Kinetic Theory and Burnett Order Constitutive Relations for a Smooth Granular Gas

A Thesis
Submitted for the Degree of
MASTER OF SCIENCE (ENGINEERING)

by

VINAY KUMAR GUPTA

ENGINEERING MECHANICS UNIT
JAWAHARLAL NEHRU CENTRE FOR ADVANCED SCIENTIFIC RESEARCH
(A Deemed University)
Bangalore – 560 064

MAY 2011

To Dr. Premananda Bera

&

Prof. Isaac Golhirsch

Acknowledgements

Many people accompanied me through this work, during good and bad times. First and foremost it is an honour for me to express my sincere gratitude to my research supervisor, *Prof. Meheboob Alam*. I thank him for his enlightening teaching and guidance, untiring interest, patience and perhaps faith in me that I would handle the problem. When I was not able to evaluate the kinetic integrals, he always encouraged me by saying, “Only you can do it!”. His deep knowledge in Fluid Mechanics, Granular Flows, Stability Analysis, Simulations and Mathematics was always a plus point. I am fortunate to have him as my research advisor.

I express my sincere thanks to *Dr. Premananda Bera* (IIT Roorkee) for his continuous motivation throughout my research. I believe that I could come back to academics and join JNCASR because of him only.

I express my deep gratitude to my parents for their blessings and continuous support. I owe them more than I could possibly express.

I thank *Prof. Govindan Rangarajan* (IISc), *Prof. Rama Govindarajan* and all the faculty members of EMU for their valuable teachings and inspiration. I especially thank *Dr. Ganesh Subramanian* and *Dr. Santosh Ansumali* for very fruitful discussions on self-adjoint property of linearized Boltzmann operator, their comments and suggestions on my work. I thank *Prof. Aloknath Chakrabarti* (IISc) for his suggestions, which helped me in deriving the Fredholm-form of kinetic integrals.

I am thankful to my senior, *Sarvesh* sir (IITD) for his scolding. Due to that, I started preparing for GATE exam and at the end, it was the only option for me to join research. I thank *Vikas* (IITB) for his help and great support, when I was frustrated from software job and wanted to come back in research.

I thank *Satya* sir (ISRO), *Priyanka* madam and *Anubhab* dada for a lot of help and motivation.

I thank *Ujjayan* for his valuable suggestions, motivation, and discussions (academic as well as non-academic). Probably, without his help in evaluating kinetic integrals, I could not complete this work.

I thank my lab mates: *Priyanka* madam, *Satya* sir, *Ashish* bhaiya, *Snehlatha*, *Pascal*, *Ujjayan*, *Arghya*, *Reddy*, *Rohan*, *Ansari*, *Sunil* and *Ramakrishna* for their help and creating a very good friendly working environment in the lab.

I thank *Arup* (TSU), *Ratul* bhaiya, *Sumesh*, *Harish*, *Vivekanand*, *Vivek*, *Dinesh* bhaiya, *Dhiraj* bhaiya, *Rahul* bhaiya, *Ponnu*, *Srikanth* and all other friends of EMU for their help in various ways.

I thank *Ram Keval* (IITR), *Jagbandhu* (IITK), *Nishant* (RRI), *Rajesh* (IISc), *Alok* (IISc), *Asha* madam (Italy), *Dinesh* bhaiya (Seoul) and *Abhisek* (USA) for providing me the papers, which were not accessible in JNC.

I am very thankful to BARC for the financial support from 01.11.2010 upto 30.04.2011 under the debit head BARC/MA/4216.

I am ever grateful to my institute, **JNCASR** for providing the necessary infrastructure and financial support.

This thesis is dedicated to *Dr. Premananda Bera* (IIT Roorkee) and in the memory of *Prof. Isaac Goldhirsch*.

Abstract

The work of Sela & Goldhirsch (1998) is generalized by including a body force (gravity) term in the Boltzmann equation. The constitutive relations: stress tensor, heat flux and collisional dissipation, have been derived till Burnett order (up to second-order in small parameters) for a smooth granular gas. To derive these relations, the approach of Sela & Goldhirsch (1998) is followed. The pertinent Boltzmann equation is perturbatively solved by performing the (generalized) Chapman-Enskog expansion: an expansion in both the small parameters, "Knudsen number", K and "degree of inelasticity", $\epsilon = 1 - e^2$, where $0 \leqslant e \leqslant 1$ is the coefficient of restitution. The expansion used in this work is similar to that used in Sela & Goldhirsch (1998) but is in different form to avoid any confusion. The inclusion of body force term in the Boltzmann equation has not affected the results and thus the constitutive relations obtained in this work are same as that in Sela & Goldhirsch (1998) except for few minor corrections in the expression of heat flux at Burnett order. The expression for heat flux at Burnett order derived in this work matches with that in Chapman & Cowling (1970). The contribution of $O(\epsilon^3)$ correction to collisional dissipation has also been derived, while it is ignored in Sela & Goldhirsch (1998). Using these constitutive relations, for a 2-D Poiseuille flow of smooth inelastic hard disks under gravity, it is shown that the normal stress difference is a Burnett-order effect which agrees with Tij & Santos (2004).

Contents

Chapter 1

Introduction

1.1 Granular Matter

A *granular material* is a conglomeration of discrete macroscopic (>1 mm in size) solid particles or grains. Granular materials are commonplace in nature and industry. According to Richard *et al.* (2005), "Granular materials are ubiquitous in nature and are the second-most manipulated material in industry (the first one is water)". They occur in all shapes and sizes ranging from powders, sand, cement, food grains, seeds, sugar, capsules and pills, gravels, fertilizers to large chunks of coal, sand dunes, planetary rings, asteroids, etc. Figure 1.1 shows few examples of granular materials. Many industries rely on handling, conveying and storage of grains.

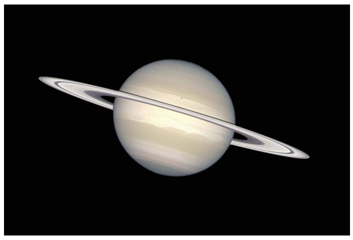
(a) Food grains

(b) Capsules and pills

(c) Gravels

(d) Coal

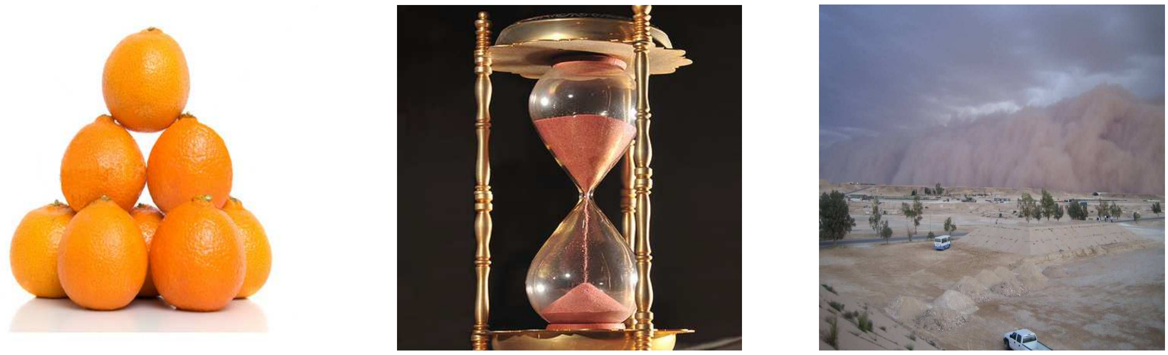
(e) Sand dunes



(f) Saturn ring

Figure 1.1: Few examples of granular materials.

Granular materials may stay at rest like a solid, flow like a liquid, or behave like a gas, depending on the rate of energy input (see figure 1.2). Of course, they can behave differently than these states too. All these states of granular materials are discussed in detail in the review article by Jaeger *et al.* (1996). In addition to practical and natural importance, they exhibit many interesting phenomena: jamming (Corwin *et al.* 2005), clustering (Kudrolli *et al.* 1997), mixing and segregation (Ottino & Khakhar 2000), Brazil nut and reverse Brazil nut phenomena (Trujillo *et al.* 2003; Alam *et al.* 2006), pattern formation (Umbanhowar *et al.* 1996; Alam *et al.* 2009), granular Leidenfrost effect (Eshuis *et al.* 2010), etc. Despite their great importance and applications, the mechanics of granular materials is not well understood at present. Nevertheless, some significant progress has been made during last few decades.



(a) (b) (c)

Figure 1.2: Behaviour of granular materials: (a) like a solid (heap of oranges), (b) like a liquid (flow of sand in an hour-glass), and (c) like a gas (dust storm).

A material is called *dry* granular material if the interstitial fluid between the grains is a gas, which is usually air. In dry granular media, the effects of interstitial fluid are negligible and the dominant interactions are inelastic collisions and friction, which are short-range and non-cohesive. On the other hand, if the voids between grains are completely filled with a liquid, e.g., water, the material is called *saturated* granular material. If some of the voids are filled with a liquid and rest of them are filled with a gas, the material is called *partially saturated*. Both saturated and partially saturated materials are called *wet* granular materials. In wet granular media, the interactions are cohesive due to surface tension. For the dynamics of wet granular materials, see Herminghaus (2005). Although, in the real world, we often see the wet granular materials, e.g., beach sand, the present work is confined to dry granular media and due to the mathematical complicacy, similar work for wet granular materials is left for future.

1.2 Rapid Granular Flows and Kinetic Theory

Depending on the interaction between the grains, the flow of granular materials can be classified into two regimes: (*i*) *slow* flow regime and (*ii*) *rapid* flow regime. In the slow flow regime the particles are densely packed, i.e., the volume fraction is high; the contacts between particles occur via sliding and rolling motion of the particles, and the particles stay in contact for a long period of time—therefore the particles move in the form of blocks of several particles relative to each other; the momentum and energy transfer usually occur due to frictional forces. On the other hand, in rapid flow regime, the volume fraction is relatively small; each particle moves randomly and the contacts between them occur via *instantaneous* collisions[†]—therefore collisions between the particles can be considered as *binary*; the momentum and energy transfer take place due to the instantaneous binary collisions between particles (see Campbell 1990).

A rapid granular flow resembles the classical picture of molecular gas and therefore fluidised state of grains is termed as "granular gas" (Goldhirsch & Zanetti 1993). Due to this resemblance, one can think that the methods used in the domain of statistical mechanics of molecular gases may be useful in analysing the rapid granular flows. Despite the similarity of a granular gas to

[†]By "instantaneous" collisions, we mean that the time of contact between particles during collisions is much smaller than the mean free time.

a (classical) molecular gas, there are two significant differences between them: (*i*) the grains are of macroscopic sizes and (*ii*) the collisions between grains are inelastic (for detailed comparison between granular gases and molecular gases, see Haff 1983). The effects of these differences are given below.

Due to the macroscopic size of the grains, the energy required for the movement of the grains is much higher than the thermal energy ($\sim k_B T$[‡], where k_B is the Boltzmann constant and T is the thermodynamic temperature). Therefore, to set the particles of a granular system into motion, the energy must be supplied by some external forcing, e.g., shearing the particles between plates, vibrating the bed of granular particles, etc. (Jaeger *et al.* 1996). Hence (ordinary) thermodynamic temperature T plays no role for the granular flows. Nevertheless, in anology with thermodynamic temperature and thermal energy of a molecular gas, (for a monodisperse collection of grains) one can define the *granular temperature* from *fluctuating kinetic energy* (sometimes referred to as *pseudothermal energy*) by (see e.g., Brilliantov & Pöschel 2004)

$$\frac{\dim}{2}\,\Theta = \left\langle \frac{1}{2} m \boldsymbol{u}^2 \right\rangle, \tag{1.1}$$

where dim is the dimension of the problem (dim = 2 for disks and dim = 3 for spheres), Θ is granular temperature, m is the mass of a particle, $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{V}$ is the fluctuating velocity of an individual particle, $\boldsymbol{v}$ is the instantaneous velocity of that particle, $\boldsymbol{V}$ is the mean (flow) velocity and $\langle\ \rangle$ represents an appropriate average (defined in §2.2).

Since collisions between the particles of granular materials are inelastic in nature, the energy associated with granular temperature continuously dissipates by collisions between the particles. That means if the system is left as it is, the granular temperature will drop to zero and the system will go to a dead state (i.e, all the particles will be at rest) after some finite time. It can easily be seen in a simple experiment—if one shakes a box containg particles (say, rice grains) and then stops shaking, the particles will almost instantaneously cease all their motion and come to rest. Hence to maintain a steady motion (or to maintain a steady granular temperature) of the particles, energy must be injected continuously into the system so that the energy lost due to dissipative collisions can be balanced with this injected energy. Therefore the only equilibrium state for a granular material is a dead state (until external energy is not fed into the system).

In the rapid flow regime, the qualitative properties of a granular gas can be described by the Boltzmann equation (suitably modified by taking inelastic collisions into account). One important objective in kinetic theory of granular gases is to derive the hydrodynamic equations from the (inelastic) Boltzmann equation. As we shall see latter (in §2.2) that pressure tensor, heat flux and collisional dissipation appear in the hydrodynamic equations in form of ensemble averages, therefore in order to complete the hydrodynamic description, *constitutive relations* through which pressure tensor (or stress tensor), heat flux and collisional dissipation are expressed in terms of the hydrodynamic fields (number density or mass density, mean flow velocity and granular temperature) need to be established. There are many methods to derive constitutive relations from the Boltzmann equation. Few commonly used methods are as follows:

[‡]The same order of energy is sufficient to set the molecules in (Brownian) motion in a molecular gas.

1. **Chapman-Enskog expansion**: The Chapman-Enskog expansion of the Boltzmann equation is the most successful systematic method for obtaining the distribution function (Goldhirsch 2003). The key idea of this method is that the time dependence of the distribution function can come only through the hydrodynamic fields. The details of this method will be explained in the next chapter. Several authors, e.g., Chapman & Cowling (1970), Goldshtein & Shapiro (1995), Sela *et al.* (1996), Sela & Goldhirsch (1998), etc. have used this method.

2. **Grad's method of moment** (Grad 1949): In this method the distribution function is approximated by a Maxwellian distribution function times a polynomial in the components of the fluctuating velocity. This approximation is substituted back into the Boltzmann equation and this substitution leads to equations of motion for the coefficients of these polynomials. The latter are then solved by using certain assumptions (e.g., a steady state). Kogan (1969), Jenkins & Richman (1985*a*), Ramírez *et al.* (2000), etc. have used this method.

3. **Bhatnagar-Gross-Krook (BGK) model for the Boltzmann collision operator** (Bhatnagar *et al.* 1954): The main complexity in dealing with the Boltzmann equation arises due to collision term (referred to as the Boltzmann collision operator). For moderate calculation effort, the Boltzmann collision term is replaced by a BGK model. Santos *et al.* (1998), Ansumali *et al.* (2003), Tij & Santos (2004), etc. have used this model.

1.3 Present Work and Organisation of Thesis

1.3.1 Present Work

In present work, the rapid flow of monodisperse collection of smooth inelastic hard spheres of diameter d under gravitational force in three dimensions is considered. The mass m of the particle is normalized to unity. Since the particles have been considered smooth in this work, the angular velocity of each particle (about its own axes) remains unchanged, no matter what other changes the system undergoes. Hence the angular velocities are irrelevant in the present analysis. Also, the particles have been considered as hard spheres so that the coefficient of (normal) restitution would be taken as a constant (note that for particles, which are not hard, coefficient of restitution is a function of impact velocity (Brilliantov & Pöschel 2004)). Following Sela & Goldhirsch (1998), the factor 1/3 is removed from the definition of granular temperature (cf. eq. (1.1))—of course, it does not affect the analysis. Considering the above arguments, the granular temperature in present study is given by $\Theta = \langle \boldsymbol{u}^2 \rangle$. The main goal of the present work is to derive the constitutive relations for the above mentioned problem by following a (generalized) Chapman-Enskog expansion of the Boltzmann equation. For this purpose, mostly, the approach of Sela & Goldhirsch (1998) is followed.

