(V_{2\tau}\left[h(A_\tau), \tilde{Y}^{(n)} - \tilde{Y}^{(m)}\right] - V_2\left[h(A), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right]\right). \tag{3.2.19}$$

But on the other hand,

$$\left[h(A), \tilde{Y_0}^{(n)}\right] = \int_b^{\infty}\int_b^{\infty} [h(\lambda) - h(\mu)]E_A(d\lambda)\tilde{Y_0}^{(n)}E_A(d\mu)$$
$$= \int_b^{\infty}\int_b^{\infty} [h(\lambda) - h(\mu)](\lambda - \text{i})^{-1}(\mu - \text{i})^{-1}E_A(d\lambda)Y_0^{(n)}E_A(d\mu)$$
$$= \int_b^{\infty}\int_b^{\infty} \frac{h(\lambda) - h(\mu)}{(\lambda - \text{i})^{-1} - (\mu - \text{i})^{-1}}(\lambda - \text{i})^{-1}(\mu - \text{i})^{-1}E_A(d\lambda)\left[(A - \text{i})^{-1}Y_0^{(n)} - Y_0^{(n)}(A - \text{i})^{-1}\right]E_A(d\mu)$$

$$= \int_b^\infty \int_b^\infty \frac{h(\lambda) - h(\mu)}{(\lambda - \mathrm{i})^{-1} - (\mu - \mathrm{i})^{-1}} (\lambda - \mathrm{i})^{-1} (\mu - \mathrm{i})^{-1} E_A(d\lambda) V_2^{(n)} E_A(d\mu)$$

and hence

$$\left[h(A), \tilde{Y_0}^{(n)}\right] = -\int_b^\infty \int_b^\infty \frac{h(\lambda) - h(\mu)}{\lambda - \mu} \, E_A(d\lambda) V_2^{(n)} E_A(d\mu).$$

Therefore

$$\mathrm{Tr}\left(V_2 \left[h(A), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right]\right) = -\int_b^\infty \int_b^\infty \frac{h(\lambda) - h(\mu)}{\lambda - \mu} \, \mathrm{Tr}\left(V_2 E_A(d\lambda) \left[V_2^{(n)} - V_2^{(m)}\right] E_A(d\mu)\right) \tag{3.2.20}$$

and hence as in Birman-Solomyak ([2],[3]) and in [7] ,

$$\left|\mathrm{Tr}\left(V_2 \left[h(A), \tilde{Y_0}^{(n)} - \tilde{Y_0}^{(m)}\right]\right)\right| \leq \|f\|_\infty \, \|\psi\|_{L^1} \, \|V\|_2 \left\|\left[V_2^{(n)} - V_2^{(m)}\right]\right\|_2$$

and hence

$$\sup_{f \in L^\infty(\mathbb{R})} \frac{\left|\int_{-\infty}^\infty f(\lambda)\psi(\lambda) \left[\eta_\tau^{(n)}(\lambda) - \eta_\tau^{(m)}(\lambda)\right] d\lambda\right|}{\|f\|_\infty} \leq \|\psi\|_{L^1} \, \|V\|_2 \, \left(\left\|\left[V_2^{(n)} - V_2^{(m)}\right]\right\|_2 + \left\|\left[V_{2\tau}^{(n)} - V_{2\tau}^{(m)}\right]\right\|_2\right)$$

i.e.

$$\|\eta_\tau^{(n)} - \eta_\tau^{(m)}\|_{L^1(\mathbb{R}, \psi(\lambda)d\lambda)} \leq \|\psi\|_{L^1} \, \|V\|_2 \, \left(\left\|\left[V_2^{(n)} - V_2^{(m)}\right]\right\|_2 + \left\|\left[V_{2\tau}^{(n)} - V_{2\tau}^{(m)}\right]\right\|_2\right),$$

which converges to 0 as $m, n \longrightarrow \infty$ and $\forall \, \tau \in [0, 1]$. Therefore

$$\|\eta^{(n)} - \eta^{(m)}\|_{L^1(\mathbb{R}, \psi(\lambda)d\lambda)} \leq \int_0^1 ds \int_0^s d\tau \, \|\eta_\tau^{(n)} - \eta_\tau^{(m)}\|_{L^1(\mathbb{R}, \psi(\lambda)d\lambda)},$$

which converges to 0 as $m, n \longrightarrow \infty$, by the bounded convergence theorem and hence $\{\eta^{(n)}\}$ is a Cauchy sequence in $L^1(\mathbb{R}, \psi(\lambda)d\lambda)$ and thus there exists a function $\eta \in L^1(\mathbb{R}, \psi(\lambda)d\lambda)$ such that $\|\eta^{(n)} - \eta\|_{L^1(\mathbb{R}, \psi(\lambda)d\lambda)} \to 0$ as $n \longrightarrow \infty$. Therefore by using the Dominated Convergence theorem as well as Fubini's theorem, from (3.2.17) we conclude that

$$\mathrm{Tr}\left[\phi(A + V) - \phi(A) - D^{(1)}\phi(A) \bullet V - \frac{1}{2} D^{(2)}\phi(A) \bullet (V, V)\right] = \int_{-\infty}^\infty \phi'''(\lambda)\eta(\lambda)d\lambda.$$

□

The proof of the uniqueness and the real-valued nature of η is postponed till after the corollary 3.2.4, which is the counterpart of Corollary 3.1.6 for the unbounded case.

Corollary 3.2.4. *Let A be an unbounded self-adjoint operator in a Hilbert space $\mathcal{H}$ with $\sigma(A) \subseteq [b, \infty)$ for some $b \in \mathbb{R}$ and V be a self-adjoint operator such that $V \in \mathcal{B}_2(\mathcal{H})$. Then the function $\eta \in L^1(\mathbb{R}, \psi(\lambda)d\lambda)$ obtained as in Theorem 3.2.3 satisfies the following equation*

$$\begin{aligned}&\int_{-\infty}^{\infty} f(\lambda)\psi(\lambda)\eta(\lambda)d\lambda \\ &= \int_0^1 ds \int_0^s d\tau \int_b^{\infty} \int_b^{\infty} \frac{h(\lambda)-h(\mu)}{\lambda-\mu} \operatorname{Tr}\left[VE_A(d\lambda)VE_A(d\mu) - VE_{A_\tau}(d\lambda)VE_{A_\tau}(d\mu)\right],\end{aligned} \tag{3.2.21}$$

where $f(\lambda),\ g(\lambda),\ h(\lambda)$ and $\psi(\lambda)$ are as in the proof of the Theorem 3.2.3. Moreover

$$\|\eta\|_{L^1(\mathbb{R},\psi(\lambda)d\lambda)} \le \|\psi\|_{L^1}\|V\|_2^2.$$

Proof. By Fubini's theorem we have that

$$\int_{-\infty}^{\infty} f(\lambda)\psi(\lambda)\eta^{(n)}(\lambda)d\lambda = \int_0^1 ds \int_0^s d\tau \int_b^{\infty} f(\lambda)\psi(\lambda)\left[\eta_{1\tau}(\lambda) + \eta_{2\tau}^{(n)}(\lambda)\right]d\lambda.$$

But

$$\int_b^{\infty} f(\lambda)\psi(\lambda)\eta_{1\tau}(\lambda)d\lambda = \int_b^{\infty} -g'(\lambda)\operatorname{Tr}\left[V_1^2E_A(\lambda) - V_{1\tau}^2E_{A_\tau}(\lambda)\right]d\lambda,$$

which by integrating by-parts and observing that the boundary terms vanishes, leads to

$$\begin{aligned}&-g(\lambda)\operatorname{Tr}\left[V_1^2E_A(\lambda) - V_{1\tau}^2E_{A_\tau}(\lambda)\right]d\lambda\Big|_{\lambda=a}^{\infty} + \int_b^{\infty} g(\lambda)\operatorname{Tr}\left[V_1^2E_A(d\lambda) - V_{1\tau}^2E_{A_\tau}(d\lambda)\right] \\ &= \int_b^{\infty} g(\lambda)\operatorname{Tr}\left[V_1^2E_A(d\lambda) - V_{1\tau}^2E_{A_\tau}(d\lambda)\right] = \operatorname{Tr}\left[V_1^2h'(A) - V_{1\tau}^2h'(A_\tau)\right], \quad \text{which}\end{aligned}$$

by Lemma 3.1.3 (ii) is equal to

$$\int_b^{\infty}\int_b^{\infty} \frac{h(\lambda)-h(\mu)}{\lambda-\mu} \operatorname{Tr}\left[V_1E_A(d\lambda)V_1E_A(d\mu) - V_{1\tau}E_{A_\tau}(d\lambda)V_{1\tau}E_{A_\tau}(d\mu)\right]. \tag{3.2.22}$$

Again by repeating the same calculations as in the proof of the Theorem 3.2.3, we conclude that

$$\int_b^{\infty} f(\lambda)\psi(\lambda)\eta_{2\tau}^{(n)}(\lambda)d\lambda = \int_b^{\infty}\int_b^{\infty} \frac{h(\lambda)-h(\mu)}{\lambda-\mu} \operatorname{Tr}\left[V_2E_A(d\lambda)V_2^{(n)}E_A(d\mu) - V_{2\tau}E_{A_\tau}(d\lambda)V_{2\tau}^{(n)}E_{A_\tau}(d\mu)\right]. \tag{3.2.23}$$

Combining (3.2.22) and (3.2.23) we have,

$$\begin{aligned}&\int_{-\infty}^{\infty} f(\lambda)\psi(\lambda)\eta^{(n)}(\lambda)d\lambda\\ &= \int_0^1 ds \int_0^s d\tau \int_b^\infty \int_b^\infty \frac{h(\lambda)-h(\mu)}{\lambda-\mu} \operatorname{Tr}[(V_1 \oplus V_2)E_A(d\lambda)\left(V_1 \oplus V_2^{(n)}\right) E_A(d\mu)\\ &\qquad - (V_{1\tau} \oplus V_{2\tau})E_{A_\tau}(d\lambda)\left(V_{1\tau} \oplus V_{2\tau}^{(n)}\right) E_{A_\tau}(d\mu)].\end{aligned} \tag{3.2.24}$$

But by definition $V_2^{(n)}, V_{2\tau}^{(n)}$ converges to $V_2, V_{2\tau}$ respectively in $\|.\|_2$ and we have already proved that $\eta^{(n)}$ converges to η in $L^1(\mathbb{R}, \psi(\lambda)d\lambda)$ .Hence by taking limit on both sides of (3.2.24) we get that

$$\begin{aligned}&\int_{-\infty}^{\infty} f(\lambda)\psi(\lambda)\eta(\lambda)d\lambda\\ &= \int_0^1 ds \int_0^s d\tau \int_b^\infty \int_b^\infty \frac{h(\lambda)-h(\mu)}{\lambda-\mu} \operatorname{Tr}\left[VE_A(d\lambda)VE_A(d\mu) - VE_{A_\tau}(d\lambda)VE_{A_\tau}(d\mu)\right],\end{aligned} \tag{3.2.25}$$

where we have used the fact that

$\operatorname{Var}\left(\mathcal{G}_2^{(n)} - \mathcal{G}_2\right) \leq \|V\|_2 \left(\|V_{2\tau}^{(n)} - V_{2\tau}\|_2 + \|V_2 - V_2^{(n)}\|\right) \longrightarrow 0$, and that $\|h\|_{\text{Lip}} \leq \|g\|_\infty \leq \|f\|_\infty \|\psi\|_{L^1}$, where $\mathcal{G}_2^{(n)}(\Delta\times\delta) = \operatorname{Tr}\left[VE_{A_\tau}(\Delta)\left(V_{1\tau} \oplus V_{2\tau}^{(n)}\right)E_{A_\tau}(\delta) - VE_A(\Delta)\left(V_1 \oplus V_2^{(n)}\right)E_A(\delta)\right]$ and $\mathcal{G}_2(\Delta \times \delta) = \operatorname{Tr}\left[VE_{A_\tau}(\Delta)VE_{A_\tau}(\delta) - VE_A(\Delta)VE_A(\delta)\right]$ are complex measures on $\mathbb{R}^2$ and $\operatorname{Var}\left(\mathcal{G}_2^{(n)} - \mathcal{G}_2\right)$ is the variation of $\left(\mathcal{G}_2^{(n)} - \mathcal{G}_2\right)$. Again from (3.2.25) we have the following estimate (as in Birman-Solomyak ([2],[3]) and in [7])

$$\left|\int_{-\infty}^{\infty} f(\lambda)\psi(\lambda)\eta(\lambda)d\lambda\right| \leq \int_0^1 ds \int_0^s d\tau\ 2\ \|h\|_{\text{Lip}}\ \|V\|_2^2 \leq \|\psi\|_{L^1}\ \|f\|_\infty\ \|V\|_2^2$$

and hence

$$\sup_{f\in L^\infty(\mathbb{R})} \frac{\left|\int_{-\infty}^{\infty} f(\lambda)\psi(\lambda)\eta(\lambda)d\lambda\right|}{\|f\|_\infty} \leq \|\psi\|_{L^1}\ \|V\|_2^2 \quad \text{i.e.} \quad \|\eta\|_{L^1(\mathbb{R},\psi(\lambda)d\lambda)} \leq \|\psi\|_{L^1}\ \|V\|_2^2.$$

□

Proof of uniqueness and reality property of η***:*** For uniqueness in Theorem 3.2.3, let us assume that there exists $\eta_1, \eta_2 \in L^1(\mathbb{R}, \psi(\lambda)d\lambda)$ such that

$$\text{Tr}\left[\phi(A+V) - \phi(A) - D^{(1)}\phi(A) \bullet V - \frac{1}{2}D^{(2)}\phi(A) \bullet (V,V)\right] = \int_{-\infty}^{\infty} \phi'''(\lambda)\eta_j(\lambda)d\lambda$$

for $j = 1, 2$. Then using Corollary 3.2.4 we conclude that

$$\int_{\mathbb{R}} f(\lambda)\psi(\lambda)\eta_j(\lambda)d\lambda = \int_0^1 ds \int_0^s d\tau \int_b^{\infty} \int_b^{\infty} \frac{h(\lambda) - h(\mu)}{\lambda - \mu} \text{Tr}\left[VE_A(d\lambda)VE_A(d\mu) - VE_{A_\tau}(d\lambda)VE_{A_\tau}(d\mu)\right],$$

for $j = 1, 2$ and for all $f \in L^\infty(\mathbb{R})$. Hence

$$\int_{\mathbb{R}} f(\lambda)\; \psi(\lambda)\; \eta(\lambda)\; d\lambda = 0 \;\; \forall \;\; f \;\; \in L^\infty(\mathbb{R}), \tag{3.2.26}$$

where $\eta(\lambda) \equiv \eta_1(\lambda) - \eta_2(\lambda) \in L^1(\mathbb{R}, \psi(\lambda)d\lambda)$. Since (3.2.26) is true for all $f \in L^\infty(\mathbb{R})$, in particular it is true for all real valued $f \in L^\infty(\mathbb{R})$ i.e.

$$\int_{\mathbb{R}} f(\lambda)\; \psi(\lambda)\; \eta(\lambda)\; d\lambda = 0 \;\; \forall \quad \text{real valued} \quad f \;\; \in L^\infty(\mathbb{R}). \tag{3.2.27}$$

Let $\eta(\lambda) = \eta_{Rel}(\lambda) + \mathrm{i}\; \eta_{Img}(\lambda)$, where $\eta_{Rel}(\lambda)$ and $\eta_{Img}(\lambda)$ are real valued $L^1(\mathbb{R}, \psi(\lambda)d\lambda)$-function. Hence from (3.2.27) , we conclude that

$$\int_{\mathbb{R}} f(\lambda)\; \psi(\lambda)\; \eta_{Rel}(\lambda)\; d\lambda = 0 = \int_{\mathbb{R}} f(\lambda)\; \psi(\lambda)\; \eta_{Img}(\lambda)\; d\lambda \;\; \forall \quad \text{real valued} \quad f \;\; \in L^\infty(\mathbb{R}). \tag{3.2.28}$$

In particular if we consider $f(\lambda) = sgn\; \eta_{Rel}(\lambda)$, where $sgn\; \eta_{Rel}(\lambda) = 0 \;\forall\; \lambda$ such that $\eta_{Rel}(\lambda) = 0$; $sgn\; \eta_{Rel}(\lambda) = 1 \;\forall\; \lambda$ such that $\eta_{Rel}(\lambda) > 0$; $sgn\; \eta_{Rel}(\lambda) = -1 \;\forall\; \lambda$ such that $\eta_{Rel}(\lambda) < 0$. Then $f = sgn\; \eta_{Rel} \in L^\infty(\mathbb{R})$ and hence

$$\int_{\mathbb{R}} |\eta_{Rel}(\lambda)|\, |\psi(\lambda)|\; d\lambda = \int_{\mathbb{R}} sgn\; \eta_{Rel}(\lambda)\; \eta_{Rel}(\lambda)\; \psi(\lambda)d\lambda = 0,$$

which implies that $|\eta_{Rel}(\lambda)|\, |\psi(\lambda)| = 0$ a.e. and hence $\eta_{Rel}(\lambda) = 0$ a.e.. Similarly by the same above argument we conclude that $\eta_{Img}(\lambda) = 0$ a.e. and hence $\eta(\lambda) = 0$ a.e.. Therefore $\eta_1(\lambda) = \eta_2(\lambda)$a.e..

Next we want to show that η is real valued. Suppose $\eta(\lambda) = \eta_{Rel}(\lambda) + \mathrm{i}\; \eta_{Img}(\lambda)$, where $\eta_{Rel}(\lambda)$ and $\eta_{Img}(\lambda)$ are real valued $L^1(\mathbb{R}, \psi(\lambda)d\lambda)$-function. Then from (3.2.21) , we conclude that

$$\int_{-\infty}^{\infty} f(\lambda)\psi(\lambda)\eta(\lambda)d\lambda = \int_{-\infty}^{\infty} f(\lambda)\left[\eta_{Rel}(\lambda) + \mathrm{i}\,\eta_{Img}(\lambda)\right]\psi(\lambda)d\lambda$$
$$= \int_{-\infty}^{\infty} f(\lambda)\eta_{Rel}(\lambda)\psi(\lambda)d\lambda + \mathrm{i}\int_{-\infty}^{\infty} f(\lambda)\,\eta_{Img}(\lambda)\psi(\lambda)d\lambda$$
$$= \int_0^1 ds \int_0^s d\tau \int_b^{\infty}\int_b^{\infty} \frac{h(\lambda)-h(\mu)}{\lambda-\mu}\ \mathrm{Tr}\left[VE_A(d\lambda)VE_A(d\mu) - VE_{A_\tau}(d\lambda)VE_{A_\tau}(d\mu)\right], \tag{3.2.29}$$

But the right hand side of (3.2.29) is real valued for all real valued $f \in L^\infty(\mathbb{R})$ and hence

$$\int_{\mathbb{R}} f(\lambda)\,\psi(\lambda)\,\eta_{Img}(\lambda)\,d\lambda = 0\ \ \forall \quad \text{real valued} \quad f\ \in L^\infty(\mathbb{R}).$$

In particular if we consider $f(\lambda) = sgn\ \eta_{Img}(\lambda)$, where $sgn\ \eta_{Img}(\lambda) = 0\ \forall\ \lambda$ such that $\eta_{Img}(\lambda) = 0$; $sgn\ \eta_{Img}(\lambda) = 1\ \forall\ \lambda$ such that $\eta_{Img}(\lambda) > 0$; $sgn\ \eta_{Img}(\lambda) = -1\ \forall\ \lambda$ such that $\eta_{Img}(\lambda) < 0$. Then $f = sgn\ \eta_{Img} \in L^\infty(\mathbb{R})$ and hence

$$\int_{\mathbb{R}} |\eta_{Img}(\lambda)|\,|\psi(\lambda)|\,d\lambda = \int_{\mathbb{R}} sgn\ \eta_{Img}(\lambda)\ \eta_{Img}(\lambda)\ \psi(\lambda)d\lambda = 0,$$

which implies that $|\eta_{Img}(\lambda)|\,|\psi(\lambda)| = 0$ a.e. and hence $\eta_{Img}(\lambda) = 0$ a.e., which implies that η is real valued.

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Chapter 4

Some Results for Commuting n-Tuples

We have already discussed various trace theorems and their proofs using final dimensional approximations in Chapters 1-3 for single operator variable case. In this chapter we present some preliminary results towards obtaining some trace formula for two or more generally n-operator variables.

4.1 Approximation Results for n-Tuples

The main result in this section (Theorem 4.1.2) is an adaptation from the proof of Weyl-von Neumaan-Berg theorem [2] . First we need a simple lemma.

Lemma 4.1.1. *Let* $A \in \mathcal{B}(\mathcal{H})$ *be such that* $0 \leq A \leq I$*. Now consider the spectral projections* $E_k = E_A\left(\bigcup_{j=1}^{2^{k-1}} \left(2^{-k}(2j-1), 2^{-k}(2j)\right]\right)$ *for* $k \geq 1$*. Then*

$$A = \sum_{k=1}^{\infty} 2^{-k} E_k, \tag{4.1.1}$$

where the right hand side of (4.1.1) *converges in operator norm.*

Proof. We want to show that

$$A = \sum_{k=1}^{\infty} 2^{-k} E_k = \sum_{k=1}^{\infty} 2^{-k} E_A \left(\bigcup_{j=1}^{2^{k-1}} \left(2^{-k}(2j-1), 2^{-k}(2j)\right] \right)$$
$$= \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{2^{k-1}} E_A \left(2^{-k}(2j-1), 2^{-k}(2j)\right],$$

since $\left(2^{-k}(2i-1), 2^{-k}(2i)\right] \bigcap \left(2^{-k}(2j-1), 2^{-k}(2j)\right] = \emptyset$ for $i \neq j$ and $1 \leq i, j \leq 2^{k-1}$. Let

$$S_N \equiv \sum_{k=1}^{N} 2^{-k} E_k = \sum_{k=1}^{N} 2^{-k} \sum_{j=1}^{2^{k-1}} E_A \left(2^{-k}(2j-1), 2^{-k}(2j)\right]$$

Claim:

$$S_N = \sum_{m=1}^{2^N-1} m.2^{-N} E_A \left(m.2^{-N}, (m+1).2^{-N}\right].$$

Applying induction on N, we will prove the above claim.

For $N = 1$, $S_1 = 2^{-1} E_A \left(2^{-1}.1, 2^{-1}(2)\right] = 2^{-1} E_A \left(2^{-1}, 1\right] = 1.2^{-1} E_A \left(1.2^{-1}, (1+1).2^{-1}\right]$ and hence the result is true for $N = 1$

Next assume that the result is true for $N = l$ i.e.

$$S_l = \sum_{m=1}^{2^l-1} m.2^{-l} E_A \left(m.2^{-l}, (m+1).2^{-l}\right].$$

Now by using induction hypothesis, we have

$$\begin{aligned}
S_{l+1} &= \sum_{k=1}^{l+1} 2^{-k} \sum_{j=1}^{2^{k-1}} E_A \left(2^{-k}(2j-1), 2^{-k}(2j)\right] \\
&= \sum_{k=1}^{l} 2^{-k} \sum_{j=1}^{2^{k-1}} E_A \left(2^{-k}(2j-1), 2^{-k}(2j)\right] + 2^{-(l+1)} \sum_{j=1}^{2^l} E_A \left(2^{-(l+1)}(2j-1), 2^{-(l+1)}(2j)\right] \\
&= \sum_{m=1}^{2^l-1} m.2^{-l} E_A \left(m.2^{-l}, (m+1).2^{-l}\right] + 2^{-(l+1)} \sum_{j=1}^{2^l} E_A \left(2^{-(l+1)}(2j-1), 2^{-(l+1)}(2j)\right] \\
&= \sum_{m=1}^{2^l-1} m.2^{-l} E_A \left(m.2^{-l}, (m+1).2^{-l}\right] + 2^{-(l+1)} E_A \left(2^{-(l+1)}, 2^{-l}\right] \\
&\qquad + \sum_{j=2}^{2^l} 2^{-(l+1)} E_A \left(2^{-(l+1)}(2j-1), 2^{-(l+1)}(2j)\right].
\end{aligned} \tag{4.1.2}$$

But on the other hand

$$\sum_{m=1}^{2^l-1} m.2^{-l} E_A\left(m.2^{-l},(m+1).2^{-l}\right] = \sum_{m=1}^{2^l-1} 2m.2^{-(l+1)} E_A\left(2m.2^{-(l+1)},2(m+1).2^{-(l+1)}\right]$$
$$= \sum_{m=1}^{2^l-1} 2m.2^{-(l+1)} E_A\{\left(2m.2^{-(l+1)},2m+1.2^{-(l+1)}\right] \bigcup \left(2m+1.2^{-(l+1)},2(m+1).2^{-(l+1)}\right]\}$$
$$= \sum_{m=1}^{2^l-1} 2m.2^{-(l+1)} E_A\left(2m.2^{-(l+1)},2m+1.2^{-(l+1)}\right]$$
$$+ \sum_{m=1}^{2^l-1} 2m.2^{-(l+1)} E_A\left(2m+1.2^{-(l+1)},2(m+1).2^{-(l+1)}\right]$$
$$= \sum_{m=1}^{2^l-1} 2m.2^{-(l+1)} E_A\left(2m.2^{-(l+1)},2m+1.2^{-(l+1)}\right]$$
$$+ \sum_{j=2}^{2^l} 2(j-1).2^{-(l+1)} E_A\left(2j-1.2^{-(l+1)},2j.2^{-(l+1)}\right], \tag{4.1.3}$$

by substituting the index $j = m+1$ in second summation in last equality. Combining (4.1.2) and (4.1.3), we conclude that

$$S_{l+1} = \sum_{m=1}^{2^l-1} 2m.2^{-(l+1)} E_A\left(2m.2^{-(l+1)},2m+1.2^{-(l+1)}\right] + 2^{-(l+1)} E_A\left(2^{-(l+1)},2^{-l}\right]$$
$$+ \sum_{j=2}^{2^l} 2j-1.2^{-(l+1)} E_A\left(2j-1.2^{-(l+1)},2j.2^{-(l+1)}\right]$$
$$= \sum_{m=1}^{2^l-1} 2m.2^{-(l+1)} E_A\left(2m.2^{-(l+1)},2m+1.2^{-(l+1)}\right] + 2^{-(l+1)} E_A\left(2^{-(l+1)},2^{-l}\right]$$
$$+ \sum_{m=1}^{2^l-1} 2m+1.2^{-(l+1)} E_A\left(2m+1.2^{-(l+1)},2(m+1).2^{-(l+1)}\right]$$
$$= \sum_{m=1}^{2^{(l+1)}-1} m.2^{-(l+1)} E_A\left(m.2^{-(l+1)},(m+1).2^{-(l+1)}\right],$$

by substituting the index $j = m+1$ in second summation in second equality and hence by induction the result follows.