1.3.2 Organisation of Thesis

Chapter 2

The chapter starts with collision rule for binary collision of two smooth inelastic spheres. After this, the Boltzmann equation for granular flows and the equations of motion for the hydrodynamic fields are briefly introduced. The constitutive relations to zeroth-order are derived. The generalized version of Chapman-Enskog expansion method is described in detail. Using generalized Chapman-Enskog expansion, the method to obtain higher-order constitutive relation is explained.

Chapter 3

With the help of (perturbed) Boltzmann equation, the correction terms at first-order in small parameters are obtained. Using these corrections to distribution function, the constitutive relations at first-order in small parameters are derived.

Chapter 4

In this chapter, it is shown that the linearized Boltzmann collision operator is self-adjoint. With the help of this self-adjoint property of the linearized Boltzmann collision operator, the constitutive relations at second-order in small parameters are derived without evaluating the correction terms at corresponding orders.

Chapter 5

The results obtained are summarized in this chapter. Normal stress difference in 2-D granular Poiseuille flow of smooth inelastic hard spheres is discussed at Euler, Navier-Stokes and Burnett order and it has been concluded that normal stress difference is Burnett order effect for this problem. After this, the possible future work is mentioned.

Chapter 2

Boltzmann Equation and Method of Solution

As discussed in §1.3.1, in the present problem, the inelastic collisions between the spheres are characterized by constant coefficient of restitution e, where $0 \leqslant e \leqslant 1$ with $e = 0$ and $e = 1$ for the collision between perfectly inelastic and perfectly elastic spheres respectively. The binary collision between spheres labeled 1 and 2 leads to the following velocity transformation:

$$\boldsymbol{v}_1 = \boldsymbol{v}_1' - \frac{(1+e)}{2}(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}')\hat{\boldsymbol{k}}, \tag{2.1a}$$

$$\boldsymbol{v}_2 = \boldsymbol{v}_2' + \frac{(1+e)}{2}(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}')\hat{\boldsymbol{k}}, \tag{2.1b}$$

where $\boldsymbol{v}_1'$ and $\boldsymbol{v}_2'$ are the velocities of the spheres 1 and 2 respectively, prior to the collision, and $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the corresponding velocities of the spheres 1 and 2 respectively, after the collision, $\boldsymbol{v}_{12}' \equiv \boldsymbol{v}_1' - \boldsymbol{v}_2'$, and $\hat{\boldsymbol{k}}$ is a unit vector pointing from centre of sphere 2 to that of sphere 1 at the time of contact. It does not matter if $\hat{\boldsymbol{k}}$ is defined from centre of sphere 1 to that of sphere 2 (Goldhirsch 2003).

Substracting eq. (2.1b) from eq. (2.1a) and taking the dot product of the resulting equation with $\hat{\boldsymbol{k}}$, we get

$$\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12} = \hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}' - (1+e)(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}')$$

or

$$\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12} = -e(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}'). \tag{2.2}$$

This equation shows the relation between the normal component of relative velocities before and after the collision. Also, on squaring and adding eqs. (2.1a) and (2.1b), we get

$$v_1^2 + v_2^2 = v_1'^2 + v_2'^2 + \frac{(1+e)^2}{2}(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}')^2 - (1+e)(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}')^2$$

or

$$v_1'^2 + v_2'^2 = v_1^2 + v_2^2 - (1+e)(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}')^2 \left\{ \frac{(1+e)}{2} - 1 \right\} = v_1^2 + v_2^2 + \frac{(1-e^2)}{2}(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}')^2.$$

Using eq. (2.2),

$$v_1'^2 + v_2'^2 = v_1^2 + v_2^2 + \frac{(1-e^2)}{2e^2}(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12})^2.$$

Let us define a small parameter "degree of inelasticity (ϵ)" by $\epsilon = 1 - e^2$ and in terms of ϵ, above equation can be written as

$$v_1'^2 + v_2'^2 = v_1^2 + v_2^2 + \frac{\epsilon}{2(1-\epsilon)}(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12})^2$$
$$= v_1^2 + v_2^2 + \frac{1}{2}\left(\epsilon + \epsilon^2 + \epsilon^3 + \ldots\right)(\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12})^2. \tag{2.3}$$

This equation will be used in interchanging the total kinetic energy of the particles before collision into the total kinetic energy of the particles after collision and vice-versa.

2.1 Boltzmann Equation

As discussed in §1.2, the qualitative properties of the system (assuming it is dilute enough) can be described by the (inelastic) Boltzmann equation (see Brilliantov & Pöschel 2004; Rao & Nott 2008, for derivation of the Boltzmann equation):

$$\frac{\partial f}{\partial t} + \boldsymbol{v}_1 \cdot \boldsymbol{\nabla} f + \boldsymbol{g} \cdot \boldsymbol{\nabla}_v f = d^2 \int_{\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12} > 0} \mathrm{d}\boldsymbol{v}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12}) \left(\frac{1}{e^2} f\left(\boldsymbol{v}_1'\right) f\left(\boldsymbol{v}_2'\right) - f\left(\boldsymbol{v}_1\right) f\left(\boldsymbol{v}_2\right) \right), \tag{2.4}$$

where $f \equiv f(\boldsymbol{v}, \boldsymbol{r}, t)$ is the single-particle distribution function defined in such a way that $f(\boldsymbol{v}, \boldsymbol{r}, t)\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}$ gives the number of particles at time t in an infinitesimal volume $\mathrm{d}\boldsymbol{r}$ located at $\boldsymbol{r}$ whose velocities belong to an infinitesimal volume in velocity space $\mathrm{d}\boldsymbol{v}$ centred around $\boldsymbol{v}$, $\boldsymbol{\nabla}$ is the gradient with respect to the spatial coordinate $\boldsymbol{r}$ and $\boldsymbol{\nabla}_v$ is the gradient with respect to the velocity space coordinate $\boldsymbol{v}$. The other variables are defined in eqs. (2.1a), (2.1b) and the text following them. The constraint $\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{12} > 0$ takes care for the impending collisions. Eq. (2.4) is defined in such a way that **precollisional velocities depend on coefficient of restitution, while postcollisional velocities do not depend on the same**. Note that the right-hand side of eq. (2.4) depends on coefficient of restitution explicitly as well as implicitly via relations (2.1) between the postcollisional and precollisional velocities.

2.2 Hydrodynamic Variables and the Equations of Motion

The hydrodynamic fields: the number density $n(\boldsymbol{r}, t)$, the mean flow velocity $\boldsymbol{V}(\boldsymbol{r}, t)$, and the granular temperature $\Theta(\boldsymbol{r}, t)$ are given by (Grad 1949; Chapman & Cowling 1970; Tij & Santos 2004; Rao & Nott 2008):

$$n(\boldsymbol{r}, t) \equiv \int \mathrm{d}\boldsymbol{v}\, f(\boldsymbol{v}, \boldsymbol{r}, t), \tag{2.5}$$

$$\boldsymbol{V}(\boldsymbol{r}, t) \equiv \frac{1}{n} \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f(\boldsymbol{v}, \boldsymbol{r}, t), \tag{2.6}$$

$$\Theta(\boldsymbol{r}, t) \equiv \frac{1}{n} \int \mathrm{d}\boldsymbol{v}\, (\boldsymbol{v} - \boldsymbol{V})^2 f(\boldsymbol{v}, \boldsymbol{r}, t). \tag{2.7}$$

The macroscopic balance equations for these hydrodynamic fields can be derived by multiplying the Boltzmann equation (eq. (2.4)) by 1, $\boldsymbol{v}_1$ and v_1^2 respectively, and integrating over $\boldsymbol{v}_1$ (Lun *et al.* 1984; Jenkins & Richman 1985*a*,*b*; Rao & Nott 2008). They are:

$$\frac{\mathrm{D}n}{\mathrm{D}t} + n\frac{\partial V_i}{\partial r_i} = 0, \tag{2.8}$$

$$n\frac{\mathrm{D}V_i}{\mathrm{D}t} + \frac{\partial P_{ij}}{\partial r_j} = ng_i, \tag{2.9}$$

$$n\frac{\mathrm{D}\Theta}{\mathrm{D}t} + 2\frac{\partial V_i}{\partial r_j}P_{ij} + 2\frac{\partial Q_i}{\partial r_i} = -n\Gamma, \tag{2.10}$$

where

$$P_{ij} \equiv n\langle u_i u_j\rangle = \int \mathrm{d}\boldsymbol{v}\, u_i u_j f, \tag{2.11}$$

are the components of pressure tensor (or stress tensor),

$$Q_i \equiv \frac{1}{2}n\langle u^2 u_i\rangle = \frac{1}{2}\int \mathrm{d}\boldsymbol{v}\, u^2 u_i f, \tag{2.12}$$

are the components of heat flux, $\boldsymbol{u}$ is the fluctuating velocity defined in §1.2, $\langle\ \rangle$ is an average with respect to f defined by $\langle\psi\rangle = \frac{1}{n}\int \mathrm{d}\boldsymbol{v}\,\psi f(\boldsymbol{v},\boldsymbol{r},t)$, $\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}$ is the material derivative and Γ is the collisional dissipation, given by

$$\Gamma \equiv \frac{\pi\epsilon d^2}{8n}\int \mathrm{d}\boldsymbol{v}_1 \mathrm{d}\boldsymbol{v}_2\, v_{12}^3 f(\boldsymbol{v}_1)f(\boldsymbol{v}_2). \tag{2.13}$$

Note that the expression for Γ contains a prefactor ϵ. Now, the goal is to determine the constitutive relations (pressure tensor P_{ij}, heat flux Q_i and collisional dissipation Γ). Clearly, the expressions of P_{ij}, Q_i and Γ (eqs. (2.11)-(2.13)) contain the distribution function f, which is unknown and it is not easy to obtain an exact solution for f. Therefore, in order to determine the constitutive relation, a perturbative expansion for f is assumed (see below), following Sela & Goldhirsch (1998).

2.3 Method of Solution

The classical Chapman-Enskog expansion for molecular gases (Cercignani 1969; Kogan 1969; Chapman & Cowling 1970; Harris 2004) involves a perturbative expansion of the distribution function in powers of the spatial gradients of the hydrodynamic fields; the method is used for the systems that have an equilibrium solution, which serves the zeroth-order solution of the expansion. But as discussed in §1.2, granular systems do not have an equilibrium solution and therefore the classical Chapman-Enskog expansion technique can not be applied directly to granular systems. Nevertheless, we shall see below that one can obtain a zeroth-order solution in elastic and zero Knudsen number (defined below) limit. An appropriate generalization of Chapman-Enskog expansion (Sela & Goldhirsch 1998) along with this zeroth-order solution can be employed in case of granular systems.

Let us define a small parameter *Knudsen number* (K), by $K \equiv \ell/L$, where ℓ is the mean free path given by $\ell \equiv 1/(\pi n d^2)$ and L is a macroscopic length scale in the problem. In the present work, gravity is serving as a body force, so one can define the macroscopic length scale as $L = \frac{v_{th}^2}{g}$, where $v_{th} = \left(\frac{2\Theta}{3}\right)^{1/2}$ is the thermal speed and g is the magnitude of the gravitational acceleration, hence the macroscopic length scale $L = \frac{2\Theta}{3g}$. Next, we perform a rescaling of the Boltzmann equation (eq. (2.4)), as follows: the rescaled fluctuating velocity is $\tilde{\boldsymbol{u}} \equiv \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(\boldsymbol{v}-\boldsymbol{V})$, the rescaled gravity is $\tilde{\boldsymbol{g}} \equiv \boldsymbol{g}/g$, and $f \equiv n\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\tilde{f}(\tilde{u})$. Hence the Boltzmann equation (2.4) changes to

$$\frac{\partial}{\partial t}\left\{n\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\tilde{f}\right\} + \boldsymbol{v}_1\cdot\boldsymbol{\nabla}\left\{n\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\tilde{f}\right\} + g\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\left\{n\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\tilde{f}\right\}$$
$$= d^2\int_{\hat{\boldsymbol{k}}\cdot\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{u}}_{12}>0}\left\{\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\mathrm{d}\tilde{\boldsymbol{u}}_2\right\}\mathrm{d}\hat{\boldsymbol{k}}\left\{\hat{\boldsymbol{k}}\cdot\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{u}}_{12}\right\}$$
$$\times\left[\frac{1}{e^2}\left\{n\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\tilde{f}(\tilde{\boldsymbol{u}}_1')\right\}\left\{n\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\tilde{f}(\tilde{\boldsymbol{u}}_2')\right\}-\left\{n\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\tilde{f}(\tilde{\boldsymbol{u}}_1)\right\}\left\{n\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\tilde{f}(\tilde{\boldsymbol{u}}_2)\right\}\right]$$

or

$$\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\left\{n\frac{\partial\tilde{f}}{\partial t}+\tilde{f}\frac{\partial n}{\partial t}\right\}+n\tilde{f}\,\frac{3}{2}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(-\frac{3}{2\Theta^2}\right)\frac{\partial\Theta}{\partial t}+\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\left\{n(\boldsymbol{v}_1\cdot\boldsymbol{\nabla}\tilde{f})\right.$$
$$\left.+\tilde{f}(\boldsymbol{v}_1\cdot\boldsymbol{\nabla}n)\right\}+n\tilde{f}\,\frac{3}{2}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(-\frac{3}{2\Theta^2}\right)(\boldsymbol{v}_1\cdot\boldsymbol{\nabla}\Theta)+n\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}g\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\tilde{f}$$
$$= n^2d^2\frac{3}{2\Theta}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2')-\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right)$$

or

$$n\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\left[\left\{\frac{\partial\tilde{f}}{\partial t}+\frac{\tilde{f}}{n}\frac{\partial n}{\partial t}\right\}-\tilde{f}\,\frac{3}{2\Theta}\frac{\partial\Theta}{\partial t}+\left\{\boldsymbol{v}_1\cdot\boldsymbol{\nabla}\tilde{f}+\frac{\tilde{f}}{n}(\boldsymbol{v}_1\cdot\boldsymbol{\nabla}n)\right\}-\tilde{f}\,\frac{3}{2\Theta}(\boldsymbol{v}_1\cdot\boldsymbol{\nabla}\Theta)\right.$$
$$\left.+\,g\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\tilde{f}\right]=n^2d^2\frac{3}{2\Theta}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2')-\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right)$$

or

$$\frac{1}{\pi nd^2}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left[\left\{\frac{\partial\tilde{f}}{\partial t}+\boldsymbol{v}_1\cdot\boldsymbol{\nabla}\tilde{f}\right\}+\frac{\tilde{f}}{n}\left\{\frac{\partial n}{\partial t}+\boldsymbol{v}_1\cdot\boldsymbol{\nabla}n\right\}-\tilde{f}\,\frac{3}{2\Theta}\left\{\frac{\partial\Theta}{\partial t}+\boldsymbol{v}_1\cdot\boldsymbol{\nabla}\Theta\right\}\right.$$
$$\left.+\,g\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\tilde{f}\right]=\frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2')-\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right).$$

Now, using the definitions of mean free path and Knudsen number, $\frac{1}{\pi nd^2}=\ell=K\frac{2\Theta}{3g}$. Therefore

$$K\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left[\left(\frac{\partial}{\partial t}+\boldsymbol{v}_1\cdot\boldsymbol{\nabla}\right)\tilde{f}+\tilde{f}\left(\frac{\partial}{\partial t}+\boldsymbol{v}_1\cdot\boldsymbol{\nabla}\right)\ln n-\frac{3}{2}\tilde{f}\left(\frac{\partial}{\partial t}+\boldsymbol{v}_1\cdot\boldsymbol{\nabla}\right)\ln\Theta\right.$$
$$\left.+\,g\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\tilde{f}\right]=\frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2')-\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right)$$

or

$$\tilde{\mathscr{D}}\tilde{f}+\tilde{f}\tilde{\mathscr{D}}\left(\ln n-\frac{3}{2}\ln\Theta\right)+K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\tilde{f}=\mathscr{B}(\tilde{f},\tilde{f},e), \tag{2.14}$$

where

$$\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e) = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right) \tag{2.15}$$

and

$$\tilde{\mathscr{D}} = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t} + \boldsymbol{v}_1\cdot\boldsymbol{\nabla}\right). \tag{2.16}$$

Eq. (2.14) is another form of the Boltzmann equation in terms of rescaled quantities. Here onwards, eq. (2.14) will be referred as the rescaled Boltzmann equation. Note that $\tilde{\mathscr{D}}$ is not a material derivative since the velocity $\boldsymbol{v}_1$ is not the hydrodynamic velocity, rather it is the velocity of the particle. Clearly, in the double limit $\epsilon \to 0$ and $K \to 0$, eq. (2.14) changes to $\tilde{\mathscr{B}}(\tilde{f},\tilde{f},1) = 0$, the solution of which is a Maxwellian distribution function (see Chapman & Cowling 1970; Harris 2004; Rao & Nott 2008, for a derivation) given by

$$\tilde{f}_0(\tilde{u}) = \pi^{-3/2}\,\mathrm{e}^{-\tilde{u}^2}. \tag{2.17}$$

Hence, for $K \ll 1$ and $\epsilon \ll 1$, $\tilde{f}$ can be expressed as $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1+\Phi)$, where Φ is a 'small' perturbation. $\tilde{f}_0(\tilde{u})$ will serve as zeroth-order solution in this work.