Thus for $f \in \mathcal{H}$,

$$\begin{aligned}\left(\sum_{k=1}^{\infty} 2^{-k} E_k\right) f &= \lim_{N \longrightarrow \infty} \left(\sum_{k=1}^{N} 2^{-k} E_k\right) f \\ &= \lim_{N \longrightarrow \infty} \left(\sum_{m=1}^{2^N-1} m.2^{-N} E_A \left(m.2^{-N}, (m+1).2^{-N}\right]\right) f \\ &= \int \lambda E_A(d\lambda) f,\end{aligned}$$

by using the definition of spectral integral of A (see [1]). Therefore

$$A = \int \lambda E_A\,(d\lambda) = \sum_{k=1}^{\infty} 2^{-k} E_k.$$

Hence the proof. □

Weyl-von Neumann [3] proved for one self-adjoint operator A that given $\epsilon > 0$, $\exists B \in \mathcal{B}_2(\mathcal{H})$ such that $\|B\|_2 < \epsilon$ and $A + B$ is diagonal. Later Berg extended this to an n-tuple of bounded commuting self-adjoint operators $(A_1, A_2, \ldots, A_n)$, which says that given $\epsilon > 0$, $\exists \, \{B_j\}_{j=1}^n$ of compact operators such that $\|B_j\| < \epsilon \; \forall j$ and $A_j + B_j$ is diagonal $\forall j$. We extend the ideas from Berg's proof in the next Theorem.

Theorem 4.1.2. *Let $A_1, A_2, \ldots, A_n$ be n-commuting bounded self-adjoint operators in an infinite dimensional separable Hilbert space $\mathcal{H}$. Then there exists a sequence $\{P_N\}$ of finite rank projections increases strongly to I (i.e. $P_N \uparrow I$) and $\|[A_i, P_N]\|_p \to 0$ as $N \to \infty$ for $p > n$ and for all $1 \le i \le n$.*

Proof. Without loss of generality we can assume that $0 \le A_i \le I$ for $1 \le i \le n$. For each A_i, consider the spectral projections

$$E_k^{(i)} = E_{A_i}\left(\bigcup_{j=1}^{2^{k-1}} \left(2^{-k}(2j-1), 2^{-k}(2j)\right]\right) \qquad \text{for } k \ge 1.$$

Hence by using Lemma 4.1.1 we conclude that

$$A_i = \sum_{k=1}^{\infty} 2^{-k} E_k^{(i)} \qquad \text{for} \quad 1 \le i \le n.$$

Now consider the commuting family of projections $\Omega = \{E_k^{(i)} : k \in \mathbb{N}; 1 \leq i \leq n\}$. Let $\{f_1, f_2, f_3, \ldots, f_n, \ldots\}$ be an countable orthonormal basis for $\mathcal{H}$ (as $\mathcal{H}$ is separable). Denote $\left(E_k^{(i)}\right)^1 \equiv E_k^{(i)}$ and $\left(E_k^{(i)}\right)^{-1} \equiv I - E_k^{(i)}$. Define

$$\mathcal{L}_N = span\{\left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j : \ 1 \leq j \leq N \ ; \ \epsilon_k^{(i)} = \pm 1\}.$$

Thus $\mathcal{L}_N$ is a linear subspace of $\mathcal{H}$ and it has the following properties.

$$(i) \quad \dim(\mathcal{L}_N) \leq N2^{nN}; \quad (ii) \quad \overline{\left(\bigcup_{N=1}^{\infty} \mathcal{L}_N\right)} = \mathcal{H}; \quad (iii) \quad \mathcal{L}_N \subsetneqq \mathcal{L}_{N+1}.$$

(i) From the definition of $\mathcal{L}_N$, we conclude that $\mathcal{L}_N$ is spanned by at most $N.2^{nN}$ number of distinct elements of the form $\left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j$ and hence $\dim(\mathcal{L}_N) \leq N2^{nN}$.

(ii) Now $\left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j \in \mathcal{L}_N$ for $1 \leq j \leq N$,

i.e. $\left(E_N^{(n)}\right)^{\epsilon_N^{(n)}} \left[\prod_{k=1}^{N-1}\prod_{i=1}^{n-1}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j \in \mathcal{L}_N$ for $1 \leq j \leq N$,

i.e. $\left(I - E_N^{(n)}\right) \left[\prod_{k=1}^{N-1}\prod_{i=1}^{n-1}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j \in \mathcal{L}_N$ and

$$\left(E_N^{(n)}\right) \left[\prod_{k=1}^{N-1}\prod_{i=1}^{n-1}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j \in \mathcal{L}_N \quad \text{for} \quad 1 \leq j \leq N,$$

which implies that $\left[\prod_{k=1}^{N-1}\prod_{i=1}^{n-1}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j \in \mathcal{L}_N$ for $1 \leq j \leq N$. By repeating the above argument we conclude that $\left(E_1^{(1)}\right)^{\epsilon_1^{(1)}} f_j \in \mathcal{L}_N$ for $1 \leq j \leq N$, i.e. $E_1^{(1)} f_j, \left(I - E_1^{(1)}\right) f_j \in \mathcal{L}_N$ for $1 \leq j \leq N$ and hence $\{f_1, f_2, f_3, \ldots, f_N\} \subset \mathcal{L}_N$, proving that $\overline{\left(\bigcup_{N=1}^{\infty} \mathcal{L}_N\right)} = \mathcal{H}$.

(iii) Again $\left[\prod_{k=1}^{N+1}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j \in \mathcal{L}_{N+1}$ for $1 \leq j \leq N+1$. But on the other hand

$$\left[\prod_{k=1}^{N+1}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j = \prod_{i=1}^{n}\left(E_{N+1}^{(i)}\right)^{\epsilon_{N+1}^{(i)}}\left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j$$

$= \left(E_{N+1}^{(n)}\right)^{\epsilon_{N+1}^{(n)}}\prod_{i=1}^{n-1}\left(E_{N+1}^{(i)}\right)^{\epsilon_{N+1}^{(i)}}\left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j$ and hence

$$\left(E_{N+1}^{(n)}\right)\prod_{i=1}^{n-1}\left(E_{N+1}^{(i)}\right)^{\epsilon_{N+1}^{(i)}}\left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j,$$

$$\left(I - E_{N+1}^{(n)}\right)\prod_{i=1}^{n-1}\left(E_{N+1}^{(i)}\right)^{\epsilon_{N+1}^{(i)}}\left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j \in \mathcal{L}_{N+1} \quad \text{for} \quad 1 \leq j \leq N+1.$$

Therefore, $\prod_{i=1}^{n-1}\left(E_{N+1}^{(i)}\right)^{\epsilon_{N+1}^{(i)}}\left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j \in \mathcal{L}_{N+1} \quad \text{for} \quad 1 \leq j \leq N+1.$

By repeating the above argument we conclude that

$\left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j \in \mathcal{L}_{N+1}$ for $1 \leq j \leq N+1$ and hence $\mathcal{L}_N \subsetneqq \mathcal{L}_{N+1}$

(since $f_{N+1} \in \mathcal{L}_{N+1}$ but $f_{N+1} \notin \mathcal{L}_N$). Let $P_N : \mathcal{H} \longrightarrow \mathcal{L}_N$ be the orthogonal projection onto $\mathcal{L}_N$. Then using the properties of $\mathcal{L}_N$, we conclude that $\{P_N\}_{N=1}^{\infty}$ is a sequence of finite rank projections increases strongly to I (i.e. $P_N \uparrow I$). Moreover,

$$E_k^{(i)}(\mathcal{L}_N) \subseteq \mathcal{L}_N \quad \text{for} \quad 1 \leq k \leq N \;;\; 1 \leq i \leq n, \quad \text{since}$$

$$E_k^{(i)}\left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j = \left[\prod_{k=1}^{N}\prod_{i=1}^{n}\left(E_k^{(i)}\right)^{\epsilon_k^{(i)}}\right] f_j \quad \text{or} \quad 0$$

and hence $\mathcal{L}_N$ is a reducing subspace for $E_k^{(i)}$ $(1 \leq i \leq n \;;\; 1 \leq k \leq N)$. Therefore

$$E_k^{(i)} P_N = P_N E_k^{(i)} \quad \text{for} \quad 1 \leq k \leq N \;;\; 1 \leq i \leq n. \tag{4.1.4}$$

Now for $1 \leq i \leq n$; $A_i = \sum_{k=1}^{\infty} 2^{-k} E_k^{(i)}$ and using (4.1.4), we have

$$[A_i, P_N] = \sum_{k=1}^{\infty} 2^{-k}\left[E_k^{(i)}, P_N\right] = \sum_{k=N+1}^{\infty} 2^{-k}\left[E_k^{(i)}, P_N\right]$$

and hence using $\|P_N\|_p = \{\mathrm{Tr}(P_N)\}^{1/p} = \{\dim(\mathcal{L}_N)\}^{1/p} \leq \left[N2^{nN}\right]^{1/p}$, we get

$$\begin{aligned}\|[A_i, P_N]\|_p &\leq \sum_{k=N+1}^{\infty} 2^{-k} \left\|\left[E_k^{(i)}, P_N\right]\right\|_p \leq 2 \sum_{k=N+1}^{\infty} 2^{-k} \|P_N\|_p \\ &\leq 2\left[N2^{nN}\right]^{1/p} \sum_{k=N+1}^{\infty} 2^{-k} = 2N^{1/p} 2^{-N(1-n/p)},\end{aligned}$$

which converges to 0 as $N \longrightarrow \infty$ for $p > n$, which completes the proof. □

4.2 Trace Approximations For Bounded Commuting Tuples

The next Theorem shows how Theorem 4.1.2 can be used for finite dimensional reduction. For simplicity we take n=2. It will be clear from the proof that the same proof will go through for any n. But before going to the main theorem, first we will state some simple lemmas.

Lemma 4.2.1. *Let A, B be two bounded self-adjoint operators in $\mathcal{H}$ such that $B - A \equiv V \in \mathcal{B}_2(\mathcal{H})$. Suppose $\|[A, P_N]\|_p \longrightarrow 0$ as $N \longrightarrow \infty$ for $p > 2$, where $\{P_N\}$ is a sequence of projections such that $P_N \uparrow I$. Then $\|[B, P_N]\|_2 \longrightarrow 0$ as $N \longrightarrow \infty$ for $p > 2$.*

Proof. Since $V \in \mathcal{B}_2(\mathcal{H})$ and $P_N \uparrow I$, then we have that $\|[V, P_N]\|_2 \longrightarrow 0$ as $N \longrightarrow \infty$ and hence using property (iv) of Schatten p-ideals in Chapter 1, we get

$$\begin{aligned}\|[B, P_N]\|_p &\leq \|[A, P_N]\|_p + \|[V, P_N]\|_p \\ &\leq \|[A, P_N]\|_p + (2.\|V\|)^{(1-2/p)} \|[V, P_N]\|_2^{2/p},\end{aligned}$$

which converges to 0 as $N \longrightarrow \infty$ for $p > 2$ (by hypothesis). □

Lemma 4.2.2. *Let A be a bounded self-adjoint operator in $\mathcal{H}$ such that $\|[A, P_N]\|_p \longrightarrow 0$ as $N \longrightarrow \infty$ for $p > 2$, where $\{P_N\}$ is a sequence of finite rank projections such that $P_N \uparrow \infty$. Then*

(i) $\|[A^r, P_N]\|_p \longrightarrow 0$ as $N \longrightarrow \infty$ for $r \in \mathbb{N}$.

(ii) If furthermore B is another bounded operator such that $\|[B, P_N]\|_p \longrightarrow 0$ as $N \longrightarrow \infty$ for $p > 2$, then $\|P_N\left[A^r B^s - (P_N A P_N)^r (P_N B P_N)^s\right] P_N\|_{p/2} \longrightarrow 0$ as $N \longrightarrow \infty$, for $r, s \in \mathbb{N}$.

Proof. (i) We will apply induction on r. If $r = 1$, then $\|[A^r, P_N]\|_p = \|[A, P_N]\|_p \longrightarrow 0$ as $N \longrightarrow \infty$ (by hypothesis). Assume that the result is true for $r \leq k$.

But $[A^{k+1}, P_N] = A[A^k, P_N] + [A, P_N]A^k$ and hence

$$\begin{aligned} \|[A^{k+1}, P_N]\|_p &\leq \|A[A^k, P_N]\|_p + \|[A, P_N]A^k\|_p \\ &\leq \|A\|\|[A^k, P_N]\|_p + \|[A, P_N]\|_p\|A\|^k, \end{aligned}$$

which converges to 0 as $N \longrightarrow \infty$ (by induction hypothesis). Therefore by induction $\|[A^r, P_N]\|_p \longrightarrow 0$ as $N \longrightarrow \infty$ for $r \in \mathbb{N}$.

(ii) The case $r = s = 0$ is trivial. If one of r or s is equal to 1 and other is equal to 0, then the expression itself is equal to 0. Next if $r = s = 1$, then

$$\begin{aligned} P_N \left[A^r B^s - (P_N A P_N)^r (P_N B P_N)^s\right] P_N &= P_N \left[AB - (P_N A P_N)(P_N B P_N)\right] P_N \\ &= P_N A P_N^{\perp} B P_N = P_N[P_N, A][B, P_N]P_N \end{aligned}$$

and hence using the properties of Schatten p-ideals (see Chapter1), we get

$$\|P_N \left[A^r B^s - (P_N A P_N)^r (P_N B P_N)^s\right] P_N\|_{p/2} \leq \|[P_N, A]\|_p \|[B, P_N]\|_p,$$

which converges to 0 as $N \longrightarrow \infty$ (by hypothesis).

Therefore we assume that either r or s ≥ 2. But on the other hand

$$\begin{aligned} &P_N \left[A^r B^s - (P_N A P_N)^r (P_N B P_N)^s\right] P_N \\ &\quad = \left[P_N A^r P_N - (P_N A P_N)^r\right] P_N B^s P_N + (P_N A P_N)^r \left[P_N B^s P_N - (P_N B P_N)^s\right] \\ &\qquad + P_N A^r P_N^{\perp} B^s P_N \end{aligned} \tag{4.2.1}$$

The equation (4.2.1) shows that the cases $(r = 0, s \geq 2)$ and $(s = 0, r \geq 2)$ are identical. Similarly if either $(r = 1, s \geq 2)$ or $(s = 1, r \geq 2)$, one of the first two factors in the equation (4.2.1) vanishes identically.

Therefore without loss of generality we assume that $r \geq 2$. Now using the properties of

Schatten p-ideals (see Chapter1), we have

$$\begin{aligned}
\|P_N(A)^r P_N - (P_N A P_N)^r\|_{\frac{p}{2}} &= \|P_N\left[(A)^r - (P_N A P_N)^r\right] P_N\|_{\frac{p}{2}} \\
&= \left\| P_N \left(\sum_{j=1}^{r} (A)^{r-j}(A - P_N A P_N)(P_N A P_N)^{j-1} \right) P_N \right\|_{\frac{p}{2}} \\
&= \left\| \sum_{j=1}^{r-1} P_N (A)^{r-j}(A - P_N A P_N)(P_N A P_N)^{j-1} P_N \right\|_{\frac{p}{2}} \\
&= \left\| \sum_{j=1}^{r-1} P_N (A)^{r-j} P_N^{\perp} A P_N (P_N A P_N)^{j-1} P_N \right\|_{\frac{p}{2}} \\
&= \left\| \sum_{j=1}^{r-1} P_N [P_N, A^{r-j}][A, P_N] P_N (P_N A P_N)^{j-1} P_N \right\|_{\frac{p}{2}} \\
&\leq \sum_{j=1}^{r-1} \|[P_N, A^{r-j}][A, P_N]\|_{p/2} \|A\|^{j-1} \\
&\leq \sum_{j=1}^{r-1} \|[P_N, A^{r-j}]\|_p \|[A, P_N]\|_p \|A\|^{j-1},
\end{aligned}$$

which converges to 0 as $N \longrightarrow \infty$, by hypothesis and using (i). Similarly if $s \geq 2$,

$$\|P_N(B)^s P_N - (P_N B P_N)^s\|_{p/2} \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty.$$

Furthermore for the third term in the right hand side of (4.2.1) , we have

$$\begin{aligned}
\|P_N A^r P_N^{\perp} B^s P_N\|_{p/2} &= \|P_N [P_N, A^r][B^s, P_N] P_N\|_{p/2} \\
&\leq \|[P_N, A^r]\|_p \|[B^s, P_N]\|_p,
\end{aligned}$$

which again converges to 0 as $N \longrightarrow \infty$, by hypothesis and using (i). Therefore each term in the right hand side of (4.2.1) converges to 0 as $N \longrightarrow \infty$ in $\|.\|_{\frac{p}{2}}$ and hence

$$\|P_N\left[A^r B^s - (P_N A P_N)^r (P_N B P_N)^s\right] P_N\|_{p/2} \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty.$$

□

Now we are in a position to prove the main theorem in this section.

Theorem 4.2.3. *Let (H_1, H_2) and (H_1^0, H_2^0) be two commuting pairs of bounded self-adjoint operators in an infinite dimensional Hilbert space $\mathcal{H}$ such that $H_1 - H_1^0 \equiv V_1 \in \mathcal{B}_2(\mathcal{H})$ and $H_2 - H_2^0 \equiv V_2 \in \mathcal{B}_2(\mathcal{H})$. Let $p_1(\lambda, \mu) = \sum\limits_{r,s\geq 0\ ;\ r+s\leq n} c(r,s)\lambda^r \mu^s$ and $p_2(x,y) = \sum\limits_{i,j\geq 0\ ;\ i+j\leq m} d(i,j)x^i y^j$. Then there exists a sequence $\{P_N\}$ of finite rank projections such that $P_N \uparrow I$ and*

$$\begin{aligned}
&\mathrm{Tr}\{\left[p_1\left(H_1, H_2\right) - p_1\left(H_1^0, H_2^0\right)\right]\left[p_2\left(H_1, H_2\right) - p_2\left(H_1^0, H_2^0\right)\right]\} \\
&\quad = \lim_{N\longrightarrow\infty} \mathrm{Tr}\{P_N\left[p_1\left(P_N H_1 P_N, P_N H_2 P_N\right) - p_1\left(P_N H_1^0 P_N, P_N H_2^0 P_N\right)\right] \\
&\qquad\qquad P_N\left[p_2\left(P_N H_1 P_N, P_N H_2 P_N\right) - p_2\left(P_N H_1^0 P_N, P_N H_2^0 P_N\right)\right] P_N\}.
\end{aligned}$$

Remark 4.2.4. In the statement of the above Theorem 4.2.3, though (H_1, H_2) and (H_1^0, H_2^0) are two commuting pairs of self-adjoint operators but $(P_N H_1 P_N, P_N H_2 P_N)$ and $(P_N H_1^0 P_N, P_N H_2^0 P_N)$ may not be a commuting pairs of self-adjoint operators. In that case the meaning of $p_i\left(P_N H_1 P_N, P_N H_2 P_N\right)$ and $p_i\left(P_N H_1^0 P_N, P_N H_2^0 P_N\right)$ for $i = 1, 2$ are as follows:

$$\begin{aligned}
p_1\left(P_N H_1 P_N, P_N H_2 P_N\right) &= \sum_{r,s\geq 0\ ;\ r+s\leq n} c(r,s)(P_N H_1 P_N)^r (P_N H_2 P_N)^s\ ; \\
p_1\left(P_N H_1^0 P_N, P_N H_2^0 P_N\right) &= \sum_{r,s\geq 0\ ;\ r+s\leq n} c(r,s)(P_N H_1^0 P_N)^r (P_N H_2^0 P_N)^s\ ; \\
p_2\left(P_N H_1 P_N, P_N H_2 P_N\right) &= \sum_{i,j\geq 0\ ;\ i+j\leq m} d(i,j)(P_N H_1 P_N)^i (P_N H_2 P_N)^j \quad \text{and} \\
p_2\left(P_N H_1^0 P_N, P_N H_2^0 P_N\right) &= \sum_{i,j\geq 0\ ;\ i+j\leq m} d(i,j)(P_N H_1^0 P_N)^i (P_N H_2^0 P_N)^j.
\end{aligned}$$

Proof of Theorem 4.2.3: Applying Theorem 4.1.2 for the pair (H_1^0, H_2^0) and using Lemma 4.2.1, we conclude that there exists a sequence $\{P_N\}$ of finite rank projections such that $P_N \uparrow I$ and $\|[H_1^0, P_N]\|_p, \|[H_2^0, P_N]\|_p, \|[H_1, P_N]\|_p, \|[H_2, P_N]\|_p \longrightarrow 0$ as $N \longrightarrow \infty$ for $p > 2$. Now

$$\begin{aligned}
p_1\left(H_1, H_2\right) - p_1\left(H_1^0, H_2^0\right) &= \sum_{r,s\geq 0\ ;\ r+s\leq n} c(r,s)\left[\left(H_1\right)^r\left(H_2\right)^s - \left(H_1^0\right)^r\left(H_2^0\right)^s\right] \\
&= \sum_{r,s\geq 0\ ;\ r+s\leq n} c(r,s)\{\left[\left(H_1\right)^r - \left(H_1^0\right)^r\right]\left(H_2\right)^s + \left(H_1^0\right)^r\left[\left(H_2\right)^s - \left(H_2^0\right)^s\right]\} \in \mathcal{B}_2(\mathcal{H}),
\end{aligned} \tag{4.2.2}$$

since $H_1 - H_1^0, H_2 - H_2^0 \in \mathcal{B}_2(\mathcal{H})$ and hence

$$P_N\left[p_1\left(H_1, H_2\right) - p_1\left(H_1^0, H_2^0\right)\right] P_N \longrightarrow \left[p_1\left(H_1, H_2\right) - p_1\left(H_1^0, H_2^0\right)\right] \quad \text{in} \quad \|.\|_2$$

as $N \longrightarrow \infty$. Similarly we conclude that

$$P_N\left[p_2\left(H_1, H_2\right)-p_2\left(H_1^0, H_2^0\right)\right] P_N \longrightarrow\left[p_2\left(H_1, H_2\right)-p_2\left(H_1^0, H_2^0\right)\right] \quad \text{in} \quad \|.\|_2$$

as $N \longrightarrow \infty$. Therefore

$$\begin{aligned} &\operatorname{Tr}\{\left[p_1\left(H_1, H_2\right)-p_1\left(H_1^0, H_2^0\right)\right]\left[p_2\left(H_1, H_2\right)-p_2\left(H_1^0, H_2^0\right)\right]\} \\ &=\lim_{N \longrightarrow \infty} \operatorname{Tr}\{P_N\left[p_1\left(H_1, H_2\right)-p_1\left(H_1^0, H_2^0\right)\right] P_N\left[p_2\left(H_1, H_2\right)-p_2\left(H_1^0, H_2^0\right)\right] P_N\}. \end{aligned} \tag{4.2.3}$$

Again

$$\begin{aligned} & p_1\left(H_1, H_2\right)-p_1\left(H_1^0, H_2^0\right) \\ & \quad=\{c(1,0)\left[H_1-H_1^0\right]+c(0,1)\left[H_2-H_2^0\right]\} \\ & \quad\quad+\sum_{r, s \geq 0\ ;\ 2 \leq r+s \leq n} c(r, s)\left[\left(H_1\right)^r\left(H_2\right)^s-\left(H_1^0\right)^r\left(H_2^0\right)^s\right] \end{aligned}$$

and

$$\begin{aligned} & p_2\left(H_1, H_2\right)-p_2\left(H_1^0, H_2^0\right) \\ & \quad=\{d(1,0)\left[H_1-H_1^0\right]+d(0,1)\left[H_2-H_2^0\right]\} \\ & \quad\quad+\sum_{i, j \geq 0\ ;\ 2 \leq i+j \leq m} d(i, j)\left[\left(H_1\right)^i\left(H_2\right)^j-\left(H_1^0\right)^i\left(H_2^0\right)^j\right] \end{aligned}$$

and hence

$$\begin{aligned} & \left[p_1\left(H_1, H_2\right)-p_1\left(H_1^0, H_2^0\right)\right]\left[p_2\left(H_1, H_2\right)-p_2\left(H_1^0, H_2^0\right)\right] \\ & =c(1,0) d(1,0)\left[H_1-H_1^0\right]\left[H_1-H_1^0\right]+c(0,1) d(1,0)\left[H_2-H_2^0\right]\left[H_1-H_1^0\right] \\ & \quad+c(1,0) d(0,1)\left[H_1-H_1^0\right]\left[H_2-H_2^0\right]+c(0,1) d(0,1)\left[H_2-H_2^0\right]\left[H_2-H_2^0\right] \\ & \quad+\sum_{r, s \geq 0\ ;\ 2 \leq r+s \leq n} \sum_{i, j \geq 0\ ;\ 2 \leq i+j \leq m} c(r, s) d(i, j)\{\left[\left(H_1\right)^r\left(H_2\right)^s-\left(H_1^0\right)^r\left(H_2^0\right)^s\right] \\ & \qquad\qquad\left[\left(H_1\right)^i\left(H_2\right)^j-\left(H_1^0\right)^i\left(H_2^0\right)^j\right]\}. \end{aligned}$$