Since the equilibrium distribution function, f_0, is defined in such a way that the hydrodynamic fields are velocity moments of f_0, i.e.,

$$n = \int \mathrm{d}\boldsymbol{v}\, f_0, \quad \boldsymbol{V} = \frac{1}{n}\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v} f_0, \quad \Theta = \frac{1}{n}\int \mathrm{d}\boldsymbol{v}\,(\boldsymbol{v}-\boldsymbol{V})^2 f_0.$$

Therefore, it follows from eqs. (2.5)-(2.7) and the relation $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1+\Phi)$,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{f}_0(\tilde{u})\Phi = \int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{\boldsymbol{u}}\tilde{f}_0(\tilde{u})\Phi = \int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}^2\tilde{f}_0(\tilde{u})\Phi = 0. \tag{2.18}$$

These are called the orthogonality conditions and they should hold at all orders in Chapman-Enskog expansion.

Next, we shall substitute $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1+\Phi)$ in eq. (2.14) and for that, let us first simplify the first and third terms of eq. (2.14), separately. (Here onwards, wherever there is no subscript in $\tilde{\boldsymbol{u}}$ or $\boldsymbol{v}$, they will be read as $\tilde{\boldsymbol{u}}_1$ or $\boldsymbol{v}_1$ respectively, until $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{u}}_1$ or $\boldsymbol{v}$ and $\boldsymbol{v}_1$ appear simultaneously in integrals.)

First Term:

$$\begin{aligned}
\tilde{\mathscr{D}}\tilde{f} &= \tilde{\mathscr{D}}\left\{\tilde{f}_0(\tilde{u})(1+\Phi)\right\} = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\left\{\tilde{f}_0(\tilde{u})(1+\Phi)\right\} \\
&= (1+\Phi)\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\left(\pi^{-3/2}\,\mathrm{e}^{-\tilde{u}^2}\right) + \tilde{f}_0(\tilde{u})\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\Phi \\
&= \left(\pi^{-3/2}\,\mathrm{e}^{-\tilde{u}^2}\right)(1+\Phi)\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)(-\tilde{u}^2) + \tilde{f}_0(\tilde{u})\tilde{\mathscr{D}}\Phi \\
&= -\tilde{f}_0(\tilde{u})(1+\Phi)\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\left\{\frac{3}{2\Theta}(\boldsymbol{v}-\boldsymbol{V})^2\right\} + \tilde{f}_0(\tilde{u})\tilde{\mathscr{D}}\Phi
\end{aligned}$$

or

$$\begin{aligned}
\tilde{\mathscr{D}}\tilde{f} &= -\tilde{f}_0(\tilde{u})(1+\mathit{\Phi})\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left[-\frac{3}{2\Theta^2}(\boldsymbol{v}-\boldsymbol{V})^2\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\Theta\right.\\
&\quad\left.+\frac{3}{2\Theta}2(\boldsymbol{v}-\boldsymbol{V})\cdot\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)(-\boldsymbol{V})\right]+\tilde{f}_0(\tilde{u})\tilde{\mathscr{D}}\mathit{\Phi}\\
&= -\tilde{f}_0(\tilde{u})(1+\mathit{\Phi})\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left[-\tilde{u}^2\frac{1}{\Theta}\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\Theta\right.\\
&\quad\left.-2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{u}}\cdot\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{V}\right]+\tilde{f}_0(\tilde{u})\tilde{\mathscr{D}}\mathit{\Phi}\\
&= \tilde{f}_0(\tilde{u})(1+\mathit{\Phi})\left[\tilde{u}^2\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\ln\Theta\right.\\
&\quad\left.+2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\right)V_i\right]+\tilde{f}_0(\tilde{u})\tilde{\mathscr{D}}\mathit{\Phi}\\
&= \tilde{f}_0(\tilde{u})(1+\mathit{\Phi})\left[\tilde{u}^2\tilde{\mathscr{D}}\ln\Theta+2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}V_i\right]+\tilde{f}_0(\tilde{u})\tilde{\mathscr{D}}\mathit{\Phi}\\
&= \tilde{f}_0(\tilde{u})\left[(1+\mathit{\Phi})\left\{2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}V_i+\tilde{u}^2\tilde{\mathscr{D}}\ln\Theta\right\}+\tilde{\mathscr{D}}\mathit{\Phi}\right].
\end{aligned}$$

Third Term:

$$\begin{aligned}
&K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\tilde{f}\\
&= K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\tilde{f}_0(\tilde{u})(1+\mathit{\Phi})\\
&= (1+\mathit{\Phi})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\left(\pi^{-3/2}\,\mathrm{e}^{-\tilde{u}^2}\right)+\tilde{f}_0(\tilde{u})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\mathit{\Phi}\\
&= \left(\pi^{-3/2}\,\mathrm{e}^{-\tilde{u}^2}\right)(1+\mathit{\Phi})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\left(-\tilde{u}^2\right)+\tilde{f}_0(\tilde{u})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\mathit{\Phi}\\
&= -\tilde{f}_0(\tilde{u})(1+\mathit{\Phi})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\left(\frac{3}{2\Theta}(\boldsymbol{v}-\boldsymbol{V})^2\right)+\tilde{f}_0(\tilde{u})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\mathit{\Phi}\\
&= -\tilde{f}_0(\tilde{u})(1+\mathit{\Phi})K\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\left\{(\boldsymbol{v}-\boldsymbol{V})^2\right\}+\tilde{f}_0(\tilde{u})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\mathit{\Phi}\\
&= -\tilde{f}_0(\tilde{u})(1+\mathit{\Phi})K\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\{2(\boldsymbol{v}-\boldsymbol{V})\}\,\tilde{f}_0(\tilde{u})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\mathit{\Phi}\\
&= -2\tilde{f}_0(\tilde{u})(1+\mathit{\Phi})K\tilde{\boldsymbol{g}}\cdot\tilde{\boldsymbol{u}}+\tilde{f}_0(\tilde{u})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\mathit{\Phi}\\
&= -2\tilde{f}_0(\tilde{u})(1+\mathit{\Phi})K\tilde{g}_i\tilde{u}_i+\tilde{f}_0(\tilde{u})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\mathit{\Phi},
\end{aligned}$$

where we have used the following,

$$\begin{aligned}\boldsymbol{\nabla}_v \left\{(\boldsymbol{v}-\boldsymbol{V})^2\right\} &= \sum_{j=1}^{3} \hat{\boldsymbol{\delta}}_j \frac{\partial}{\partial v_j}\left\{\sum_{i=1}^{3}(v_i - V_i)^2\right\} = \sum_{j=1}^{3} \hat{\boldsymbol{\delta}}_j \left\{\sum_{i=1}^{3} 2\,(v_i - V_i)\left(\frac{\partial v_i}{\partial v_j}\right)\right\} \\ &= \sum_{j=1}^{3} \hat{\boldsymbol{\delta}}_j \left\{\sum_{i=1}^{3} 2\,(v_i - V_i)\,\delta_{ij}\right\} = 2\sum_{j=1}^{3} \hat{\boldsymbol{\delta}}_j\,(v_j - V_j) = 2(\boldsymbol{v}-\boldsymbol{V}).\end{aligned}$$

Substituting the above values of the first and third terms along with $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1+\Phi)$ in eq. (2.14), we get

$$\begin{aligned}&\tilde{f}_0(\tilde{u})\left[(1+\Phi)\left\{2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}V_i + \tilde{u}^2\tilde{\mathscr{D}}\ln\Theta\right\} + \tilde{\mathscr{D}}\Phi\right] + \tilde{f}_0(\tilde{u})(1+\Phi)\tilde{\mathscr{D}}\left(\ln n - \frac{3}{2}\ln\Theta\right) \\ &- 2\tilde{f}_0(\tilde{u})(1+\Phi)K\tilde{g}_i\tilde{u}_i + \tilde{f}_0(\tilde{u})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\Phi = \tilde{\mathscr{B}}(\tilde{f},\tilde{f},e)\end{aligned}$$

or

$$\begin{aligned}&(1+\Phi)\left[\tilde{\mathscr{D}}\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathscr{D}}\ln\Theta - 2K\tilde{g}_i\tilde{u}_i\right] + \tilde{\mathscr{D}}\Phi \\ &+ K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\Phi = \frac{1}{\tilde{f}_0(\tilde{u})}\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e). \qquad (2.19)\end{aligned}$$

Here onwards, eq. (2.19) will be referred as the perturbed Boltzmann equation. The action of operator $\tilde{\mathscr{D}}$ on the hydrodynamic fields can be evaluated with the help of eq. (2.16) and eqs. (2.8)-(2.10) as following.

$$\tilde{\mathscr{D}}\ln n = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\ln n = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{n}\left(\frac{\partial n}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}n\right),$$

but from eq. (2.8), $\left\{\frac{\partial n}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})n\right\} + (\boldsymbol{\nabla}\cdot\boldsymbol{V})n = 0$ or $\frac{\partial n}{\partial t} = -(\boldsymbol{V}\cdot\boldsymbol{\nabla})n - n(\boldsymbol{\nabla}\cdot\boldsymbol{V})$, hence

$$\begin{aligned}\tilde{\mathscr{D}}\ln n &= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{n}\left\{-(\boldsymbol{V}\cdot\boldsymbol{\nabla})n - n(\boldsymbol{\nabla}\cdot\boldsymbol{V}) + (\boldsymbol{v}\cdot\boldsymbol{\nabla})n\right\} \\ &= \frac{K}{g}\frac{2\Theta}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{1}{n}\left\{(\boldsymbol{v}-\boldsymbol{V})\cdot\boldsymbol{\nabla}n - n(\boldsymbol{\nabla}\cdot\boldsymbol{V})\right\} \\ &= K\frac{2\Theta}{3g}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(\boldsymbol{v}-\boldsymbol{V})\cdot\frac{1}{n}\boldsymbol{\nabla}n - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\boldsymbol{\nabla}\cdot\boldsymbol{V}\right\} \\ &= K\frac{2\Theta}{3g}\left\{\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}n - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\boldsymbol{\nabla}\cdot\boldsymbol{V}\right\} \\ &= K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial\ln n}{\partial r_i} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}, \qquad (2.20)\end{aligned}$$

$$\tilde{\mathscr{D}}V_i = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)V_i,$$

but from eq. (2.9), $n\left\{\frac{\partial V_i}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})V_i\right\} + \frac{\partial P_{ij}}{\partial r_j} = ng_i$ or $\frac{\partial V_i}{\partial t} = g_i - (\boldsymbol{V}\cdot\boldsymbol{\nabla})V_i - \frac{1}{n}\frac{\partial P_{ij}}{\partial r_j}$, hence

$$\begin{aligned}
\tilde{\mathscr{D}}V_i &= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left\{g_i - (\boldsymbol{V}\cdot\boldsymbol{\nabla})V_i - \frac{1}{n}\frac{\partial P_{ij}}{\partial r_j} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})V_i\right\}\\
&= \frac{K}{g}\frac{2\Theta}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{(\boldsymbol{v}-\boldsymbol{V})\cdot\boldsymbol{\nabla}V_i - \frac{1}{n}\frac{\partial P_{ij}}{\partial r_j}\right\} + \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}g_i\\
&= K\frac{2\Theta}{3g}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(\boldsymbol{v}-\boldsymbol{V})\cdot\boldsymbol{\nabla}V_i - \frac{1}{n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial P_{ij}}{\partial r_j}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{g_i}{g}\\
&= K\frac{2\Theta}{3g}\left\{\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}V_i - \frac{1}{n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial P_{ij}}{\partial r_j}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\\
&= K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial V_i}{\partial r_j} - \frac{1}{n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial P_{ij}}{\partial r_j}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i, \qquad (2.21)
\end{aligned}$$

and

$$\tilde{\mathscr{D}}\ln\Theta = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\ln\Theta = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{\Theta}\left\{\frac{\partial\Theta}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\Theta\right\},$$

but from eq. (2.10), $n\left\{\frac{\partial\Theta}{\partial t} + (\boldsymbol{V}\cdot\boldsymbol{\nabla})\Theta\right\} + 2\frac{\partial V_i}{\partial r_j}P_{ij} + 2\frac{\partial Q_i}{\partial r_i} = -n\Gamma$ or $\frac{\partial\Theta}{\partial t} = -(\boldsymbol{V}\cdot\boldsymbol{\nabla})\Theta - \frac{2}{n}\frac{\partial V_i}{\partial r_j}P_{ij} - \frac{2}{n}\frac{\partial Q_i}{\partial r_i} - \Gamma$, hence

$$\begin{aligned}
\tilde{\mathscr{D}}\ln\Theta &= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{\Theta}\left\{-(\boldsymbol{V}\cdot\boldsymbol{\nabla})\Theta - \frac{2}{n}\frac{\partial V_i}{\partial r_j}P_{ij} - \frac{2}{n}\frac{\partial Q_i}{\partial r_i} - \Gamma + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\Theta\right\}\\
&= \frac{K}{g}\frac{2\Theta}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{1}{\Theta}\left\{(\boldsymbol{v}-\boldsymbol{V})\cdot\boldsymbol{\nabla}\Theta - \frac{2}{n}\frac{\partial V_i}{\partial r_j}P_{ij} - \frac{2}{n}\frac{\partial Q_i}{\partial r_i}\right\} - \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{\Theta}\Gamma\\
&= K\frac{2\Theta}{3g}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(\boldsymbol{v}-\boldsymbol{V})\cdot\frac{1}{\Theta}\boldsymbol{\nabla}\Theta - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}P_{ij} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial Q_i}{\partial r_i}\right\}\\
&\quad - \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{\Theta}\Gamma\\
&= K\frac{2\Theta}{3g}\left\{\tilde{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\ln\Theta - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}P_{ij} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial Q_i}{\partial r_i}\right\} - \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{\Theta}\Gamma.
\end{aligned}$$