Therefore

$$\begin{aligned} & P_N\left[p_1\left(H_1, H_2\right)-p_1\left(H_1^0, H_2^0\right)\right] P_N\left[p_2\left(H_1, H_2\right)-p_2\left(H_1^0, H_2^0\right)\right] P_N \\ & \quad-\{P_N\left[p_1\left(P_N H_1 P_N, P_N H_2 P_N\right)-p_1\left(P_N H_1^0 P_N, P_N H_2^0 P_N\right)\right] \\ & \qquad P_N\left[p_2\left(P_N H_1 P_N, P_N H_2 P_N\right)-p_2\left(P_N H_1^0 P_N, P_N H_2^0 P_N\right)\right] P_N\} \\ & =\sum_{r, s \geq 0\ ;\ 2 \leq r+s \leq n} \sum_{i, j \geq 0\ ;\ 2 \leq i+j \leq m} c(r, s) d(i, j) \\ & \quad(\ \{P_N[\left(H_1\right)^r\left(H_2\right)^s-\left(H_1^0\right)^r\left(H_2^0\right)^s] P_N[\left(H_1\right)^i\left(H_2\right)^j-\left(H_1^0\right)^i\left(H_2^0\right)^j] P_N\} \\ & \qquad-\{P_N[(P_N H_1 P_N)^r(P_N H_2 P_N)^s-(P_N H_1^0 P_N)^r(P_N H_2^0 P_N)^s] \\ & \qquad\quad P_N[(P_N H_1 P_N)^i(P_N H_2 P_N)^j-(P_N H_1^0 P_N)^i(P_N H_2^0 P_N)^j] P_N\}). \end{aligned}$$

But on the other hand

$$\begin{aligned}
&\|P_N[(H_1)^r (H_2)^s - \left(H_1^0\right)^r \left(H_2^0\right)^s]P_N[(H_1)^i (H_2)^j - \left(H_1^0\right)^i \left(H_2^0\right)^j]P_N \\
&\qquad - \{P_N[(P_N H_1 P_N)^r(P_N H_2 P_N)^s - (P_N H_1^0 P_N)^r(P_N H_2^0 P_N)^s] \\
&\qquad\qquad P_N[(P_N H_1 P_N)^i(P_N H_2 P_N)^j - (P_N H_1^0 P_N)^i(P_N H_2^0 P_N)^j]P_N\}\|_1 \\
&\leq \|P_N[(H_1)^r (H_2)^s - \left(H_1^0\right)^r \left(H_2^0\right)^s]P_N \\
&\qquad - P_N[(P_N H_1 P_N)^r(P_N H_2 P_N)^s - (P_N H_1^0 P_N)^r(P_N H_2^0 P_N)^s]P_N\|_2 \\
&\qquad\qquad \left\|P_N[(H_1)^i (H_2)^j - \left(H_1^0\right)^i \left(H_2^0\right)^j]P_N\right\|_2 \\
&+ \left\|P_N[(P_N H_1 P_N)^r(P_N H_2 P_N)^s - (P_N H_1^0 P_N)^r(P_N H_2^0 P_N)^s]P_N\right\|_2 \\
&\qquad \|P_N[(H_1)^i (H_2)^j - \left(H_1^0\right)^i \left(H_2^0\right)^j]P_N \\
&\qquad\qquad - P_N[(P_N H_1 P_N)^i(P_N H_2 P_N)^j - (P_N H_1^0 P_N)^i(P_N H_2^0 P_N)^j]P_N\|_2.
\end{aligned} \tag{4.2.4}$$

Again

$$\begin{aligned}
&\left\|P_N[(H_1)^i (H_2)^j - \left(H_1^0\right)^i \left(H_2^0\right)^j]P_N\right\|_2 \\
&\quad \leq \left\|[(H_1)^i - \left(H_1^0\right)^i]\right\|_2 \|H_2\|^j + \|H_1^0\|^i \left\|[(H_2)^j - \left(H_2^0\right)^j]\right\|_2 \\
&\quad \leq \|V_1\|_2 \left(\sum_{l=0}^{i-1} \|H_1\|^{i-l-1}\|H_1^0\|^l\right) \|H_2\|^j + \|H_1^0\|^i\|V_2\|_2 \left(\sum_{l=0}^{j-1} \|H_2\|^{j-l-1}\|H_2^0\|^l\right) \equiv C_1(\text{say})
\end{aligned} \tag{4.2.5}$$

and similarly

$$\begin{aligned}
&\left\|P_N[(P_N H_1 P_N)^r(P_N H_2 P_N)^s - (P_N H_1^0 P_N)^r(P_N H_2^0 P_N)^s]P_N\right\|_2 \\
&\leq \|V_1\|_2 \left(\sum_{l=0}^{r-1} \|H_1\|^{r-l-1}\|H_1^0\|^l\right) \|H_2\|^s + \|H_1^0\|^r\|V_2\|_2 \left(\sum_{l=0}^{s-1} \|H_2\|^{s-l-1}\|H_2^0\|^l\right) \equiv C_2(\text{say}).
\end{aligned} \tag{4.2.6}$$

Therefore combining (4.2.5) and (4.2.6), the right hand side of (4.2.4) is less than or equal to

$$\begin{aligned}
&C_1\|P_N[(H_1)^r (H_2)^s - \left(H_1^0\right)^r \left(H_2^0\right)^s]P_N \\
&\qquad - P_N[(P_N H_1 P_N)^r(P_N H_2 P_N)^s - (P_N H_1^0 P_N)^r(P_N H_2^0 P_N)^s]P_N\|_2 \\
&+ C_2\|P_N[(H_1)^i (H_2)^j - \left(H_1^0\right)^i \left(H_2^0\right)^j]P_N \\
&\qquad - P_N[(P_N H_1 P_N)^i(P_N H_2 P_N)^j - (P_N H_1^0 P_N)^i(P_N H_2^0 P_N)^j]P_N\|_2,
\end{aligned}$$

which by using property (iv) of Schatten p-ideals in Chapter 1 is less than or equal to

$$\begin{aligned} C_1 & \left[2(\|H_1\|^r\|H_2\|^s + \|H_1^0\|^r\|H_2^0\|^s)\right]^{(1-\frac{p}{4})} \\ & \qquad \|P_N[(H_1)^r\,(H_2)^s - \left(H_1^0\right)^r\left(H_2^0\right)^s]P_N \\ & \qquad\qquad - P_N[(P_NH_1P_N)^r(P_NH_2P_N)^s - (P_NH_1^0P_N)^r(P_NH_2^0P_N)^s]P_N\|_{\frac{p}{2}}^{\frac{p}{4}} \\ + C_2 & \left[2(\|H_1\|^i\|H_2\|^j + \|H_1^0\|^i\|H_2^0\|^j)\right]^{(1-\frac{p}{4})} \\ & \qquad \|P_N[(H_1)^i\,(H_2)^j - \left(H_1^0\right)^i\left(H_2^0\right)^j]P_N \\ & \qquad\qquad - P_N[(P_NH_1P_N)^i(P_NH_2P_N)^j - (P_NH_1^0P_N)^i(P_NH_2^0P_N)^j]P_N\|_{\frac{p}{2}}^{\frac{p}{4}}, \end{aligned} \tag{4.2.7}$$

provided $p/2 \leq 2$ and hence by using part (ii) of Lemma 4.2.2 we conclude that (4.2.7) converges to 0 as $N \longrightarrow \infty$. Therefore we have proved that

$$\begin{aligned} \|P_N & \left[p_1\left(H_1,H_2\right) - p_1\left(H_1^0,H_2^0\right)\right] P_N \left[p_2\left(H_1,H_2\right) - p_2\left(H_1^0,H_2^0\right)\right] P_N \\ & - \{P_N\left[p_1\left(P_NH_1P_N,P_NH_2P_N\right) - p_1\left(P_NH_1^0P_N,P_NH_2^0P_N\right)\right] \\ & \qquad P_N\left[p_2\left(P_NH_1P_N,P_NH_2P_N\right) - p_2\left(P_NH_1^0P_N,P_NH_2^0P_N\right)\right] P_N\}\|_1 \end{aligned} \tag{4.2.8}$$

converges to 0 as $N \longrightarrow \infty$ and hence combining (4.2.3) and (4.2.8), we get that

$$\begin{aligned} & \mathrm{Tr}\{\left[p_1\left(H_1,H_2\right) - p_1\left(H_1^0,H_2^0\right)\right]\left[p_2\left(H_1,H_2\right) - p_2\left(H_1^0,H_2^0\right)\right]\} \\ & = \lim_{N\to\infty} \mathrm{Tr}\{P_N\left[p_1\left(P_NH_1P_N,P_NH_2P_N\right) - p_1\left(P_NH_1^0P_N,P_NH_2^0P_N\right)\right] \\ & \qquad P_N\left[p_2\left(P_NH_1P_N,P_NH_2P_N\right) - p_2\left(P_NH_1^0P_N,P_NH_2^0P_N\right)\right] P_N\}. \end{aligned} \tag{4.2.9}$$

□

4.3 Trace Approximations For unbounded Commuting Tuples

We begin with lemma which is an extension of Theorem 4.1.2.

Lemma 4.3.1. *Let* (H_1, H_2) *and* (H_1^0, H_2^0) *be two commuting pairs of unbounded self-adjoint operators in an infinite dimensional Hilbert space* $\mathcal{H}$ *such that* $H_1 - H_1^0 \equiv V_1 \in \mathcal{B}_2(\mathcal{H})$ *and* $H_2 - H_2^0 \equiv V_2 \in \mathcal{B}_2(\mathcal{H})$. *Then given* $\epsilon > 0,\ \exists$ *a projection* P *of finite rank such that for* $i = 1, 2$ *and* $p > 2$,

$$(i) \quad \|[H_i^0, P]\|_p \leq \epsilon, \qquad\qquad (ii) \quad \|[H_i, P]\|_p \leq \epsilon.$$

Proof. Let $E_{H_1}(.), E_{H_2}(.), E_{H_1^0}(.)$ and $E_{H_2^0}(.)$ be the spectral families of the operators H_1, H_2, H_1^0 and H_2^0 respectively. Define $F^0(\triangle) = E_{H_1^0}(\triangle_1)E_{H_2^0}(\triangle_2)$, where $\triangle = \triangle_1 \times \triangle_2 \subseteq \text{Borel}(\mathbb{R}^2)$. Then $F^0(.)$ is a spectral measure on $\mathbb{R}^2$, since $E_{H_1^0}(.)$ commutes with $E_{H_2^0}(.)$. Thus

$$F^0\left((-a,a] \times (-a,a]\right) = E_{H_1^0}\left((-a,a]\right) E_{H_2^0}\left((-a,a]\right) \longrightarrow I \quad \text{strongly as} \quad \mathbb{R} \ni a \longrightarrow \infty.$$

Denote $F^0_{(a)} \equiv F^0\left((-a,a] \times (-a,a]\right)$ and hence

$$(F^0_{(a)})^\perp \longrightarrow 0 \quad \text{strongly as} \quad a \longrightarrow \infty. \tag{4.3.1}$$

Next we note that

$$\text{Ran}\left(F^0_{(a)}\right) \subseteq \text{Dom}\left(H_i^0\right) \quad \text{for} \quad i = 1, 2. \tag{4.3.2}$$

Indeed, for $f \in \mathcal{H}$

$$\begin{aligned}\int_{-\infty}^{\infty} \lambda^2 \|E_{H_1^0}(d\lambda)F^0_{(a)}f\|^2 &= \int_{-\infty}^{\infty} \lambda^2 \|E_{H_1^0}(d\lambda)E_{H_1^0}((-a,a])E_{H_2^0}((-a,a])f\|^2 \\ &= \int_{-a}^{a} \lambda^2 \|E_{H_1^0}(d\lambda)E_{H_2^0}((-a,a])f\|^2 < \infty.\end{aligned}$$

Moreover, since $V_1, V_2 \in \mathcal{B}_2(\mathcal{H})$, then

$$\|(F^0_{(a)})^\perp V_i\|_2 \longrightarrow 0 \quad \text{as} \quad a \longrightarrow \infty \qquad \text{for} \quad i = 1, 2. \tag{4.3.3}$$

Therefore given $\epsilon > 0$, $\exists \quad \tilde{a} \in \mathbb{N}$ such that

$$2\|V_i\|^{(1-\frac{2}{p})}\|(F^0_{(\tilde{a})})^\perp V_i\|_2^{\frac{2}{p}} \quad \le \frac{\epsilon}{4} \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad p > 2. \tag{4.3.4}$$

Having chosen $\tilde{a}$, consider the operators $H^0_{1(\tilde{a})} \equiv H_1^0 F^0_{(\tilde{a})}$ and $H^0_{2(\tilde{a})} \equiv H_2^0 F^0_{(\tilde{a})}$. Also note that $H^0_{1(\tilde{a})}$ commutes with $H^0_{2(\tilde{a})}$. Therefore $\left(H^0_{1(\tilde{a})}, H^0_{2(\tilde{a})}\right)$ is a pair of commuting bounded self-adjoint operators in $\mathcal{H}$. Without loss of generality we assume that $0 \le H^0_{i(\tilde{a})} \le I$ for $i = 1, 2$. For each $H^0_{i(\tilde{a})}$, consider the spectral projections

$$E^{(i)0}_{k(\tilde{a})} = E_{H^0_{i(\tilde{a})}}\left(\bigcup_{j=1}^{2^{k-1}} \left(2^{-k}(2j-1), 2^{-k}(2j)\right]\right) \quad \text{for} \quad k \ge 1.$$