As we have already seen in eq. (2.13), the expression for Γ contains a prefactor ϵ, we shall take out the factor ϵ and replace other factors in the last term of above equation by $\tilde{\Gamma}$, so that $\tilde{\Gamma}$ does not depend on ϵ explicitly. Hence

$$\tilde{\mathscr{D}}\ln\Theta = K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}P_{ij} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial Q_i}{\partial r_i}\right\} - \epsilon\tilde{\Gamma}, \qquad (2.22)$$

where

$$\epsilon\tilde{\varGamma} = \frac{K}{g}\left(\frac{2\varTheta}{3}\right)^{\frac{1}{2}}\frac{1}{\varTheta}\varGamma \tag{2.23}$$

or

$$\varGamma = \frac{\epsilon g\varTheta}{K}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\tilde{\varGamma} = \frac{\epsilon g\varTheta}{\ell}\times\frac{2\varTheta}{3g}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\tilde{\varGamma} = \frac{\epsilon\varTheta}{\ell}\left(\frac{2\varTheta}{3}\right)^{\frac{1}{2}}\tilde{\varGamma}. \tag{2.24}$$

Also, substituting the value of $\varGamma$ from eq. (2.13),

$$\begin{aligned}
\epsilon\tilde{\varGamma} &= \frac{K}{g}\left(\frac{2\varTheta}{3}\right)^{\frac{1}{2}}\frac{1}{\varTheta}\varGamma = \frac{K}{g}\left(\frac{2\varTheta}{3}\right)^{\frac{1}{2}}\frac{1}{\varTheta}\frac{\pi\epsilon d^2}{8n}\int \mathrm{d}\boldsymbol{v}_1\mathrm{d}\boldsymbol{v}_2\, v_{12}^3 f(\boldsymbol{v}_1)f(\boldsymbol{v}_2)\\
&= \left(\frac{1}{\pi n d^2}\times\frac{3g}{2\varTheta}\right)\frac{1}{g}\left(\frac{2\varTheta}{3}\right)^{\frac{1}{2}}\frac{1}{\varTheta}\frac{\pi\epsilon d^2}{8n}\int\left\{\left(\frac{2\varTheta}{3}\right)^{\frac{3}{2}}\mathrm{d}\tilde{\boldsymbol{u}}_1\right\}\left\{\left(\frac{2\varTheta}{3}\right)^{\frac{3}{2}}\mathrm{d}\tilde{\boldsymbol{u}}_2\right\}\\
&\quad\times\left\{\left(\frac{2\varTheta}{3}\right)^{\frac{3}{2}}\tilde{u}_{12}^3\right\}\left\{n\left(\frac{3}{2\varTheta}\right)^{\frac{3}{2}}\tilde{f}(\tilde{\boldsymbol{u}}_1)\right\}\left\{n\left(\frac{3}{2\varTheta}\right)^{\frac{3}{2}}\tilde{f}(\tilde{\boldsymbol{u}}_2)\right\}\\
&= \epsilon\left(\frac{1}{8n^2}\right)\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\frac{1}{\varTheta}\times n^2\left(\frac{2\varTheta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\\
&= \frac{\epsilon}{12}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)
\end{aligned}$$

or

$$\tilde{\varGamma} = \frac{1}{12}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2). \tag{2.25}$$

Next, following Sela & Goldhirsch (1998), $\varPhi$ is expanded in both small parameters, K and ϵ (a generalization of classical Chapman-Enskog expansion for granular flows) as following.

$$\varPhi = \varPhi_K + \varPhi_\epsilon + \varPhi_{KK} + \varPhi_{K\epsilon} + \varPhi_{\epsilon\epsilon} + \ldots \tag{2.26}$$

or

$$\varPhi = \varPhi_K + \epsilon\varphi_1^{(1)} + \varPhi_{KK} + \epsilon\varphi_K^{(1)} + \epsilon^2\varphi_1^{(2)} + \ldots. \tag{2.27}$$

In eq. (2.27), $\varPhi$ is written in another form to avoid any confusion. From eqs. (2.26) and (2.27), $\varPhi_\epsilon = \epsilon\varphi_1^{(1)}$, $\varPhi_{K\epsilon} = \epsilon\varphi_K^{(1)}$, $\varPhi_{\epsilon\epsilon} = \epsilon^2\varphi_1^{(2)}$ and so on. In eqs. (2.26) and (2.27), and here onwards subscripts indicate the order of the corresponding terms in the small parameters, e.g. $\varPhi_K = O(K)$ and bracketed superscripts in eq. (2.27) are just to indicate the ϵ order of corresponding terms in eq. (2.26).

Similar to the expansion of $\varPhi$ in small parameters, $\tilde{\mathscr{D}}\psi$ (where ψ is function of the field variables) can be expanded as: $\tilde{\mathscr{D}}\psi = \tilde{\mathscr{D}}_K\psi + \tilde{\mathscr{D}}_\epsilon\psi + \tilde{\mathscr{D}}_{KK}\psi + \tilde{\mathscr{D}}_{K\epsilon}\psi + \ldots$, where e.g. $\tilde{\mathscr{D}}_{K\epsilon}\psi$ is the $O(K\epsilon)$ term in the expansion of $\tilde{\mathscr{D}}\psi$ in powers of K and ϵ.

For a non-negative integer n, $O(\epsilon^n)$ corrections to the single particle distribution function are the Euler order terms, $O(K\epsilon^n)$ corrections are the Navier-Stokes or Chapman-Enskog order terms and $O(K^2\epsilon^n)$ corrections are Burnett order terms.

2.3.1 Zeroth-order Constitutive Relations

Pressure Tensor

Writing eq. (2.11) in terms of rescaled quantities, we have

$$P_{ij} = \int \mathrm{d}\boldsymbol{v}\, u_i u_j f = \int \left\{ \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \mathrm{d}\tilde{\boldsymbol{u}} \right\} \left\{ \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{u}_i \right\} \left\{ \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{u}_j \right\} \left\{ n \left(\frac{3}{2\Theta}\right)^{\frac{3}{2}} \tilde{f}(\tilde{\boldsymbol{u}}) \right\}$$
$$= \frac{2n\Theta}{3} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i \tilde{u}_j \tilde{f}(\tilde{\boldsymbol{u}}).$$

Using relation $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1+\Phi)$ along with eqs. (2.17) and (2.26),

$$P_{ij} = \frac{2n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i \tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2} \left(1 + \Phi_K + \Phi_\epsilon + \Phi_{KK} + \Phi_{K\epsilon} + \ldots\right)$$
$$= P_{ij}^0 + \frac{2n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i \tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2} \left(\Phi_K + \Phi_\epsilon + \Phi_{KK} + \Phi_{K\epsilon} + \ldots\right), \tag{2.28}$$

where P_{ij}^0 is the zeroth order term in the expansion of P_{ij}, given by

$$P_{ij}^0 = \frac{2n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i \tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2}.$$

Using eq. (F.9a),

$$P_{ij}^0 = \frac{2n\Theta}{3\pi^{3/2}} \times \frac{4\pi}{3}\delta_{ij} \int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\, \tilde{u}^4 = \frac{2n\Theta}{3\pi^{3/2}} \times \frac{4\pi}{3}\delta_{ij} \times \frac{3}{8}\sqrt{\pi}.$$

This implies

$$\boxed{P_{ij}^0 = \frac{1}{3} n\Theta\delta_{ij}} \tag{2.29}$$

Heat Flux

Writing eq. (2.12) in terms of rescaled quantities, we have

$$Q_i = \frac{1}{2}\int \mathrm{d}\boldsymbol{v}\, u^2 u_i f = \frac{1}{2}\int \left\{ \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \mathrm{d}\tilde{\boldsymbol{u}} \right\} \left\{ \left(\frac{2\Theta}{3}\right) \tilde{u}^2 \right\} \left\{ \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{u}_i \right\} \left\{ n \left(\frac{3}{2\Theta}\right)^{\frac{3}{2}} \tilde{f}(\tilde{\boldsymbol{u}}) \right\}$$
$$= \frac{n}{2}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{u}_i \tilde{f}(\tilde{\boldsymbol{u}}).$$

Using relation $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1+\Phi)$ along with eqs. (2.17) and (2.26),

$$Q_i = \frac{n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2} \left(1 + \Phi_K + \Phi_\epsilon + \Phi_{KK} + \Phi_{K\epsilon} + \ldots\right)$$
$$= Q_i^0 + \frac{n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2} \left(\Phi_K + \Phi_\epsilon + \Phi_{KK} + \Phi_{K\epsilon} + \ldots\right), \tag{2.30}$$

where Q_i^0 is the zeroth order term in the expansion of Q_i, given by

$$Q_i^0 = \frac{n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, .$$

In the above integral, the integrand is an odd function of components of $\tilde{\boldsymbol{u}}$. Therefore

$$\boxed{Q_i^0 = 0} \tag{2.31}$$

Collisional Dissipation

From eq. (2.25) (along with relation $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1+\Phi)$ and eq. (2.17)), we have

$$\begin{aligned}
\tilde{\Gamma} &= \frac{1}{12}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3 \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\\
&= \frac{1}{12}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3 \left\{\tilde{f}_0(\tilde{u}_1)\left(1+\Phi(\tilde{\boldsymbol{u}}_1)\right)\right\}\left\{\tilde{f}_0(\tilde{u}_2)\left(1+\Phi(\tilde{\boldsymbol{u}}_2)\right)\right\}\\
&= \frac{1}{12}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3 \tilde{f}_0(\tilde{u}_1)\tilde{f}_0(\tilde{u}_2)\{1+\Phi_K(\tilde{\boldsymbol{u}}_1)+\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)+\Phi_{KK}(\tilde{\boldsymbol{u}}_1)+\Phi_{K\epsilon}(\tilde{\boldsymbol{u}}_1)+\ldots\}\\
&\quad \times\{1+\Phi_K(\tilde{\boldsymbol{u}}_2)+\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)+\Phi_{KK}(\tilde{\boldsymbol{u}}_2)+\Phi_{K\epsilon}(\tilde{\boldsymbol{u}}_2)+\ldots\}\\
&= \frac{1}{12}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3 \left(\pi^{-3/2}\,\mathrm{e}^{-\tilde{u}_1^2}\right)\left(\pi^{-3/2}\,\mathrm{e}^{-\tilde{u}_2^2}\right)\{1+\Phi_K(\tilde{\boldsymbol{u}}_1)+\Phi_K(\tilde{\boldsymbol{u}}_2)\\
&\quad +\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)+\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)+\Phi_{KK}(\tilde{\boldsymbol{u}}_1)+\Phi_{KK}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_1)\,\Phi_K(\tilde{\boldsymbol{u}}_2)+\ldots\}\\
&= \tilde{\Gamma}_0 + \frac{1}{12\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\{\Phi_K(\tilde{\boldsymbol{u}}_1)+\Phi_K(\tilde{\boldsymbol{u}}_2)+\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)+\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)\\
&\quad +\Phi_{KK}(\tilde{\boldsymbol{u}}_1)+\Phi_{KK}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_1)\,\Phi_K(\tilde{\boldsymbol{u}}_2)+\ldots\}\,,
\end{aligned} \tag{2.32}$$

where $\tilde{\Gamma}_0$ is the zeroth order term in the expansion of $\tilde{\Gamma}$, given by

$$\tilde{\Gamma}_0 = \frac{1}{12\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}$$

and using eq. (G.2), it simplifies to

$$\boxed{\tilde{\Gamma}_0 = \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}} \tag{2.33}$$

Now, from eqs. (2.24) and (2.32), we have

$$\begin{aligned}\Gamma &= \frac{\epsilon\Theta}{\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\Bigg[\tilde{\Gamma}_0 + \frac{1}{12\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \{\Phi_K(\tilde{\boldsymbol{u}}_1) + \Phi_K(\tilde{\boldsymbol{u}}_2) \\ &\quad + \Phi_\epsilon(\tilde{\boldsymbol{u}}_1) + \Phi_\epsilon(\tilde{\boldsymbol{u}}_2) + \Phi_{KK}(\tilde{\boldsymbol{u}}_1) + \Phi_{KK}(\tilde{\boldsymbol{u}}_2) + \Phi_K(\tilde{\boldsymbol{u}}_1)\,\Phi_K(\tilde{\boldsymbol{u}}_2) + \ldots\}\Bigg] \\ &= \Gamma_\epsilon + \frac{\epsilon\Theta}{12\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \{\Phi_K(\tilde{\boldsymbol{u}}_1) + \Phi_K(\tilde{\boldsymbol{u}}_2) \\ &\quad + \Phi_\epsilon(\tilde{\boldsymbol{u}}_1) + \Phi_\epsilon(\tilde{\boldsymbol{u}}_2) + \Phi_{KK}(\tilde{\boldsymbol{u}}_1) + \Phi_{KK}(\tilde{\boldsymbol{u}}_2) + \Phi_K(\tilde{\boldsymbol{u}}_1)\,\Phi_K(\tilde{\boldsymbol{u}}_2) + \ldots\}\,, \end{aligned} \tag{2.34}$$

where

$$\Gamma_\epsilon = \frac{\epsilon\Theta}{\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\Gamma}_0 = \frac{\epsilon\Theta}{\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \times \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}$$

or

$$\boxed{\Gamma_\epsilon = \frac{\epsilon}{\ell}\left(\frac{16}{27\pi}\right)^{1/2}\Theta^{3/2}} \tag{2.35}$$

Thus, we see from eqs. (2.29) and (2.31) that at zeroth-order pressure tensor is just the mean pressure times an identity matrix and heat flux is zero. Due to the prefactor ϵ in the expression of Γ (cf. eq. (2.13)), the lowest-order contribution to Γ is of order ϵ and given by eq. (2.35).

2.3.2 Method for obtaining Higher-order Constitutive Relations

As it is clear from eqs. (2.28), (2.30) and (2.34), that to evaluate the higher-order constitutive relations, higher-order correction terms (i.e., Φ_K, Φ_ϵ, etc.) should be known. But we shall realize latter that it is extremely complicated to evalute the correction terms, even to second-order in small parameters.

Therefore we shall follow Sela & Goldhirsch (1998) and evaluate the correction terms, to first-order in small parameters using the Boltzmann equation. Then we shall use the first-order correction terms along with the self-adjoint property of the linearized Boltzmann operator (defined in §4.1) to evaluate the constitutive relations directly, without obtaining the correction terms at second-order in small parameters. For this purpose, we shall substitute the value of Φ from eqs. (2.26) and (2.27) in eq. (2.19) as following.

In the right-hand side (let us say, it is equal to A) of eq. (2.19), we shall replace Φ by its expansion given in eq. (2.27). A is further simplified by using Taylor-series expansion (for its implicit dependence on ϵ). The details are given in Appendix A and the final form of A is given in eq. (A.1). Now, in the left-hand side of eq. (2.19), let us replace Φ by its expansion given in eq. (2.26) and $\tilde{\mathscr{D}}$ by its expansion given above, and A by its expression given in eq. (A.1). Thus, eq. (2.19) changes to

$$
\begin{aligned}
&(1+\Phi_K+\Phi_\epsilon+\Phi_{KK}+\Phi_{K\epsilon}+\ldots)\Big[\left(\tilde{\mathscr{D}}_K+\tilde{\mathscr{D}}_\epsilon+\tilde{\mathscr{D}}_{KK}+\tilde{\mathscr{D}}_{K\epsilon}+\ldots\right)\ln n\\
&+2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\left(\tilde{\mathscr{D}}_K+\tilde{\mathscr{D}}_\epsilon+\tilde{\mathscr{D}}_{KK}+\tilde{\mathscr{D}}_{K\epsilon}+\ldots\right)V_i\\
&+\left(\tilde{u}^2-\frac{3}{2}\right)\left(\tilde{\mathscr{D}}_K+\tilde{\mathscr{D}}_\epsilon+\tilde{\mathscr{D}}_{KK}+\tilde{\mathscr{D}}_{K\epsilon}+\ldots\right)\ln\Theta-2K\tilde{g}_i\tilde{u}_i\Big]\\
&+\left(\tilde{\mathscr{D}}_K+\tilde{\mathscr{D}}_\epsilon+\tilde{\mathscr{D}}_{KK}+\tilde{\mathscr{D}}_{K\epsilon}+\ldots\right)(\Phi_K+\Phi_\epsilon+\Phi_{KK}+\Phi_{K\epsilon}+\ldots)\\
&+K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,(\Phi_K+\Phi_\epsilon+\Phi_{KK}+\Phi_{K\epsilon}+\ldots)\\
&=\tilde{\mathscr{L}}(\Phi_K)+\epsilon\,\tilde{\mathscr{L}}(\varphi_1^{(1)})+\frac{\epsilon}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right)\\
&\quad+\epsilon\,\tilde{\mathscr{L}}(\varphi_K^{(1)})+\epsilon\tilde{\Xi}(\Phi_K)+\epsilon\tilde{\Lambda}(\Phi_K)+\epsilon\tilde{\Omega}(\Phi_K,\varphi_1^{(1)})+\tilde{\mathscr{L}}(\Phi_{KK})+\frac{1}{2}\tilde{\Omega}(\Phi_K,\Phi_K)\\
&\quad+\epsilon^2\tilde{\mathscr{L}}(\varphi_1^{(2)})+\epsilon^2\tilde{\Xi}(\varphi_1^{(1)})+\epsilon^2\tilde{\Lambda}(\varphi_1^{(1)})+\frac{1}{2}\epsilon^2\tilde{\Omega}(\varphi_1^{(1)},\varphi_1^{(1)})\\
&\quad+\frac{\epsilon^2}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1-(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2+\frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right)+\text{h.o.t.},
\end{aligned}
\tag{2.36}
$$

where

$$
\tilde{\mathscr{L}}(\Phi)\equiv\frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\{\Phi(\tilde{\boldsymbol{u}}_1')+\Phi(\tilde{\boldsymbol{u}}_2')-\Phi(\tilde{\boldsymbol{u}}_1)-\Phi(\tilde{\boldsymbol{u}}_2)\},\tag{2.37}
$$

$$
\begin{aligned}
\tilde{\Omega}(\Phi,\psi)\equiv\frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}&\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\{\Phi(\tilde{\boldsymbol{u}}_1')\psi(\tilde{\boldsymbol{u}}_2')+\Phi(\tilde{\boldsymbol{u}}_2')\psi(\tilde{\boldsymbol{u}}_1')\\
&-\Phi(\tilde{\boldsymbol{u}}_1)\psi(\tilde{\boldsymbol{u}}_2)-\Phi(\tilde{\boldsymbol{u}}_2)\psi(\tilde{\boldsymbol{u}}_1)\},
\end{aligned}
\tag{2.38}
$$

$$
\tilde{\Xi}(\Phi)\equiv\frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(1-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right)\mathrm{e}^{-\tilde{u}_2^2}\{\Phi(\tilde{\boldsymbol{u}}_1')+\Phi(\tilde{\boldsymbol{u}}_2')\}\tag{2.39}
$$

and

$$
\tilde{\Lambda}(\Phi)=\frac{1}{\pi^{5/2}}\lim_{\epsilon\to0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\{\Phi(\tilde{\boldsymbol{u}}_1')+\Phi(\tilde{\boldsymbol{u}}_2')\}.\tag{2.40}
$$

The operator $\tilde{\mathscr{L}}$ in eq. (2.37) is the (standard) rescaled linearized Boltzmann operator defined for elastically colliding particles. The operators $\tilde{\Omega}$ and $\tilde{\Xi}$ in eqs. (2.38) and (2.39) respectively, are also defined for elastically colliding particles (see Appendix A for details). Here onwards, eq. (2.36) will be referred as the expanded Boltzmann equation.

2.4 Summary

The hydrodynamic equations for granular flows are the consequences of the Boltzmann equation. The method to obtain the constitutive relations is presented in detail. Zeroth-order (Euler level) constitutive relations are derived. At this order, pressure tensor turns out to be mean pressure time an identity matrix, i.e.,

$$P_{ij}^0 = \frac{1}{3} n \Theta \delta_{ij},$$

heat flux vanishes, i.e.,

$$Q_i^0 = 0,$$

and contribution to collisional dissipation is of $O(\epsilon)$ due to eq. (2.13) and given by

$$\Gamma_\epsilon = \frac{\epsilon}{\ell} \left(\frac{16}{27\pi} \right)^{1/2} \Theta^{3/2}.$$

Expanded form of the Boltzmann equation (eq. (2.36)) is derived to obtain the higher-order constitutive relations.

Chapter 3

Constitutive Relations at First-order in small parameters

With the help of the expanded Boltzmann equation (2.36) and the equations of motion for hydrodynamic fields (eqs. (2.8)-(2.10)), first-order correction terms are obtained. Using these correction terms, the constitutive relations at first-order in small parameters are derived.

3.1 Solution at $O(K)$

Collecting $O(K)$ terms in eq. (2.36), we have

$$\tilde{\mathscr{D}}_K \ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{u}_i \tilde{\mathscr{D}}_K V_i + \left(\tilde{u}^2 - \frac{3}{2}\right) \tilde{\mathscr{D}}_K \ln \Theta - 2K\tilde{g}_i\tilde{u}_i = \tilde{\mathscr{L}}(\Phi_K). \tag{3.1}$$

The operation of $\tilde{\mathscr{D}}_K$ on the hydrodynamic fields can be obtained as follows:
From eq. (2.20),

$$\tilde{\mathscr{D}}_K \ln n = K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial \ln n}{\partial r_i} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}. \tag{3.2}$$

From eq. (2.21),

$$\tilde{\mathscr{D}}_K V_i = K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial V_i}{\partial r_j} - \frac{1}{n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial P^0_{ij}}{\partial r_j}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i$$

but from eq. (2.29), $P^0_{ij} = \frac{1}{3}n\Theta\delta_{ij} \Rightarrow \frac{\partial P^0_{ij}}{\partial r_j} = \frac{1}{3}\frac{\partial}{\partial r_j}(n\Theta\delta_{ij}) = \frac{1}{3}\frac{\partial(n\Theta)}{\partial r_i}$, therefore

$$\begin{aligned}\tilde{\mathscr{D}}_K V_i &= K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial V_i}{\partial r_j} - \frac{1}{n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{1}{3}\frac{\partial(n\Theta)}{\partial r_i}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i \\ &= K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial V_i}{\partial r_j} - \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{n\Theta}\frac{\partial(n\Theta)}{\partial r_i}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\end{aligned}$$

or

$$\tilde{\mathscr{D}}_K V_i = K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial V_i}{\partial r_j} - \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial \ln(n\Theta)}{\partial r_i}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i. \tag{3.3}$$

Clearly, the term $\epsilon\tilde{\Gamma}$ in eq. (2.22) can not contribute $O(K)$ terms. Hence from eq. (2.22),

$$\tilde{\mathscr{D}}_K \ln \Theta = K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}P^0_{ij} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial Q^0_i}{\partial r_i}\right\},$$

but from eqs. (2.29) and (2.31), $P^0_{ij} = \frac{1}{3}n\Theta\delta_{ij}$ and $Q^0_i = 0$ respectively, therefore

$$\tilde{\mathscr{D}}_K \ln\Theta = K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}. \tag{3.4}$$

Substitution of eqs. (3.2)-(3.4) in eq. (3.1) gives

$$\begin{aligned}
\tilde{\mathscr{L}}(\Phi_K) &= K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial\ln n}{\partial r_i} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}\\
&\quad + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial V_i}{\partial r_j} - \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\ln(n\Theta)}{\partial r_i}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\right]\\
&\quad + \left(\tilde{u}^2 - \frac{3}{2}\right)\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}\right] - 2K\tilde{g}_i\tilde{u}_i\\
&= K\frac{2\Theta}{3g}\left[\tilde{u}_i\frac{\partial\ln n}{\partial r_i} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i} + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{u}_j\frac{\partial V_i}{\partial r_j} - \tilde{u}_i\frac{\partial\ln(n\Theta)}{\partial r_i}\right.\\
&\quad \left. + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}^2\frac{\partial V_i}{\partial r_i} + \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right] + 2K\tilde{g}_i\tilde{u}_i - 2K\tilde{g}_i\tilde{u}_i\\
&= K\frac{2\Theta}{3g}\left[2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{u}_j\frac{\partial V_i}{\partial r_j} - \tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i} + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}^2\frac{\partial V_j}{\partial r_j}\right]\\
&= K\frac{2\Theta}{3g}\left[2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\tilde{u}_i\tilde{u}_j - \frac{1}{3}\tilde{u}^2\delta_{ij}\right\}\frac{\partial V_i}{\partial r_j} + \left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\right]\\
&= K\frac{2\Theta}{3g}\left[2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\frac{\tilde{u}_i\tilde{u}_j + \tilde{u}_j\tilde{u}_i}{2} - \frac{1}{3}\tilde{u}_k^2\delta_{ij}\right\}\frac{\partial V_i}{\partial r_j} + \left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\right]
\end{aligned}$$

or

$$\tilde{\mathscr{L}}(\Phi_K) = 2K\frac{2\Theta}{3g}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} + K\frac{2\Theta}{3g}\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}, \tag{3.5}$$

where the overline denotes a symmetric traceless tensor (sometimes called as deviatoric tensor) defined by $\overline{A_{ij}} \equiv \frac{1}{2}(A_{ij} + A_{ji}) - \frac{1}{3}A_{kk}\delta_{ij}$. Note that eq. (3.5) is identical to that obtained in the classical Chapman-Enskog expansion (of elastic systems) to first order in spatial gradients.

To solve eq. (3.5), note that the left-hand side is linear in Φ_K, and the right-hand side is linear in gradients of velocity ($\boldsymbol{V}$) and granular temperatute (Θ). From the theory of linear equations, the solution of eq. (3.5) is the sum of the most general solution of the corresponding homogeneous equation $\tilde{\mathscr{L}}(\Phi_K) = 0$ and a particular solution of eq. (3.5). From the expression for $\tilde{\mathscr{L}}$ (eq. (2.37)), it is clear that

$$\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2') - \Phi_K(\tilde{\boldsymbol{u}}_1) - \Phi_K(\tilde{\boldsymbol{u}}_2) = 0$$

satisfies the homogeneous equation; also, it is a necessary condition on functions, which qualify to become solutions of homogeneous equation (Chapman & Cowling 1970). Consequently, the linear combination of all the functions, which satisfy the above condition, is the most general solution of the homogeneous equation. A function which satisfies the above condition is called summational (or collisional) invariant. Since $\tilde{\mathscr{L}}$ is defined for elastically colliding particles (see Appendix A) and we know that the mass, momentum and energy are the conserved quantities in the elastic collisions; consequently, 1, $\tilde{\boldsymbol{u}}$ and $\tilde{u}^2$ are the summational invariants for elastically colliding particles of same mass. It can be shown that they are the only independent summational invariants in this case (Chapman & Cowling 1970; Rao & Nott 2008). Therefore the most general solution of the homogeneous equation is:

$$\Phi_{K_{hom}} = a_1 + \boldsymbol{a}_2 \cdot \tilde{\boldsymbol{u}} + a_3 \tilde{u}^2,$$

where subscript *hom* in Φ_K denotes that it is the solution of homogeneous equation, a_1 and a_3 are scalar constant, and $\boldsymbol{a}_2$ is a vector constant. Since $\frac{\partial V_i}{\partial r_j}$ and $\frac{\partial \ln \Theta}{\partial r_i}$ are independently variable, the particular solution must depend linearly on them, i.e.

$$\Phi_{K_{par}} = 2K \frac{2\Theta}{3g} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} A_{ji} \frac{\partial V_i}{\partial r_j} + K \frac{2\Theta}{3g} B_i \frac{\partial \ln \Theta}{\partial r_i},$$

where subscript *par* in Φ_K denotes that it is a particular solution of eq. (3.5), A_{ji} are the components of a second-order tensor and B_i are components of a vector. Therefore the general solution of eq. (3.5) is given by:

$$\Phi_K(\tilde{\boldsymbol{u}}) = a_1 + \boldsymbol{a}_2 \cdot \tilde{\boldsymbol{u}} + a_3 \tilde{u}^2 + 2K \frac{2\Theta}{3g} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} A_{ji} \frac{\partial V_i}{\partial r_j} + K \frac{2\Theta}{3g} B_i \frac{\partial \ln \Theta}{\partial r_i}, \tag{3.6}$$

where a_1, $\boldsymbol{a}_2$, a_3, A_{ji} and B_i are unknowns. Substituting the above expression for Φ_K back into eq. (3.5), using the fact that hydrodynamic fields are conserved and equating the coefficients of $\frac{\partial V_i}{\partial r_j}$ and $\frac{\partial \ln \Theta}{\partial r_i}$ on both sides, we get

$$\tilde{\mathscr{L}}(A_{ji}) = \overline{\tilde{u}_i \tilde{u}_j}, \tag{3.7}$$

$$\tilde{\mathscr{L}}(B_i) = \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i. \tag{3.8}$$

The solvability conditions for these equations are

$$(i) \quad \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j} \psi = 0, \quad \text{and} \quad (ii) \quad \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i \psi = 0,$$

for $\psi = 1$, $\tilde{u}_k$ and $\tilde{u}^2$. The condition (i) with $\psi = 1$ and $\tilde{u}^2$ is satisfied using eq. (F.10), and with $\psi = \tilde{u}_k$ is satisfied because the integrand is an odd function in components of $\tilde{\boldsymbol{u}}$. The condition (ii) with $\psi = 1$ and $\tilde{u}^2$ is satisfied because the corresponding integrands are odd functions in components of $\tilde{\boldsymbol{u}}$, and with $\psi = \tilde{u}_k$, (using eq. (F.9a)) it reduces to

$$\frac{4\pi}{3} \delta_{ik} \int_0^\infty \mathrm{d}\tilde{u}\, \tilde{f}_0(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}^4 = 0,$$

which is identically satisfied on using eq. (2.17) and integrating over $\tilde{u}$. Hence eqs. (3.7) and (3.8) are solvable.