Hence by using Lemma 4.1.1 we conclude that

$$H^0_{i(\tilde{a})} = \sum_{k=1}^{\infty} 2^{-k} E^{(i)0}_{k(\tilde{a})} \quad \text{for} \quad i = 1, 2.$$

Now consider the commuting family of projections $\Omega = \{E^{(i)0}_{k(\tilde{a})} : k \in \mathbb{N} \; ; \; i = 1,2\}$. Let $\{f_1, f_2, f_3, \dots, f_n, \dots\}$ be an orthonormal basis for $\mathcal{H}$. Denote

$$\left(E^{(i)0}_{k(\tilde{a})}\right)^1 \equiv E^{(i)0}_{k(\tilde{a})} \quad \text{and} \quad \left(E^{(i)0}_{k(\tilde{a})}\right)^{-1} \equiv I - E^{(i)0}_{k(\tilde{a})} \quad \text{for} \quad i = 1,2.$$

Define

$$\tilde{\mathcal{L}}_N = span\{\left[\prod_{k=1}^{N}\left(E^{(1)0}_{k(\tilde{a})}\right)^{\epsilon_k^{(1)}}\left(E^{(2)0}_{k(\tilde{a})}\right)^{\epsilon_k^{(2)}} F^0_{(\tilde{a})}\right] f_j : 1 \le j \le N \; ; \; \epsilon_k^{(1)}, \epsilon_k^{(2)} = \pm 1\}.$$

But $F^0_{(\tilde{a})}$ commutes with $H^0_{i(\tilde{a})}$ for $i = 1,2$ and hence

$$\tilde{\mathcal{L}}_N = F^0_{(\tilde{a})}\left(span\{\left[\prod_{k=1}^{N}\left(E^{(1)0}_{k(\tilde{a})}\right)^{\epsilon_k^{(1)}}\left(E^{(2)0}_{k(\tilde{a})}\right)^{\epsilon_k^{(2)}}\right] f_j : 1 \le j \le N \; ; \; \epsilon_k^{(1)}, \epsilon_k^{(2)} = \pm 1\}\right).$$

Therefore $\tilde{\mathcal{L}}_N \subseteq \text{Ran}\left(F^0_{(\tilde{a})}\right) \subseteq \text{Dom}\,(H^0_i)$ for $i = 1,2$. Again by the same calculation as in the proof of of Theorem 4.1.2, we conclude that

$$(i)\ \dim\left(\tilde{\mathcal{L}}_N\right) \le N 2^{2N} \quad (ii)\ \overline{\left(\bigcup_{N=1}^{\infty} \tilde{\mathcal{L}}_N\right)} = Ran\left(F^0_{(\tilde{a})}\right) \quad (iii)\ \tilde{\mathcal{L}}_N \subseteqq \tilde{\mathcal{L}}_{N+1}.$$

Let $\tilde{P}_N : \mathcal{H} \longrightarrow \tilde{\mathcal{L}}_N$ be the orthogonal projection onto $\tilde{\mathcal{L}}_N$. Then using the above properties of $\tilde{\mathcal{L}}_N$, we conclude that $\{\tilde{P}_N\}_{N=1}^{\infty}$ is a sequence of finite rank projections increases strongly to $F^0_{(\tilde{a})}$ (i.e. $\tilde{P}_N \uparrow F^0_{(\tilde{a})}$). Also

$$\text{Ran}\left(\tilde{P}_N\right) \subseteq \text{Ran}\left(F^0_{(\tilde{a})}\right) \subseteq \text{Dom}\left(H^0_i\right) = \text{Dom}\,(H_i) \quad \text{for} \quad i = 1,2. \tag{4.3.5}$$

Moreover,

$$E^{(i)0}_{k(\tilde{a})}\left(\tilde{\mathcal{L}}_N\right) \subseteq \tilde{\mathcal{L}}_N \quad \text{for} \quad 1 \le k \le N \; ; \; i = 1,2, \quad \text{since}$$

$$E^{(i)0}_{k(\tilde{a})}\left[\prod_{k=1}^{N}\left(E^{(1)0}_{k(\tilde{a})}\right)^{\epsilon_k^{(1)}}\left(E^{(2)0}_{k(\tilde{a})}\right)^{\epsilon_k^{(2)}} F^0_{(\tilde{a})}\right] f_j = \left[\prod_{k=1}^{N}\left(E^{(1)0}_{k(\tilde{a})}\right)^{\epsilon_k^{(1)}}\left(E^{(2)0}_{k(\tilde{a})}\right)^{\epsilon_k^{(2)}} F^0_{(\tilde{a})}\right] f_j \quad \text{or} \quad 0$$

and hence $\tilde{\mathcal{L}}_N$ is a reducing subspace for $E^{(i)0}_{k(\tilde{a})}$ $(1 \le k \le N \; ; \; i = 1,2)$. Therefore

$$\tilde{P}_N\, E^{(i)0}_{k(\tilde{a})} = E^{(i)0}_{k(\tilde{a})}\, \tilde{P}_N \quad \text{for} \quad 1 \le k \le N \quad \text{and} \quad i = 1,2. \tag{4.3.6}$$

Again from (4.3.5) we conclude that

$$\tilde{P}_N F^0_{(\tilde{a})} = F^0_{(\tilde{a})} \tilde{P}_N = \tilde{P}_N \quad \text{for all} \quad N \in \mathbb{N}. \tag{4.3.7}$$

Therefore by using (4.3.6), we have

$$\left[H^0_{i(\tilde{a})}, \tilde{P}_N\right] = \sum_{k=1}^{\infty} 2^{-k}\left[E^{(i)0}_{k(\tilde{a})}, \tilde{P}_N\right] = \sum_{k=N+1}^{\infty} 2^{-k}\left[E^{(i)0}_{k(\tilde{a})}, \tilde{P}_N\right]$$

and hence using $\|\tilde{P}_N\|_p = \{\text{Tr}\left(\tilde{P}_N\right)\}^{1/p} = \{\text{dim}\left(\tilde{\mathcal{L}}_N\right)\}^{1/p} \leq \left[N2^{2N}\right]^{1/p}$, we get

$$\begin{aligned}\left\|\left[H^0_{i(\tilde{a})}, \tilde{P}_N\right]\right\|_p &\leq \sum_{k=N+1}^{\infty} 2^{-k}\left\|\left[E^{(i)0}_{k(\tilde{a})}, \tilde{P}_N\right]\right\|_p \leq 2\sum_{k=N+1}^{\infty} 2^{-k}\|\tilde{P}_N\|_p \\ &\leq 2\left[N2^{2N}\right]^{1/p}\sum_{k=N+1}^{\infty} 2^{-k} = 2N^{1/p}2^{-N(1-2/p)},\end{aligned}$$

which converges to 0 as $N \longrightarrow \infty$ for $p > 2$ and $i = 1, 2$.

But on the other hand, using (4.3.5) and (4.3.7) we get

$$\begin{aligned}\left[H^0_i, \tilde{P}_N\right] &= H^0_i\tilde{P}_N - \tilde{P}_N H^0_i = H^0_i F^0_{(\tilde{a})}\tilde{P}_N - \tilde{P}_N F^0_{(\tilde{a})} H^0_i \\ &= H^0_i F^0_{(\tilde{a})}\tilde{P}_N - \tilde{P}_N H^0_i F^0_{(\tilde{a})} = \left[H^0_{i(\tilde{a})}, \tilde{P}_N\right],\end{aligned}$$

since H^0_i commutes with $F^0_{(\tilde{a})}$ and hence

$$\left\|\left[H^0_i, \tilde{P}_N\right]\right\|_p = \left\|\left[H^0_{i(\tilde{a})}, \tilde{P}_N\right]\right\|_p \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty \quad \text{for} \quad p > 2 \quad \text{and} \quad i = 1, 2.$$

Therefore given $\epsilon > 0$,

$$\left\|\left[H^0_i, \tilde{P}_N\right]\right\|_p \leq \frac{\epsilon}{2} < \epsilon \quad \text{for sufficiently large} \quad N \quad \text{and for} \quad p > 2 \;\; ; \;\; i = 1, 2. \tag{4.3.8}$$

Again by using (4.3.7), we have for $i = 1, 2$,

$$\begin{aligned}\left[V_i, \tilde{P}_N\right] &= V_i\tilde{P}_N - \tilde{P}_N V_i = V_i F^0_{(\tilde{a})}\tilde{P}_N - \tilde{P}_N F^0_{(\tilde{a})} V_i \\ &= -V_i F^0_{(\tilde{a})}\tilde{P}^{\perp}_N + \tilde{P}^{\perp}_N F^0_{(\tilde{a})} V_i + V_i(F^0_{(\tilde{a})})^{\perp} - (F^0_{(\tilde{a})})^{\perp} V_i\end{aligned}$$

and hence using the properties of Schatten-p ideals (see Chapter 1), we get

$$\begin{aligned}\left\|\left[V_i, \tilde{P}_N\right]\right\|_p &\leq \|V_i F^0_{(\tilde{a})}\tilde{P}^{\perp}_N\|_p + \|\tilde{P}^{\perp}_N F^0_{(a)} V_i\|_p + \|(F^0_{(\tilde{a})})^{\perp} V_i\|_p + \|V_i(F^0_{(a)})^{\perp}\|_p \\ &\leq 2\|V_i F^0_{(\tilde{a})}\tilde{P}^{\perp}_N\|_p + 2\|V_i(F^0_{(\tilde{a})})^{\perp}\|_p \\ &\leq 2\|V_i F^0_{(\tilde{a})}\tilde{P}^{\perp}_N\|^{(1-\frac{2}{p})}\|V_i F^0_{(\tilde{a})}\tilde{P}^{\perp}_N\|_2^{\frac{2}{p}} + 2\|V_i(F^0_{(\tilde{a})})^{\perp}\|^{(1-\frac{2}{p})}\|V_i(F^0_{(\tilde{a})})^{\perp}\|_2^{\frac{2}{p}} \\ &\leq 2\|V_i\|^{(1-\frac{2}{p})}\left(\|V_i F^0_{(\tilde{a})}\tilde{P}^{\perp}_N\|_2^{\frac{2}{p}} + \|V_i(F^0_{(\tilde{a})})^{\perp}\|_2^{\frac{2}{p}}\right),\end{aligned} \tag{4.3.9}$$

provided $p > 2$. Next by using (4.3.4), the right hand side of (4.3.9) is less than or equal to

$$\frac{\epsilon}{4} + 2\|V_i\|^{(1-\frac{2}{p})}\|V_i F^0_{(\tilde{a})}\tilde{P}_N^{\perp}\|_2^{\frac{2}{p}} \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}, \tag{4.3.10}$$

for sufficiently large N (since $\tilde{P}_N \uparrow F^0_{(\tilde{a})}$) and $i = 1, 2$. Now combining (4.3.8) and (4.3.10), we have

$$\left\|\left[H_i, \tilde{P}_N\right]\right\|_p = \left\|\left[H_i^0, \tilde{P}_N\right] + \left[V_i, \tilde{P}_N\right]\right\|_p \leq \left\|\left[H_i^0, \tilde{P}_N\right]\right\|_p + \left\|\left[V_i, \tilde{P}_N\right]\right\|_p \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for sufficiently large N and $p > 2$; $i = 1, 2$, which completes the proof.

□

The following Theorem is the main Theorem in this section.

Theorem 4.3.2. *Let $\underline{H} = (H_1, H_2)$ and $\underline{H}^0 = (H_1^0, H_2^0)$ be two commuting pairs of unbounded self-adjoint operators in an infinite dimensional separable Hilbert space $\mathcal{H}$ such that $H_1 - H_1^0 \equiv V_1 \in \mathcal{B}_2(\mathcal{H})$ and $H_2 - H_2^0 \equiv V_2 \in \mathcal{B}_2(\mathcal{H})$. Then given $\epsilon > 0$, $\exists$ a projection P of finite rank such that*

$$\begin{aligned}\| &\left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0}\right]\left[\mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}^0}\right] \\ &\qquad - P\left[\mathrm{e}^{\mathrm{i}\underline{t}.P\underline{H}P} - \mathrm{e}^{\mathrm{i}\underline{t}.P\underline{H}^0P}\right]P\left[\mathrm{e}^{\mathrm{i}\underline{s}.P\underline{H}P} - \mathrm{e}^{\mathrm{i}\underline{s}.P\underline{H}^0P}\right]P\|_1 \leq \epsilon,\end{aligned}$$

uniformly over $|\underline{t} \equiv (t_1, t_2)|, |\underline{s} \equiv (s_1, s_2)| \leq T$.