It is clear from the eqs. (3.7) and (3.8) that A_{ji} and B_i, respectively, can be the functions of $\tilde{\boldsymbol{u}}$ only. From eq. (3.7), we can see that

$$\tilde{\mathscr{L}}(A_{ii}) = 0$$

and

$$\tilde{\mathscr{L}}(A_{ji} - A_{ij}) = \tilde{\mathscr{L}}(A_{ji}) - \tilde{\mathscr{L}}(A_{ij}) = 0.$$

These two equations imply that A_{ji} are components of a traceless (i.e., $A_{ii} = 0$) and symmetric (i.e, $A_{ji} = A_{ij}$) tensor and the only symmetric and traceless tensor which can be formed from $\tilde{\boldsymbol{u}}$ is $\overline{\tilde{u}_i \tilde{u}_j}$. Therefore

$$A_{ji} = \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j}, \tag{3.9}$$

where $\hat{\Phi}_v(\tilde{u})$ is unknown function of speed $\tilde{u}$ only. Similarly, the only vector which can be formed from $\tilde{\boldsymbol{u}}$ is $\tilde{\boldsymbol{u}}$ itself. Therefore

$$B_i = \hat{\Phi}_t(\tilde{u}) \tilde{u}_i, \tag{3.10}$$

where $\hat{\Phi}_t(\tilde{u})$ is the unknown function of speed $\tilde{u}$ only. Therefore eq. (3.6) changes to:

$$\Phi_K(\tilde{\boldsymbol{u}}) = a_1 + a_{2i}\tilde{u}_i + a_3 \tilde{u}^2 + 2K \frac{2\Theta}{3g} \left(\frac{3}{2\Theta} \right)^{\frac{1}{2}} \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j} \frac{\partial V_i}{\partial r_j} + K \frac{2\Theta}{3g} \hat{\Phi}_t(\tilde{u}) \tilde{u}_i \frac{\partial \ln \Theta}{\partial r_i}. \tag{3.11}$$

Now, we shall show that the coefficients a_1, a_{2i} and a_3 can be set equal to zero as following. The orthogonality conditions for Φ_K (cf. eq. (2.18)), which Φ_K must satisfy, give the following:

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u}) \psi\, \Phi_K = \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u}) \psi \left\{ a_1 + a_{2i}\tilde{u}_i + a_3 \tilde{u}^2 \right.$$
$$\left. + 2K \frac{2\Theta}{3g} \left(\frac{3}{2\Theta} \right)^{\frac{1}{2}} \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j} \frac{\partial V_i}{\partial r_j} + K \frac{2\Theta}{3g} \hat{\Phi}_t(\tilde{u}) \tilde{u}_i \frac{\partial \ln \Theta}{\partial r_i} \right\} = 0,$$

for $\psi = 1$, $\tilde{u}_k$ and $\tilde{u}^2$. Considering eq. (F.10) and symmetry arguments, one can ignore the vanishing integrals in the above equation and thus using eqs. (2.17) and (F.9a), and changing integrals into spherical polar coordinates, we get the following equations for $\psi = 1$, $\tilde{u}_k$ and $\tilde{u}^2$:

$$\int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^2\, \mathrm{e}^{-\tilde{u}^2} (a_1 + a_3 \tilde{u}^2) = 0, \tag{3.12}$$

$$\int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^4\, \mathrm{e}^{-\tilde{u}^2} \left\{ a_{2i} + K \frac{2\Theta}{3g} \hat{\Phi}_t(\tilde{u}) \frac{\partial \ln \Theta}{\partial r_i} \right\} = 0, \tag{3.13}$$

$$\int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^4\, \mathrm{e}^{-\tilde{u}^2} (a_1 + a_3 \tilde{u}^2) = 0. \tag{3.14}$$

Eqs. (3.12) and (3.14) imply $a_1 = a_3 = 0$; eq. (3.13) implies that a_{2i} is proportional to $\frac{\partial \ln \Theta}{\partial r_i}$. Therefore the term $a_{2i}\tilde{u}_i$ can be absorbed in the term proportional to $\frac{\partial \ln \Theta}{\partial r_i}$ in eq. (3.11): hence we can set $\boldsymbol{a}_2 = 0$, when eq. (3.13) becomes

$$\int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^4\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_t(\tilde{u}) = 0. \tag{3.15}$$

Summarizing above discussions, the solution of eq. (3.5) is:

$$\Phi_K(\tilde{\boldsymbol{u}}) = 2K\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial V_i}{\partial r_j} + K\frac{2\Theta}{3g}\hat{\Phi}_t(\tilde{u})\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} \tag{3.16}$$

with orthogonality condition (3.15) to be satisfied.

To match the solution obtained above with that of Sela & Goldhirsch (1998), the function $\hat{\Phi}_t$ can be redefined as

$$\hat{\Phi}_t(\tilde{u}) = \left(\tilde{u}^2 - \frac{5}{2}\right)\hat{\Phi}_c(\tilde{u}). \tag{3.17}$$

Therefore the solution of eq. (3.5) will become

$$\boxed{\Phi_K(\tilde{\boldsymbol{u}}) = 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i}} \tag{3.18}$$

with orthogonality condition (cf. eq. (3.15))

$$\int_0^\infty \mathrm{d}\tilde{u}\,\tilde{u}^4\,\mathrm{e}^{-\tilde{u}^2}\left(\tilde{u}^2 - \frac{5}{2}\right)\hat{\Phi}_c(\tilde{u}) = 0 \tag{3.19}$$

to be satisfied. With the help of eqs. (3.9), (3.10) and (3.17), eqs. (3.7) and (3.8) can be written as

$$\tilde{\mathscr{L}}\left[\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\right] = \overline{\tilde{u}_i\tilde{u}_j}, \tag{3.20}$$

$$\tilde{\mathscr{L}}\left[\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\right] = \left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i. \tag{3.21}$$

As per above discussion, $\hat{\Phi}_v(\tilde{u})$ and $\hat{\Phi}_c(\tilde{u})$ are unknown functions of the speed $\tilde{u}$ only. Usually, these functions are expanded in (truncated) series of Sonine polynomials (Cercignani 1969; Kogan 1969; Chapman & Cowling 1970; Ferziger & Kaper 1972; Brilliantov & Pöschel 2004; Harris 2004). Sela & Goldhirsch (1998) have expanded them in (truncated) series of modified Bessel functions of first kind by considering their asymptotic properties and the fact that $\hat{\Phi}_v(\tilde{u})$ and $\hat{\Phi}_c(\tilde{u})$ are even functions of speed $\tilde{u}$ (shown in Appendix I). Here we shall directly use the values of these unknown functions given in Sela & Goldhirsch (1998) to evaluate the integrals numerically (using MATHEMATICA).

3.2 Constitutive Relations at $O(K)$

Pressure Tensor

From eqs. (2.28) and (3.18), the contribution of Φ_K to the pressure tensor is given by

$$\begin{aligned} P_{ij}^K &= \frac{2n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}_i\tilde{u}_j\,\mathrm{e}^{-\tilde{u}^2}\,\Phi_K(\tilde{\boldsymbol{u}}) \\ &= \frac{2n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}_i\tilde{u}_j\,\mathrm{e}^{-\tilde{u}^2}\left\{2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_m\tilde{u}_n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_m}{\partial r_n} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_m\frac{\partial \ln\Theta}{\partial r_m}\right\}. \end{aligned}$$

The second term on the right-hand side of the above equation is zero because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}$. Hence

$$\begin{aligned} P_{ij}^K &= 2\pi^{-3/2} K \frac{2\Theta}{3g} n \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i \tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\varPhi}_v(\tilde{u}) \overline{\tilde{u}_m \tilde{u}_n} \frac{\partial V_m}{\partial r_n} \\ &= 2\pi^{-3/2} \ell n \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \frac{8\pi}{15} \overline{\frac{\partial V_i}{\partial r_j}} \int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^6\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\varPhi}_v(\tilde{u}) \quad (\text{using eq. (F.12)}) \\ &= 2 \left\{ \frac{8}{15} \left(\frac{2}{3\pi}\right)^{\frac{1}{2}} \int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^6\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\varPhi}_v(\tilde{u}) \right\} n \ell \Theta^{1/2} \overline{\frac{\partial V_i}{\partial r_j}} \end{aligned}$$

or

$$\boxed{P_{ij}^K = -2\tilde{\mu}_0 n \ell \Theta^{1/2} \overline{\frac{\partial V_i}{\partial r_j}}} \tag{3.22}$$

where

$$\tilde{\mu}_0 = -\frac{8}{15} \left(\frac{2}{3\pi}\right)^{\frac{1}{2}} \int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^6\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\varPhi}_v(\tilde{u}). \tag{3.23}$$

The integral in eq. (3.23) has been evaluated numerically. The result is $\tilde{\mu}_0 \approx 0.3249$. Clearly (from eq. (3.22)), pressure tensor is Newton's law of viscosity at this order.

Heat Flux

From eqs. (2.30) and (3.18), the contribution of $\varPhi_K$ to the heat flux is

$$\begin{aligned} Q_i^K &= \frac{n}{2\pi^{3/2}} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \varPhi_K(\tilde{\boldsymbol{u}}) \\ &= \frac{n}{2\pi^{3/2}} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2} \\ &\quad \times \left\{ 2K \frac{2\Theta}{3g} \hat{\varPhi}_v(\tilde{u}) \overline{\tilde{u}_j \tilde{u}_k} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_k} + K \frac{2\Theta}{3g} \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} \right\}. \end{aligned}$$

The first term on the right-hand side of the above equation is zero because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}$. Hence

$$\begin{aligned} Q_i^K &= \frac{n}{2\pi^{3/2}} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} K \frac{2\Theta}{3g} \frac{\partial \ln \Theta}{\partial r_j} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{u}_i \tilde{u}_j \left(\tilde{u}^2 - \frac{5}{2}\right) \hat{\varPhi}_c(\tilde{u})\, \mathrm{e}^{-\tilde{u}^2} \\ &= \frac{n}{2\pi^{3/2}} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} K \frac{2\Theta}{3g} \frac{\partial \ln \Theta}{\partial r_i} \frac{4\pi}{3} \int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^6 \left(\tilde{u}^2 - \frac{5}{2}\right) \hat{\varPhi}_c(\tilde{u})\, \mathrm{e}^{-\tilde{u}^2} \quad (\text{using eq. (F.9b)}) \\ &= \left\{ \frac{4}{9} \left(\frac{2}{3\pi}\right)^{\frac{1}{2}} \int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^6 \left(\tilde{u}^2 - \frac{5}{2}\right) \hat{\varPhi}_c(\tilde{u})\, \mathrm{e}^{-\tilde{u}^2} \right\} n \ell \Theta^{1/2} \frac{\partial \Theta}{\partial r_i} \end{aligned}$$

or

$$\boxed{Q_i^K = -\tilde{\kappa}_0 n \ell \Theta^{1/2} \frac{\partial \Theta}{\partial r_i}} \tag{3.24}$$

where

$$\tilde{\kappa}_0 = -\frac{4}{9} \left(\frac{2}{3\pi}\right)^{\frac{1}{2}} \int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^6 \left(\tilde{u}^2 - \frac{5}{2}\right) \hat{\varPhi}_c(\tilde{u})\, \mathrm{e}^{-\tilde{u}^2}. \tag{3.25}$$

The integral in eq. (3.25) has been evaluated numerically. The result is $\tilde{\kappa}_0 \approx 0.4100$. Clearly (from eq. (3.24)), heat flux is Fourier's law of conduction at this order.

Collisional Dissipation

Eq. (2.13) implies that the contribution of Φ_K to collisional dissipation Γ is of $O(K\epsilon)$. From eq. (2.34), the contribution of Φ_K to collisional dissipation Γ is given by

$$\Gamma_{K\epsilon} = \frac{\epsilon\Theta}{12\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \left\{\Phi_K(\tilde{\boldsymbol{u}}_1) + \Phi_K(\tilde{\boldsymbol{u}}_2)\right\}.$$

But since

$$\begin{aligned}\tilde{u}_{12} = |\tilde{\boldsymbol{u}}_{12}| &= \sqrt{\left\{(\tilde{u}_{1x}-\tilde{u}_{2x})^2 + (\tilde{u}_{1y}-\tilde{u}_{2y})^2 + (\tilde{u}_{1z}-\tilde{u}_{2z})^2\right\}} \\ &= \sqrt{\left\{(\tilde{u}_{2x}-\tilde{u}_{1x})^2 + (\tilde{u}_{2y}-\tilde{u}_{1y})^2 + (\tilde{u}_{2z}-\tilde{u}_{1z})^2\right\}} = |\tilde{\boldsymbol{u}}_{21}| = \tilde{u}_{21},\end{aligned}$$

therefore interchanging the integration variables (i.e., $\tilde{\boldsymbol{u}}_1 \longleftrightarrow \tilde{\boldsymbol{u}}_2$),

$$\begin{aligned}&\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_K(\tilde{\boldsymbol{u}}_1) = \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}_1\, \tilde{u}_{21}^3\, \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}_1^2)}\, \Phi_K(\tilde{\boldsymbol{u}}_2) \\ &= \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}_1\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}_1^2)}\, \Phi_K(\tilde{\boldsymbol{u}}_2) = \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_K(\tilde{\boldsymbol{u}}_2).\end{aligned}$$

Therefore

$$\Gamma_{K\epsilon} = \frac{\epsilon\Theta}{6\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_K(\tilde{\boldsymbol{u}}_1).$$

Note that if one first integrates over $\tilde{\boldsymbol{u}}_2$ then clearly, $\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-\tilde{u}_2^2}$ is an isotropic function of $\tilde{u}_1$, i.e, one which depends on the speed $\tilde{u}_1$ alone. From eq. (G.4),

$$\chi(\tilde{u}_1) = \int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-\tilde{u}_2^2} = \pi\left[\left(\tilde{u}_1^2 + \frac{5}{2}\right)\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(3 + 12\tilde{u}_1^2 + 4\tilde{u}_1^4)}{4\tilde{u}_1}\operatorname{erf}(\tilde{u}_1)\right] \tag{3.26}$$

where $\operatorname{erf}(x)$ is the error function (Abramowitz & Stegun 1965, p. 297). Note that the right-hand side of the above equation is an even function in components of $\tilde{\boldsymbol{u}}_1$. Hence

$$\begin{aligned}\Gamma_{K\epsilon} &= \frac{\epsilon\Theta}{6\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1\; \mathrm{e}^{-\tilde{u}_1^2}\, \chi(\tilde{u}_1)\, \Phi_K(\tilde{\boldsymbol{u}}_1) \\ &= \frac{\epsilon\Theta}{6\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\; \mathrm{e}^{-\tilde{u}^2}\, \chi(\tilde{u}) \\ &\quad\times\left\{2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i}\right\}.\end{aligned}$$

Clearly, the term containing temperature gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}$. Hence

$$\Gamma_{K\epsilon} = K\epsilon\frac{2\Theta}{3g}\frac{\Theta}{3\pi^3\ell}\frac{\partial V_i}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}\; \mathrm{e}^{-\tilde{u}^2}\, \chi(\tilde{u})\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}$$

and using eq. (F.10),

$$\boxed{\Gamma_{K\epsilon} = 0} \tag{3.27}$$

i.e., the collisional dissipation is zero at this order.

3.3 Solution at $O(\epsilon)$

Collecting $O(\epsilon)$ terms in eq. (2.36), we have

$$\begin{aligned} &\tilde{\mathscr{D}}_\epsilon \ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{u}_i \tilde{\mathscr{D}}_\epsilon V_i + \left(\tilde{u}^2 - \frac{3}{2}\right) \tilde{\mathscr{D}}_\epsilon \ln \Theta \\ &= \epsilon\, \tilde{\mathscr{L}}(\varphi_1^{(1)}) + \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \end{aligned}$$

or

$$\begin{aligned} \epsilon\, \tilde{\mathscr{L}}(\varphi_1^{(1)}) &= \tilde{\mathscr{D}}_\epsilon \ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{u}_i \tilde{\mathscr{D}}_\epsilon V_i + \left(\tilde{u}^2 - \frac{3}{2}\right) \tilde{\mathscr{D}}_\epsilon \ln \Theta \\ &\quad - \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right). \end{aligned} \tag{3.28}$$

It is clear from eqs. (2.20)-(2.22) that $\tilde{\mathscr{D}}_\epsilon \ln n = \tilde{\mathscr{D}}_\epsilon V_i = 0$ and $\tilde{\mathscr{D}}_\epsilon \ln \Theta = -\epsilon \tilde{\Gamma}_0$, where $\tilde{\Gamma}_0 = \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}$ from eq. (2.33). The integral on the right-hand side of eq. (3.28) is evaluated in Appendix G. Its (final) value is given in eq. (G.6). Hence, substituting the values of each term in eq. (3.28), we get

$$\epsilon\, \tilde{\mathscr{L}}(\varphi_1^{(1)}) = \left(\tilde{u}^2 - \frac{3}{2}\right) \left\{-\epsilon \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\right\} - \epsilon \left[\left(\frac{3 - 2\tilde{u}^2}{8\,\pi^{1/2}}\right) \mathrm{e}^{-\tilde{u}^2} + \frac{(5 + 4\tilde{u}^2 - 4\tilde{u}^4)\,\mathrm{erf}(\tilde{u})}{16\tilde{u}}\right]$$

or

$$\tilde{\mathscr{L}}(\varphi_1^{(1)}) = h(\tilde{u}), \tag{3.29}$$

where

$$h(\tilde{u}) = -\left[\left(\frac{2}{\pi}\right)^{1/2} \left(\frac{2}{3}\tilde{u}^2 - 1\right) + \frac{(3 - 2\tilde{u}^2)}{8\pi^{1/2}}\, \mathrm{e}^{-\tilde{u}^2} + \frac{(5 + 4\tilde{u}^2 - 4\tilde{u}^4)\,\mathrm{erf}(\tilde{u})}{16\tilde{u}}\right]. \tag{3.30}$$

The right-hand side of eq. (3.29) is orthogonal to the invariants 1, $\tilde{\boldsymbol{u}}$ and $\tilde{u}^2$ with $\tilde{f}_0(\tilde{u})$ serving as weight function because with invariant $\tilde{\boldsymbol{u}}$, the integrand is an odd function in components of $\tilde{\boldsymbol{u}}$, and with 1 and $\tilde{u}^2$, (changing the integral into spherical polar coordinates) we have $\int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^2 h(\tilde{u})\, \mathrm{e}^{-\tilde{u}^2} = \int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^4 h(\tilde{u})\, \mathrm{e}^{-\tilde{u}^2} = 0$. That means eq. (3.29) is solvable. In Appendix B, it is shown that all equations that need to be solved in the framework of the perturbation theory employed in this work are solvable.

Again, the most general solution of eq. (3.29) is the sum of the general solution of the corresponding homogeneous equation $\tilde{\mathscr{L}}(\varphi_1^{(1)}) = 0$ and a particular solution of eq. (3.29). Since $\tilde{\mathscr{L}}$ is defined for elastically colliding particles, therefore by a similar argument as in §3.1,

$$\varphi^{(1)}_{1_{hom}} = b_1 + \boldsymbol{b}_2 \cdot \tilde{\boldsymbol{u}} + b_3\tilde{u}^2,$$

where subscript *hom* in $\varphi_1^{(1)}$ denotes that it is the solution of homogeneous equation, b_1 and b_3 are scalar constant, and $\boldsymbol{b}_2$ is a vector constant. Since the right-hand side of eq. (3.29) is a function of speed $\tilde{u}$ only, the particular solution of eq. (3.29) will be a function of speed $\tilde{u}$ only, i.e.,

$$\varphi^{(1)}_{1_{par}} = \hat{\Phi}_{in}(\tilde{u}),$$

where $\hat{\Phi}_{in}(\tilde{u})$ is an unknown function of speed $\tilde{u}$ only. Therefore the most general solution of eq. (3.29) is:

$$\varphi_1^{(1)}(\tilde{\boldsymbol{u}}) = b_1 + \boldsymbol{b}_2 \cdot \tilde{\boldsymbol{u}} + b_3\tilde{u}^2 + \hat{\Phi}_{in}(\tilde{u}) = b_{2i}\tilde{u}_i + \hat{\Phi}_e(\tilde{u}), \tag{3.31}$$

where $\hat{\Phi}_e(\tilde{u}) = b_1 + b_3\tilde{u}^2 + \hat{\Phi}_{in}(\tilde{u})$ is an unknown function of speed $\tilde{u}$ only. The orthogonality conditions for Φ_ϵ (cf. eq. (2.18)), which Φ_ϵ must satisfy, give the following:

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\psi\, \Phi_\epsilon = \epsilon \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\psi\, \varphi_1^{(1)}(\tilde{\boldsymbol{u}}) = \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\psi \left\{ b_{2i}\tilde{u}_i + \hat{\Phi}_e(\tilde{u}) \right\} = 0,$$

for $\psi = 1$, $\tilde{u}_j$ and $\tilde{u}^2$. Considering symmetry arguments, one can ignore the vanishing integrals and thus using eq. (2.17), the above orthogonality condition for $\psi = 1$, $\tilde{u}_j$ and $\tilde{u}^2$ gives the following equations:

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_e(\tilde{u}) = 0, \tag{3.32}$$

$$b_{2i} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{u}_i\tilde{u}_j = 0, \tag{3.33}$$

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_e(\tilde{u}) = 0. \tag{3.34}$$

From eq. (3.33), $b_{2i} = 0$. This implies that $\boldsymbol{b}_2 = 0$. Therefore eq. (3.31) becomes

$$\varphi_1^{(1)}(\tilde{\boldsymbol{u}}) = \hat{\Phi}_e(\tilde{u}) \tag{3.35}$$

with orthogonality conditions (3.32) and (3.34) to be satisfied. Thus

$$\boxed{\Phi_\epsilon(\tilde{\boldsymbol{u}}) = \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}) = \epsilon\,\hat{\Phi}_e(\tilde{u})} \tag{3.36}$$

with $\hat{\Phi}_e(\tilde{u})$ as an unknown functions of the speed $\tilde{u}$ only. Sela & Goldhirsch (1998) have expanded $\hat{\Phi}_e(\tilde{u})$ also in (truncated) series of modified Bessel functions of first kind by considering its asymptotic properties and the fact that $\hat{\Phi}_e(\tilde{u})$ is an even functions of speed $\tilde{u}$ (shown in Appendix I). Here we shall directly use the value of this unknown function also given in Sela & Goldhirsch (1998) to evaluate the integrals numerically (using MATHEMATICA).

3.4 Constitutive Relations at $O(\epsilon)$

Pressure Tensor

From eqs. (2.28) and (3.35), the contribution of Φ_ϵ to the pressure tensor is given by

$$\begin{aligned} P^{\epsilon}_{ij} &= \frac{2n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i\tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2}\, \Phi_\epsilon(\tilde{\boldsymbol{u}}) = \frac{2n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i\tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2}\left\{\epsilon\,\hat{\Phi}_e(\tilde{u})\right\} \\ &= \frac{2\epsilon n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i\tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_e(\tilde{u}) = \frac{2\epsilon n\Theta}{3\pi^{3/2}}\frac{1}{3}\delta_{ij}\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_e(\tilde{u}). \end{aligned}$$

Using the orthogonality condition (3.32),

$$\boxed{P^{\epsilon}_{ij} = 0} \tag{3.37}$$

Heat Flux

From eqs. (2.30) and (3.35), the contribution of Φ_ϵ to the heat flux is given by

$$Q^{\epsilon}_i = \frac{n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2\tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \Phi_\epsilon(\tilde{\boldsymbol{u}}) = \frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2\tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_e(\tilde{u}).$$

Since the integrand in the above integral is an odd function in components of $\tilde{\boldsymbol{u}}$, therefore

$$\boxed{Q^{\epsilon}_i = 0} \tag{3.38}$$

That means, Φ_ϵ neither contributes to the pressure tensor nor to the heat flux.

Collisional Dissipation

Eq. (2.13) implies that the contribution of Φ_ϵ to collisional dissipation Γ is of $O(\epsilon^2)$. From eq. (2.34), the contribution of Φ_ϵ to collisional dissipation Γ is given by

$$\Gamma_{\epsilon\epsilon} = \frac{\epsilon\Theta}{12\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\left\{\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)+\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)\right\}.$$

Again, (similar as above) on interchanging $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_\epsilon(\tilde{\boldsymbol{u}}_1) = \int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_\epsilon(\tilde{\boldsymbol{u}}_2).$$

Therefore

$$\begin{aligned} \Gamma_{\epsilon\epsilon} &= \frac{\epsilon\Theta}{6\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_\epsilon(\tilde{\boldsymbol{u}}_1) \\ &= \frac{\epsilon\Theta}{6\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\, \mathrm{e}^{-\tilde{u}_1^2}\, \chi(\tilde{u}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_1) = \frac{\epsilon^2\Theta}{6\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\, \mathrm{e}^{-\tilde{u}_1^2}\, \chi(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1), \end{aligned}$$

where the value of $\chi(\tilde{u}_1)$ is given in eq. (3.26). The integral over $\tilde{\boldsymbol{u}}_1$ can be carried out by changing it to spherical coordinate system to get

$$\Gamma_{\epsilon\epsilon} = \frac{\epsilon^2}{\ell}\Theta^{3/2}\frac{\sqrt{2}}{6\pi^3\sqrt{3}} \times 4\pi \int_0^\infty \mathrm{d}\tilde{u}_1\, \tilde{u}_1^2\, \mathrm{e}^{-\tilde{u}_1^2}\, \chi(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1).$$

The integral in the above equation has been evaluated numerically. The result is

$$\boxed{\Gamma_{\epsilon\epsilon} \approx -0.0102\,\frac{\epsilon^2}{\ell}\Theta^{3/2} \approx -0.0322\,\epsilon^2 n d^2 \Theta^{3/2}} \tag{3.39}$$

3.5 Summary

The solutions for the correction terms at $O(K)$ and $O(\epsilon)$ have been obtained in terms of the unknown functions, which depend on speed $\tilde{u}$ only. With the help of these correction terms, the constitutive relations at these orders have been derived. The facts about the constitutive relations obtained are the following.

At $O(K)$:

all the analysis is of Navier-Stokes level; expression for pressure tensor, i.e,

$$P_{ij}^K = -2\tilde{\mu}_0 n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}$$

is Newton's law of viscosity; expression for heat flux, i.e,

$$Q_i^K = -\tilde{\kappa}_0 n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i}$$

is Fourier's law of conduction; the contribution to dissipation term is zero, i.e, $\Gamma_{K\epsilon} = 0$.

At $O(\epsilon)$:

all the analysis is of Euler level; pressure tensor and heat flux vanish, i.e., $P_{ij}^\epsilon = Q_i^\epsilon = 0$, and the contribution to dissipation term is of $O(\epsilon^2)$ and given by

$$\Gamma_{\epsilon\epsilon} \approx -0.0102\,\frac{\epsilon^2}{\ell}\Theta^{3/2} \approx -0.0322\,\epsilon^2 n d^2\Theta^{3/2}.$$

The numerical values of coefficients obtained in the present work and that in Sela & Goldhirsch (1998) are given in Table 3.1.

Coefficients in:	**Coefficient**	**Sela & Goldhirsch (1998)**	**Present work**
P_{ij}^K	$\tilde{\mu}_0$	0.3249	0.3249
Q_i^K	$\tilde{\kappa}_0$	0.4101	0.4100
$\Gamma_{\epsilon\epsilon}$		−0.0352	−0.0322

Table 3.1: Comparision of coefficients

Chapter 4

Constitutive Relations at Second-order in small parameters

Theoretically speaking, obtaining the solutions for the correction terms at second-order in small parameters is possible but practically, it is extremely difficult to obtain due to plethora of terms appearing in the corresponding equations, which need to be solved. In fact, after seeing the number of terms in these equations one can feel how difficult is to even write these equations on a piece of paper. Nevertheless, in order to evaluate the constitutive relations, one does not need to find out the correction terms. The self-adjoint property of the linearized Boltzmann collision operator (defined in next section) can be used to directly evaluate the constitutive relations.

4.1 Self-adjoint property of $\tilde{\mathscr{L}}$

Let us define an inner-product[†] by

$$\langle \psi, \phi \rangle = \int \mathrm{e}^{-\tilde{u}^2}\, \psi(\tilde{\boldsymbol{u}})\phi(\tilde{\boldsymbol{u}})\, \mathrm{d}\tilde{\boldsymbol{u}}. \tag{4.1}$$

Recall that the operator $\tilde{\mathscr{L}}$ is defined for the elastically colliding particles. Therefore, in this section, it is assumed that the precollisional velocities of the particles are related to postcollisional velocities via elastic collisions. From eqs. (2.37) and (4.1),

$$\begin{aligned}\langle \Psi, \tilde{\mathscr{L}}(\Phi) \rangle = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} & \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Psi(\tilde{\boldsymbol{u}}_1) \\ & \times \{\Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') - \Phi(\tilde{\boldsymbol{u}}_1) - \Phi(\tilde{\boldsymbol{u}}_2)\}. \end{aligned} \tag{4.2}$$

Interchanging the variables $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$,

$$\begin{aligned}\langle \Psi, \tilde{\mathscr{L}}(\Phi) \rangle = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{21}>0} & \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{21})\, \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}_1^2)}\, \Psi(\tilde{\boldsymbol{u}}_2) \\ & \times \{\Phi(\tilde{\boldsymbol{u}}_2') + \Phi(\tilde{\boldsymbol{u}}_1') - \Phi(\tilde{\boldsymbol{u}}_2) - \Phi(\tilde{\boldsymbol{u}}_1)\}. \end{aligned}$$

Using eq. (F.8),

$$\begin{aligned}\langle \Psi, \tilde{\mathscr{L}}(\Phi) \rangle = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} & \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Psi(\tilde{\boldsymbol{u}}_2) \\ & \times \{\Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') - \Phi(\tilde{\boldsymbol{u}}_1) - \Phi(\tilde{\boldsymbol{u}}_2)\}. \end{aligned} \tag{4.3}$$

[†]When complex-valued functions are used the complex conjugate of ψ should appear in the right-hand side of eq. (4.1); this, however, is not important here, and we restrict ourselves to real-valued functions.