Remark 4.3.3. Again, in the statement of the above Theorem 4.3.2, though $\underline{H} = (H_1, H_2)$ and $\underline{H}^0 = (H_1^0, H_2^0)$ are two commuting pairs of self-adjoint operators but $P\underline{H}P = (PH_1P, PH_2P)$ and $P\underline{H}^0P = (PH_1^0P, PH_2^0P)$ may not be a commuting pairs of self-adjoint operators. In that case the meaning of $\mathrm{e}^{\mathrm{i}\underline{t}.P\underline{H}P}$; $\mathrm{e}^{\mathrm{i}\underline{t}.P\underline{H}^0P}$; $\mathrm{e}^{\mathrm{i}\underline{s}.P\underline{H}P}$ and $\mathrm{e}^{\mathrm{i}\underline{s}.P\underline{H}^0P}$ are as follows:

$$\mathrm{e}^{\mathrm{i}\underline{t}.P\underline{H}P} = \mathrm{e}^{\mathrm{i}t_1PH_1P}\mathrm{e}^{\mathrm{i}t_2PH_2P} \quad ; \quad \mathrm{e}^{\mathrm{i}\underline{t}.P\underline{H}^0P} = \mathrm{e}^{\mathrm{i}t_1PH_1^0P}\mathrm{e}^{\mathrm{i}t_2PH_2^0P};$$

$$\mathrm{e}^{\mathrm{i}\underline{s}.P\underline{H}P} = \mathrm{e}^{\mathrm{i}s_1PH_1P}\mathrm{e}^{\mathrm{i}s_2PH_2P} \quad \text{and} \quad \mathrm{e}^{\mathrm{i}\underline{s}.P\underline{H}^0P} = \mathrm{e}^{\mathrm{i}s_1PH_1^0P}\mathrm{e}^{\mathrm{i}s_2PH_2^0P}.$$

Proof of Theorem 4.3.2: Let $E_{H_1}(.), E_{H_2}(.), E_{H_1^0}(.)$ and $E_{H_2^0}(.)$ be the spectral families of the operators H_1, H_2, H_1^0 and H_2^0 respectively. Define $F^0_{(a)} = E_{H_1^0}((-a, a])E_{H_2^0}((-a, a])$ as in the proof of Lemma 4.3.1. Then given $\epsilon > 0$, $\exists\, \tilde{a} \in \mathbb{N}$ such that

$$\begin{aligned}\| &\left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0}\right]\left[\mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}^0}\right] \\ &\qquad - F^0_{(\tilde{a})}\left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0}\right]F^0_{(\tilde{a})}\left[\mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}^0}\right]F^0_{(\tilde{a})}\|_1 \leq \epsilon,\end{aligned} \tag{4.3.11}$$

uniformly over $|\underline{t}|, |\underline{s}| \leq T$, as well as

$$2\|V_i\|^{(1-\frac{2}{p})}\|(F^0_{(\tilde{a})})^{\perp}V_i\|_2^{\frac{2}{p}} \quad \leq \frac{\epsilon}{4} \quad \text{for} \quad i=1,2 \quad \text{and} \quad p>2. \tag{4.3.12}$$

Hence from the proof of Lemma 4.3.1 we conclude that, $\exists$ a sequence $\{\tilde{P}_N\}$ of finite rank projections such that $\tilde{P}_N \uparrow F^0_{(\tilde{a})}$ and

$$\left\|\left[H_i^0, \tilde{P}_N\right]\right\|_p, \left\|\left[H_i, \tilde{P}_N\right]\right\|_p \leq \epsilon \quad \text{for sufficiently large} \quad N; \;\; i=1,2; \;\; p>2. \tag{4.3.13}$$

Therefore choose $N \in \mathbb{N}$ large enough, so that

$$\begin{aligned} \|F^0_{(\tilde{a})}&\left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0}\right] F^0_{(\tilde{a})} \left[\mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}^0}\right] F^0_{(\tilde{a})} \\ &- \tilde{P}_N \left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0}\right] \tilde{P}_N \left[\mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}^0}\right] \tilde{P}_N\|_1 \leq \epsilon, \end{aligned} \tag{4.3.14}$$

uniformly over $|\underline{t}|, |\underline{s}| \leq T$, as well as

$$\left\|\left[H_i^0, \tilde{P}_N\right]\right\|_p, \left\|\left[H_i, \tilde{P}_N\right]\right\|_p \leq \epsilon \quad \text{for} \quad i=1,2 \quad \text{and} \quad p>2. \tag{4.3.15}$$

Now consider the expression

$$\begin{aligned} &\tilde{P}_N \left[\left(\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0}\right) - \left(\mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}\tilde{P}_N} - \mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}^0\tilde{P}_N}\right)\right] \tilde{P}_N \\ &= \tilde{P}_N \left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}\tilde{P}_N}\right] \tilde{P}_N - \tilde{P}_N \left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0} - \mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}^0\tilde{P}_N}\right] \tilde{P}_N. \end{aligned} \tag{4.3.16}$$

But

$$\begin{aligned} \tilde{P}_N &\left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}\tilde{P}_N}\right] \tilde{P}_N = \tilde{P}_N \left[\mathrm{e}^{\mathrm{i}t_1H_1}\mathrm{e}^{\mathrm{i}t_2H_2} - \mathrm{e}^{\mathrm{i}t_1\tilde{P}_NH_1\tilde{P}_N}\mathrm{e}^{\mathrm{i}t_2\tilde{P}_NH_2\tilde{P}_N}\right] \tilde{P}_N \\ &= \tilde{P}_N \left[\mathrm{e}^{it_1H_1} - \mathrm{e}^{it_1\tilde{P}_NH_1\tilde{P}_N}\right] \mathrm{e}^{it_2H_2}\tilde{P}_N + \tilde{P}_N\mathrm{e}^{\mathrm{i}t_1\tilde{P}_NH_1\tilde{P}_N} \left[\mathrm{e}^{\mathrm{i}t_2H_2} - \mathrm{e}^{\mathrm{i}t_2\tilde{P}_NH_2\tilde{P}_N}\right] \tilde{P}_N \\ &= \tilde{P}_N \left[\mathrm{e}^{\mathrm{i}t_1H_1} - \mathrm{e}^{\mathrm{i}t_1\tilde{P}_NH_1\tilde{P}_N}\right] \tilde{P}_N\mathrm{e}^{\mathrm{i}t_2H_2}\tilde{P}_N + \tilde{P}_N \left[\mathrm{e}^{\mathrm{i}t_1H_1} - \mathrm{e}^{\mathrm{i}t_1\tilde{P}_NH_1\tilde{P}_N}\right] \tilde{P}_N^{\perp}\mathrm{e}^{\mathrm{i}t_2H_2}\tilde{P}_N \\ &\qquad + \tilde{P}_N\mathrm{e}^{\mathrm{i}t_1\tilde{P}_NH_1\tilde{P}_N} \left[\mathrm{e}^{\mathrm{i}t_2H_2} - \mathrm{e}^{\mathrm{i}t_2\tilde{P}_NH_2\tilde{P}_N}\right] \tilde{P}_N. \end{aligned} \tag{4.3.17}$$

In the first term of the expression (4.3.17):

$$\begin{aligned} &\left\|\tilde{P}_N \left[\mathrm{e}^{\mathrm{i}t_1H_1} - \mathrm{e}^{\mathrm{i}t_1\tilde{P}_NH_1\tilde{P}_N}\right] \tilde{P}_N\mathrm{e}^{\mathrm{i}t_2H_2}\tilde{P}_N\right\|_{\frac{p}{2}} \\ &\quad = \left\|\tilde{P}_N \left[\int_0^1 d\alpha \, \frac{d}{d\alpha}\left(\mathrm{e}^{\mathrm{i}\alpha t_1H_1}\mathrm{e}^{\mathrm{i}(1-\alpha)t_1\tilde{P}_NH_1\tilde{P}_N}\right)\right] \tilde{P}_N\mathrm{e}^{\mathrm{i}t_2H_2}\tilde{P}_N\right\|_{\frac{p}{2}} \\ &\quad = \left\|\mathrm{i}t_1 \int_0^1 d\alpha \, \tilde{P}_N\mathrm{e}^{\mathrm{i}\alpha t_1H_1}\tilde{P}_N^{\perp}H_1\tilde{P}_N\mathrm{e}^{\mathrm{i}(1-\alpha)t_1\tilde{P}_NH_1\tilde{P}_N}\tilde{P}_N\mathrm{e}^{\mathrm{i}t_2H_2}\tilde{P}_N\right\|_{\frac{p}{2}} \\ &\quad \leq \int_0^1 d\alpha \, |t_1| \left\|\tilde{P}_N\mathrm{e}^{\mathrm{i}\alpha t_1H_1}\tilde{P}_N^{\perp}\right\|_p \left\|\tilde{P}_N^{\perp}H_1\tilde{P}_N\mathrm{e}^{\mathrm{i}(1-\alpha)t_1\tilde{P}_NH_1\tilde{P}_N}\tilde{P}_N\mathrm{e}^{\mathrm{i}t_2H_2}\tilde{P}_N\right\|_p \end{aligned}$$

$$\leq \int_0^1 d\alpha \, |t_1| \left\| \tilde{P}_N e^{i\alpha t_1 H_1} \tilde{P}_N^{\perp} \right\|_p \left\| \tilde{P}_N^{\perp} H_1 \tilde{P}_N \right\|_p , \tag{4.3.18}$$

for $p > 2$.

But on the other hand for $0 \leq \alpha \leq 1$,

$$\begin{aligned}
\tilde{P}_N^{\perp} e^{i\alpha t_1 H_1} \tilde{P}_N &= \tilde{P}_N^{\perp} \left(e^{i\alpha t_1 H_1} - I \right) \tilde{P}_N = \tilde{P}_N^{\perp} \int_0^{\alpha} d\gamma \frac{d}{d\gamma} \left(e^{i\gamma t_1 H_1} \right) \tilde{P}_N \\
&= \int_0^{\alpha} \tilde{P}_N^{\perp} e^{i\gamma t_1 H_1} \left[i t_1 H_1 \right] \tilde{P}_N d\gamma \\
&= i \int_0^{\alpha} \{ \tilde{P}_N^{\perp} e^{i\gamma t_1 \underline{H}} \tilde{P}_N \left[t_1 H_1 \right] \tilde{P}_N + \tilde{P}_N^{\perp} e^{i\gamma t_1 H_1} \tilde{P}_N^{\perp} \tilde{P}_N^{\perp} \left[t_1 H_1 \right] \tilde{P}_N \} d\gamma
\end{aligned}$$

and hence

$$\begin{aligned}
\eta(\alpha) &\equiv \| \tilde{P}_N^{\perp} e^{i\alpha t_1 H_1} \tilde{P}_N \|_p \\
&\leq \int_0^{\alpha} \{ \| \tilde{P}_N^{\perp} e^{i\gamma t_1 H_1} \tilde{P}_N \|_p \| \left[t_1 H_1 \right] \tilde{P}_N \| + \| \tilde{P}_N^{\perp} e^{i\gamma t_1 H_1} \tilde{P}_N^{\perp} \| \| \tilde{P}_N^{\perp} \left[t_1 H_1 \right] \tilde{P}_N \|_p \} d\gamma \\
&\leq |t_1| \int_0^{\alpha} \{ \eta(\gamma) \| H_1 \tilde{P}_N \| + \| \tilde{P}_N^{\perp} H_1 \tilde{P}_N \|_p \} d\gamma,
\end{aligned} \tag{4.3.19}$$

for $p > 2$.

Again

$$\begin{aligned}
\| H_1 \tilde{P}_N \| &= \| H_1^0 \tilde{P}_N + V_1 \tilde{P}_N \| = \| H_1^0 F_{(\tilde{a})}^0 \tilde{P}_N + V_1 \tilde{P}_N \| \\
&\leq \| H_1^0 F_{(\tilde{a})}^0 \tilde{P}_N \| + \| V_1 \tilde{P}_N \| \leq \| H_1^0 F_{(\tilde{a})}^0 \| + \| V_1 \| \\
&\leq 2\tilde{a} + \| V_1 \|
\end{aligned} \tag{4.3.20}$$

and

$$\| \tilde{P}_N^{\perp} H_1 \tilde{P}_N \|_p = \left\| \left[H_1, \tilde{P}_N \right] \tilde{P}_N \right\|_p \leq \epsilon, \tag{4.3.21}$$

for sufficiently large N and $p > 2$. Therefore combining (4.3.19) ; (4.3.20) and (4.3.21), we have

$$\eta(\alpha) \leq \left(2\tilde{a} + \| V_1 \| \right) |t_1| \int_0^{\alpha} \eta(\gamma) d\gamma + |t_1| \epsilon \alpha \leq \left(2\tilde{a} + \| V_1 \| \right) T \int_0^{\alpha} \eta(\gamma) d\gamma + T\epsilon \tag{4.3.22}$$

for sufficently large N and $|t_1| \leq T$. We can solve this Gronwall-type inequality (4.3.22) to conclude that

$$\eta(\alpha) \equiv \| \tilde{P}_N^{\perp} e^{i\alpha t_1 H_1} \tilde{P}_N \|_p \leq T\epsilon e^{(2\tilde{a} + \| V_1 \|) \, T\alpha} \leq T\epsilon e^{(2\tilde{a} + \| V_1 \|) \, T} \tag{4.3.23}$$

for sufficiently large N and $p > 2$. Therefore combining (4.3.18) and (4.3.23), we get

$$\left\|\tilde{P}_N\left[\mathrm{e}^{\mathrm{i}t_1 H_1} - \mathrm{e}^{\mathrm{i}t_1 \tilde{P}_N H_1 \tilde{P}_N}\right]\tilde{P}_N \mathrm{e}^{\mathrm{i}t_2 H_2}\tilde{P}_N\right\|_{\frac{p}{2}} \leq T^2\epsilon^2 \mathrm{e}^{(2\tilde{a}+\|V_1\|)\,T}, \tag{4.3.24}$$

for sufficiently large N and $p > 2$ and $|\underline{t}| \leq T$. Similarly by repeating the same above calculations for the third in (4.3.17), we note that

$$\left\|\tilde{P}_N \mathrm{e}^{\mathrm{i}t_1 \tilde{P}_N H_1 \tilde{P}_N}\left[\mathrm{e}^{\mathrm{i}t_2 H_2} - \mathrm{e}^{\mathrm{i}t_2 \tilde{P}_N H_2 \tilde{P}_N}\right]\tilde{P}_N\right\|_{\frac{p}{2}} \leq T^2\epsilon^2 \mathrm{e}^{(2\tilde{a}+\|V_2\|)\,T}, \tag{4.3.25}$$

for sufficiently large N and $p > 2$ and $|\underline{t}| \leq T$. For the second term in (4.3.17) we have

$$\begin{aligned}
\left\|\tilde{P}_N\left[\mathrm{e}^{\mathrm{i}t_1 H_1} - \mathrm{e}^{\mathrm{i}t_1 \tilde{P}_N H_1 \tilde{P}_N}\right]\tilde{P}_N^{\perp} \mathrm{e}^{\mathrm{i}t_2 H_2}\tilde{P}_N\right\|_{\frac{p}{2}} &= \left\|\tilde{P}_N \mathrm{e}^{\mathrm{i}t_1 H_1}\tilde{P}_N^{\perp} \mathrm{e}^{\mathrm{i}t_2 H_2}\tilde{P}_N\right\|_{\frac{p}{2}} \\
&\leq \left\|\tilde{P}_N \mathrm{e}^{\mathrm{i}t_1 H_1}\tilde{P}_N^{\perp}\right\|_p \left\|\tilde{P}_N^{\perp} \mathrm{e}^{\mathrm{i}t_2 H_2}\tilde{P}_N\right\|_p \leq T^2\epsilon^2 \mathrm{e}^{(2\tilde{a}+\|V_1\|+\|V_2\|)\,T},
\end{aligned} \tag{4.3.26}$$

for sufficiently large N and $p > 2$ and $|\underline{t}| \leq T$. Finally combining (4.3.17) ; (4.3.24) ; (4.3.25) and (4.3.26), we conclude that

$$\left\|\tilde{P}_N\left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N \underline{H}\tilde{P}_N}\right]\tilde{P}_N\right\|_{\frac{p}{2}} \leq T^2\epsilon^2\left(\mathrm{e}^{(2\tilde{a}+\|V_1\|)\,T} + \mathrm{e}^{(2\tilde{a}+\|V_2\|)\,T} + \mathrm{e}^{(2\tilde{a}+\|V_1\|+\|V_2\|)\,T}\right), \tag{4.3.27}$$

for sufficiently large N and $p > 2$ and $|\underline{t}| \leq T$. Again by the similar calculations we conclude that

$$\left\|\tilde{P}_N\left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0} - \mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N \underline{H}^0\tilde{P}_N}\right]\tilde{P}_N\right\|_{\frac{p}{2}} \leq T^2\epsilon^2\left(\mathrm{e}^{2\tilde{a}\,T} + \mathrm{e}^{2\tilde{a}\,T} + \mathrm{e}^{4\tilde{a}\,T}\right), \tag{4.3.28}$$

for sufficiently large N and $p > 2$ and $|\underline{t}| \leq T$. Therefore by using the properties of Schatten-p ideals (see Chapter 1), we get

$$\begin{aligned}
&\|\tilde{P}_N\left[\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0}\right]\tilde{P}_N\left[\mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}^0}\right]\tilde{P}_N \\
&\qquad - \tilde{P}_N\left[\mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}\tilde{P}_N} - \mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}^0\tilde{P}_N}\right]\tilde{P}_N\left[\mathrm{e}^{\mathrm{i}\underline{s}.\tilde{P}_N\underline{H}\tilde{P}_N} - \mathrm{e}^{\mathrm{i}\underline{s}.\tilde{P}_N\underline{H}^0\tilde{P}_N}\right]\tilde{P}_N\|_1 \\
&\leq \left\|\tilde{P}_N\left[\left(\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0}\right) - \left(\mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}\tilde{P}_N} - \mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}^0\tilde{P}_N}\right)\right]\tilde{P}_N\right\|_2 \\
&\qquad\qquad \left\|\tilde{P}_N\left[\mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}^0}\right]\tilde{P}_N\right\|_2 \\
&\quad + \left\|\tilde{P}_N\left[\mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}\tilde{P}_N} - \mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}^0\tilde{P}_N}\right]\tilde{P}_N\right\|_2 \\
&\qquad\qquad \left\|\tilde{P}_N\left[\left(\mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}^0}\right) - \left(\mathrm{e}^{\mathrm{i}\underline{s}.\tilde{P}_N\underline{H}\tilde{P}_N} - \mathrm{e}^{\mathrm{i}\underline{s}.\tilde{P}_N\underline{H}^0\tilde{P}_N}\right)\right]\tilde{P}_N\right\|_2 \\
&\leq (\|V_1\|_2 + \|V_2\|_2)\left\|\tilde{P}_N\left[\left(\mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{t}.\underline{H}^0}\right) - \left(\mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}\tilde{P}_N} - \mathrm{e}^{\mathrm{i}\underline{t}.\tilde{P}_N\underline{H}^0\tilde{P}_N}\right)\right]\tilde{P}_N\right\|_2 \\
&\quad + (\|V_1\|_2 + \|V_2\|_2)\left\|\tilde{P}_N\left[\left(\mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}} - \mathrm{e}^{\mathrm{i}\underline{s}.\underline{H}^0}\right) - \left(\mathrm{e}^{\mathrm{i}\underline{s}.\tilde{P}_N\underline{H}\tilde{P}_N} - \mathrm{e}^{\mathrm{i}\underline{s}.\tilde{P}_N\underline{H}^0\tilde{P}_N}\right)\right]\tilde{P}_N\right\|_2
\end{aligned}$$

$$\leq (\|V_1\|_2 + \|V_2\|_2) 4^{(1-\frac{p}{4})} \left\| \tilde{P}_N \left[\left(e^{i\underline{t}.\underline{H}} - e^{i\underline{t}.\underline{H}^0} \right) - \left(e^{i\underline{t}.\tilde{P}_N \underline{H} \tilde{P}_N} - e^{i\underline{t}.\tilde{P}_N \underline{H}^0 \tilde{P}_N} \right) \right] \tilde{P}_N \right\|_{p/2}^{p/4}$$
$$+ (\|V_1\|_2 + \|V_2\|_2) 4^{(1-\frac{p}{4})} \left\| \tilde{P}_N \left[\left(e^{i\underline{s}.\underline{H}} - e^{i\underline{s}.\underline{H}^0} \right) - \left(e^{i\underline{s}.\tilde{P}_N \underline{H} \tilde{P}_N} - e^{i\underline{s}.\tilde{P}_N \underline{H}^0 \tilde{P}_N} \right) \right] \tilde{P}_N \right\|_{p/2}^{p/4}, \tag{4.3.29}$$

provided $\frac{p}{2} \leq 2$. Now combining (4.3.27) ; (4.3.28) and (4.3.29), we have

$$\begin{aligned} &\|\tilde{P}_N \left[e^{i\underline{t}.\underline{H}} - e^{i\underline{t}.\underline{H}^0} \right] \tilde{P}_N \left[e^{i\underline{s}.\underline{H}} - e^{i\underline{s}.\underline{H}^0} \right] \tilde{P}_N \\ &\qquad - \tilde{P}_N \left[e^{i\underline{t}.\tilde{P}_N \underline{H} \tilde{P}_N} - e^{i\underline{t}.\tilde{P}_N \underline{H}^0 \tilde{P}_N} \right] \tilde{P}_N \left[e^{i\underline{s}.\tilde{P}_N \underline{H} \tilde{P}_N} - e^{i\underline{s}.\tilde{P}_N \underline{H}^0 \tilde{P}_N} \right] \tilde{P}_N \|_1 \\ &\leq 2 \, (\|V_1\|_2 + \|V_2\|_2) 4^{(1-\frac{p}{4})} T^2 \epsilon^2 \\ &\qquad \left[\left(e^{(2\tilde{a}+\|V_1\|)\,T} + e^{(2\tilde{a}+\|V_2\|)\,T} + e^{(2\tilde{a}+\|V_1\|+\|V_2\|)\,T} \right) + \left(e^{2\tilde{a}\,T} + e^{2\tilde{a}\,T} + e^{4\tilde{a}\,T} \right) \right]^{\frac{p}{4}} \end{aligned} \tag{4.3.30}$$

for sufficiently large N and $2 < p \leq 4$ and $|\underline{t}|, |\underline{s}| \leq T$. Finally using (4.3.11) ; (4.3.14) and (4.3.30), we conclude that

$$\begin{aligned} &\| \left[e^{i\underline{t}.\underline{H}} - e^{i\underline{t}.\underline{H}^0} \right] \left[e^{i\underline{s}.\underline{H}} - e^{i\underline{s}.\underline{H}^0} \right] \\ &\qquad - \tilde{P}_N \left[e^{i\underline{t}.\tilde{P}_N \underline{H} \tilde{P}_N} - e^{i\underline{t}.\tilde{P}_N \underline{H}^0 \tilde{P}_N} \right] \tilde{P}_N \left[e^{i\underline{s}.\tilde{P}_N \underline{H} \tilde{P}_N} - e^{i\underline{s}.\tilde{P}_N \underline{H}^0 \tilde{P}_N} \right] \tilde{P}_N \|_1 \\ &\leq \| \left[e^{i\underline{t}.\underline{H}} - e^{i\underline{t}.\underline{H}^0} \right] \left[e^{i\underline{s}.\underline{H}} - e^{i\underline{s}.\underline{H}^0} \right] \\ &\qquad - F^0_{(\tilde{a})} \left[e^{i\underline{t}.\underline{H}} - e^{i\underline{t}.\underline{H}^0} \right] F^0_{(\tilde{a})} \left[e^{i\underline{s}.\underline{H}} - e^{i\underline{s}.\underline{H}^0} \right] F^0_{(\tilde{a})} \|_1 \\ &\quad + \| F^0_{(\tilde{a})} \left[e^{i\underline{t}.\underline{H}} - e^{i\underline{t}.\underline{H}^0} \right] F^0_{(\tilde{a})} \left[e^{i\underline{s}.\underline{H}} - e^{i\underline{s}.\underline{H}^0} \right] F^0_{(\tilde{a})} \\ &\qquad - \tilde{P}_N \left[e^{i\underline{t}.\underline{H}} - e^{i\underline{t}.\underline{H}^0} \right] \tilde{P}_N \left[e^{i\underline{s}.\underline{H}} - e^{i\underline{s}.\underline{H}^0} \right] \tilde{P}_N \|_1 \\ &\quad + \| \tilde{P}_N \left[e^{i\underline{t}.\underline{H}} - e^{i\underline{t}.\underline{H}^0} \right] \tilde{P}_N \left[e^{i\underline{s}.\underline{H}} - e^{i\underline{s}.\underline{H}^0} \right] \tilde{P}_N \\ &\qquad - \tilde{P}_N \left[e^{i\underline{t}.\tilde{P}_N \underline{H} \tilde{P}_N} - e^{i\underline{t}.\tilde{P}_N \underline{H}^0 \tilde{P}_N} \right] \tilde{P}_N \left[e^{i\underline{s}.\tilde{P}_N \underline{H} \tilde{P}_N} - e^{i\underline{s}.\tilde{P}_N \underline{H}^0 \tilde{P}_N} \right] \tilde{P}_N \|_1 \\ &\leq \epsilon \; + \epsilon \; + 2 \, (\|V_1\|_2 + \|V_2\|_2) 4^{(1-\frac{p}{4})} T^2 \epsilon^2 \\ &\qquad \left[\left(e^{(2\tilde{a}+\|V_1\|)\,T} + e^{(2\tilde{a}+\|V_2\|)\,T} + e^{(2\tilde{a}+\|V_1\|+\|V_2\|)\,T} \right) + \left(e^{2\tilde{a}\,T} + e^{2\tilde{a}\,T} + e^{4\tilde{a}\,T} \right) \right]^{\frac{p}{4}}, \end{aligned} \tag{4.3.31}$$

for sufficiently large N and $2 < p \leq 4$ and uniformly over $|\underline{t}|$, $|\underline{s}| \leq T$. Therefore the result follows by choosing N sufficiently large. Hence the proof.

□

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