Adding eqs. (4.2) and (4.3),

$$
\begin{aligned}
2\langle \Psi, \tilde{\mathscr{L}}(\Phi)\rangle = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} & \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\Psi(\tilde{\boldsymbol{u}}_1)+\Psi(\tilde{\boldsymbol{u}}_2)\} \\
& \times \{\Phi(\tilde{\boldsymbol{u}}_1')+\Phi(\tilde{\boldsymbol{u}}_2')-\Phi(\tilde{\boldsymbol{u}}_1)-\Phi(\tilde{\boldsymbol{u}}_2)\}.
\end{aligned} \tag{4.4}
$$

Interchanging the dummy variables $\tilde{\boldsymbol{u}}_1 \longleftrightarrow \tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2 \longleftrightarrow \tilde{\boldsymbol{u}}_2'$ in the above equation,

$$
\begin{aligned}
2\langle \Psi, \tilde{\mathscr{L}}(\Phi)\rangle = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0} & \mathrm{d}\tilde{\boldsymbol{u}}_1' \mathrm{d}\tilde{\boldsymbol{u}}_2' \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\,\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\{\Psi(\tilde{\boldsymbol{u}}_1')+\Psi(\tilde{\boldsymbol{u}}_2')\} \\
& \times \{\Phi(\tilde{\boldsymbol{u}}_1)+\Phi(\tilde{\boldsymbol{u}}_2)-\Phi(\tilde{\boldsymbol{u}}_1')-\Phi(\tilde{\boldsymbol{u}}_2')\}.
\end{aligned}
$$

Using the relations $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}' = -\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}$, $\tilde{u}_1'^2+\tilde{u}_2'^2 = \tilde{u}_1^2+\tilde{u}_2^2$ and $\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2' = \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2$ (cf. eqs. (2.2), (2.3) and (J.3) respectively),

$$
\begin{aligned}
2\langle \Psi, \tilde{\mathscr{L}}(\Phi)\rangle = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}<0} & \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\Psi(\tilde{\boldsymbol{u}}_1')+\Psi(\tilde{\boldsymbol{u}}_2')\} \\
& \times \{\Phi(\tilde{\boldsymbol{u}}_1)+\Phi(\tilde{\boldsymbol{u}}_2)-\Phi(\tilde{\boldsymbol{u}}_1')-\Phi(\tilde{\boldsymbol{u}}_2')\}.
\end{aligned}
$$

Using eq. (F.8),

$$
\begin{aligned}
2\langle \Psi, \tilde{\mathscr{L}}(\Phi)\rangle = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} & \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\Psi(\tilde{\boldsymbol{u}}_1')+\Psi(\tilde{\boldsymbol{u}}_2')\} \\
& \times \{\Phi(\tilde{\boldsymbol{u}}_1)+\Phi(\tilde{\boldsymbol{u}}_2)-\Phi(\tilde{\boldsymbol{u}}_1')-\Phi(\tilde{\boldsymbol{u}}_2')\}.
\end{aligned} \tag{4.5}
$$

Adding eqs. (4.4) and (4.5) and dividing both sides by 4, we have

$$
\begin{aligned}
\langle \Psi, \tilde{\mathscr{L}}(\Phi)\rangle = -\frac{1}{4\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} & \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\
& \times \{\Psi(\tilde{\boldsymbol{u}}_1')+\Psi(\tilde{\boldsymbol{u}}_2')-\Psi(\tilde{\boldsymbol{u}}_1)-\Psi(\tilde{\boldsymbol{u}}_2)\}\{\Phi(\tilde{\boldsymbol{u}}_1')+\Phi(\tilde{\boldsymbol{u}}_2')-\Phi(\tilde{\boldsymbol{u}}_1)-\Phi(\tilde{\boldsymbol{u}}_2)\}.
\end{aligned} \tag{4.6}
$$

The right-hand side of eq. (4.6) shows that interchange of Ψ and Φ does not change the value of the inner-product on the left-hand side. Thus eq. (4.6) implies that

$$
\langle \Psi, \tilde{\mathscr{L}}(\Phi)\rangle = \langle \Phi, \tilde{\mathscr{L}}(\Psi)\rangle, \tag{4.7}
$$

i.e., the linearized Boltzmann collision operator $\tilde{\mathscr{L}}$ is self-adjoint with $\mathrm{e}^{-\tilde{u}^2}$ serving as a weight function.

4.2 Constitutive Relations at $O(K\epsilon)$

In the following, we shall first simplify the expanded Boltzmann equation at this order.

4.2.1 Simplified form of eq. (2.36)

Collecting $O(K\epsilon)$ terms in eq. (2.36), we have

$$\begin{aligned}
&\tilde{\mathscr{D}}_{K\epsilon}\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_{K\epsilon}V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathscr{D}}_{K\epsilon}\ln\Theta \\
&+ \Phi_K\left\{\tilde{\mathscr{D}}_{\epsilon}\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_{\epsilon}V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathscr{D}}_{\epsilon}\ln\Theta\right\} \\
&+ \Phi_{\epsilon}\left\{\tilde{\mathscr{D}}_{K}\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_{K}V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathscr{D}}_{K}\ln\Theta - 2K\tilde{g}_i\tilde{u}_i\right\} \\
&+ \tilde{\mathscr{D}}_K\Phi_{\epsilon} + \tilde{\mathscr{D}}_{\epsilon}\Phi_K + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\Phi_{\epsilon} \\
&= \epsilon\,\tilde{\mathscr{L}}(\varphi_K^{(1)}) + \epsilon\tilde{\Xi}(\Phi_K) + \epsilon\tilde{\Lambda}(\Phi_K) + \epsilon\tilde{\Omega}(\Phi_K,\varphi_1^{(1)}).
\end{aligned} \tag{4.8}$$

Let us first simplify the terms containing hydrodynamic variables in eq. (4.8). Clearly, the right-hand side of eq. (2.20) does not contain any term of order $K\epsilon$, this implies that $\tilde{\mathscr{D}}_{K\epsilon}\ln n = 0$ and since, from eq. (3.37) $P_{ij}^{\epsilon} = 0$, the right-hand side of eq. (2.21) also does not contribute to order $K\epsilon$, this implies that $\tilde{\mathscr{D}}_{K\epsilon}V_i = 0$. Moreover, in the light of eqs. (3.27) (equivalent of saying, $\tilde{\Gamma}_K = 0$), (3.37) and (3.38), the right-hand side of eq. (2.22) also does not contribute to order $K\epsilon$, this implies that $\tilde{\mathscr{D}}_{K\epsilon}\ln\Theta = 0$; the quantity $\tilde{\mathscr{D}}_{\epsilon}\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_{\epsilon}V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathscr{D}}_{\epsilon}\ln\Theta$ has appeared in the right-hand side of eq. (3.28) and its value is $\left(\tilde{u}^2 - \frac{3}{2}\right)\left\{-\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\right\}$ (see the equation following eq. (3.28)); the quantity $\tilde{\mathscr{D}}_K\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_K V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathscr{D}}_K\ln\Theta - 2K\tilde{g}_i\tilde{u}_i$ is the left-hand side of eq. (3.1), whose simplified value is the right-hand side of eq. (3.5). Next, we need to simplify $\tilde{\mathscr{D}}_K\Phi_{\epsilon}$, $\tilde{\mathscr{D}}_{\epsilon}\Phi_K$ and $K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\Phi_{\epsilon}$. Let us first consider $\tilde{\mathscr{D}}_K\Phi_{\epsilon}$. Consider the operation of $\tilde{\mathscr{D}}$ on eq. (3.36). Since $\tilde{\mathscr{D}}$ is a differential operator, using chain rule with respect to $\tilde{u}^2$, we have

$$\tilde{\mathscr{D}}\Phi_{\epsilon} = \epsilon\,\hat{\Phi}_e'(\tilde{u})\tilde{\mathscr{D}}(\tilde{u}^2), \tag{4.9}$$

where prime denotes the differentiation with respect to $\tilde{u}^2$. Hence

$$\tilde{\mathscr{D}}_K\Phi_{\epsilon} = \epsilon\,\hat{\Phi}_e'(\tilde{u})\tilde{\mathscr{D}}_K(\tilde{u}^2). \tag{4.10}$$

But, from eq. (E.3),

$$\begin{aligned}
&\tilde{\mathscr{D}}_K(\tilde{u}^2) \\
&= K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial\ln(n\Theta)}{\partial r_i} - \tilde{u}^2\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i} - 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{u}_j\frac{\partial V_i}{\partial r_j} + \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}^2\frac{\partial V_i}{\partial r_i}\right\} - 2K\tilde{g}_i\tilde{u}_i \\
&= K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial\ln n}{\partial r_i} - 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(\frac{\tilde{u}_i\tilde{u}_j + \tilde{u}_j\tilde{u}_i}{2} - \frac{1}{3}\tilde{u}_k^2\delta_{ij}\right)\frac{\partial V_i}{\partial r_j} - (\tilde{u}^2 - 1)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\right\} - 2K\tilde{g}_i\tilde{u}_i \\
&= K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial\ln n}{\partial r_i} - 2\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} - (\tilde{u}^2 - 1)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\right\} - 2K\tilde{g}_i\tilde{u}_i.
\end{aligned}$$

$$\Rightarrow \quad \tilde{\mathscr{D}}_K \Phi_\epsilon = K\epsilon \frac{2\Theta}{3g} \hat{\Phi}_e'(\tilde{u}) \left\{ \tilde{u}_i \frac{\partial \ln n}{\partial r_i} - 2\overline{\tilde{u}_i \tilde{u}_j} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j} - (\tilde{u}^2 - 1)\tilde{u}_i \frac{\partial \ln \Theta}{\partial r_i} \right\} - 2K\epsilon \hat{\Phi}_e'(\tilde{u}) \tilde{g}_i \tilde{u}_i. \tag{4.11}$$

Next, let us evaluate $\tilde{\mathscr{D}}_\epsilon \Phi_K$ with the help of Appendix D. Using eq. (D.1),

$$\begin{aligned}
\tilde{\mathscr{D}}_\epsilon \Phi_K = {} & 2\frac{2\Theta}{3g} \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j} \tilde{\mathscr{D}}_\epsilon K + 2K \frac{2\Theta}{3g} \overline{\tilde{u}_i \tilde{u}_j} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j} \hat{\Phi}_v'(\tilde{u}) \tilde{\mathscr{D}}_\epsilon \left(\tilde{u}^2\right) \\
& + 2K \frac{2\Theta}{3g} \hat{\Phi}_v(\tilde{u}) \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j} \tilde{\mathscr{D}}_\epsilon \left\{\overline{\tilde{u}_i \tilde{u}_j}\right\} + 2K \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j} \frac{\partial V_i}{\partial r_j} \frac{1}{g} \tilde{\mathscr{D}}_\epsilon \left\{ \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \right\} \\
& + 2K \frac{2\Theta}{3g} \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{\mathscr{D}}_\epsilon \left\{ \frac{\partial V_i}{\partial r_j} \right\} + \frac{2\Theta}{3g} \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i \frac{\partial \ln \Theta}{\partial r_i} \tilde{\mathscr{D}}_\epsilon K \\
& + K \frac{2\Theta}{3g} \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i \frac{\partial \ln \Theta}{\partial r_i} \hat{\Phi}_c'(\tilde{u}) \tilde{\mathscr{D}}_\epsilon \left(\tilde{u}^2\right) + K \frac{2\Theta}{3g} \hat{\Phi}_c(\tilde{u}) \tilde{u}_i \frac{\partial \ln \Theta}{\partial r_i} \tilde{\mathscr{D}}_\epsilon \left(\tilde{u}^2\right) \\
& + K \frac{2\Theta}{3g} \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \frac{\partial \ln \Theta}{\partial r_i} \tilde{\mathscr{D}}_\epsilon \tilde{u}_i + K \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i \frac{2}{3g} \tilde{\mathscr{D}}_\epsilon \left\{ \frac{\partial \Theta}{\partial r_i} \right\}.
\end{aligned} \tag{4.12}$$

Now, the action of $\tilde{\mathscr{D}}_\epsilon$ on various variables can be obtained with the help of eqs. (D.2)-(D.8) and the relations, $\tilde{\mathscr{D}}_\epsilon \ln n = \tilde{\mathscr{D}}_\epsilon V_i = 0$ and $\tilde{\mathscr{D}}_\epsilon \ln \Theta = -\epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2}$. Using eq. (D.2),

$$\tilde{\mathscr{D}}_\epsilon K = -K(\tilde{\mathscr{D}}_\epsilon \ln n + \tilde{\mathscr{D}}_\epsilon \ln \Theta) = -K \left\{ -\epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} \right\} = K\epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2}. \tag{4.13}$$

Using eq. (D.3),

$$\tilde{\mathscr{D}}_\epsilon (\tilde{u}^2) = -2 \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{u}_i \, \tilde{\mathscr{D}}_\epsilon V_i - \tilde{u}^2 \tilde{\mathscr{D}}_\epsilon \ln \Theta = -\tilde{u}^2 \left\{ -\epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} \right\} = \epsilon \tilde{u}^2 \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2}. \tag{4.14}$$

Using eq. (D.4),

$$\tilde{\mathscr{D}}_\epsilon \tilde{u}_i = -\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{\mathscr{D}}_\epsilon V_i - \frac{1}{2} \tilde{u}_i \tilde{\mathscr{D}}_\epsilon \ln \Theta = -\frac{1}{2} \tilde{u}_i \left\{ -\epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} \right\} = \frac{1}{2} \tilde{u}_i \left\{ \epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} \right\}. \tag{4.15}$$

Using eq. (D.5),

$$\begin{aligned}
\tilde{\mathscr{D}}_\epsilon \left(\overline{\tilde{u}_i \tilde{u}_j}\right) &= \tilde{u}_i \, \tilde{\mathscr{D}}_\epsilon \tilde{u}_j + \tilde{u}_j \, \tilde{\mathscr{D}}_\epsilon \tilde{u}_i - \frac{2}{3} \delta_{ij} \tilde{u}_k \, \tilde{\mathscr{D}}_\epsilon \tilde{u}_k \\
&= \tilde{u}_i \frac{1}{2} \tilde{u}_j \left\{ \epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} \right\} + \tilde{u}_j \frac{1}{2} \tilde{u}_i \left\{ \epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} \right\} - \frac{2}{3} \delta_{ij} \tilde{u}_k \frac{1}{2} \tilde{u}_k \left\{ \epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} \right\} \\
&= \epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{\tilde{u}_i \tilde{u}_j + \tilde{u}_j \tilde{u}_i}{2} - \frac{1}{3} \tilde{u}_k \tilde{u}_k \delta_{ij} \right) = \epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} \overline{\tilde{u}_i \tilde{u}_j}.
\end{aligned} \tag{4.16}$$

Using eq. (D.6),

$$\tilde{\mathscr{D}}_\epsilon \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} = \frac{1}{2} \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{\mathscr{D}}_\epsilon \ln \Theta = \frac{1}{2} \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \left\{ -\epsilon \frac{2}{3} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \right\} = -\frac{\epsilon}{2} \frac{2}{3} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}. \tag{4.17}$$

Using eq. (D.7),

$$\tilde{\mathscr{D}}_\epsilon \left(\frac{\partial V_i}{\partial r_j}\right) = \frac{\partial}{\partial r_j}\left(\tilde{\mathscr{D}}_\epsilon V_i\right) + \left(\tilde{\mathscr{D}}_\epsilon V_i\right)\frac{\partial}{\partial r_j}\left(\ln n + \frac{1}{2}\ln \Theta\right) = 0. \tag{4.18}$$

Using eq. (D.8),

$$\begin{aligned}
\tilde{\mathscr{D}}_\epsilon \left(\frac{\partial \Theta}{\partial r_i}\right) &= \frac{\partial}{\partial r_i}\left(\Theta\tilde{\mathscr{D}}_\epsilon \ln\Theta\right) + \left(\Theta\tilde{\mathscr{D}}_\epsilon \ln\Theta\right)\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln \Theta\right)\\
&= \frac{\partial}{\partial r_i}\left[\Theta\left\{-\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\right\}\right] + \Theta\left\{-\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\right\}\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln\Theta\right)\\
&= -\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\left[\Theta\frac{\partial \ln\Theta}{\partial r_i} + \Theta\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln\Theta\right)\right]\\
&= -\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\Theta\frac{\partial}{\partial r_i}\left(\ln n + \frac{3}{2}\ln\Theta\right).
\end{aligned} \tag{4.19}$$

Substituting these values in eq. (4.12), we get

$$\begin{aligned}
\tilde{\mathscr{D}}_\epsilon \Phi_K &= 2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\times K\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} + 2K\frac{2\Theta}{3g}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\hat{\Phi}'_v(\tilde{u})\times\epsilon\tilde{u}^2\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\\
&+ 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\times\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\overline{\tilde{u}_i\tilde{u}_j} + 2K\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial V_i}{\partial r_j}\frac{1}{g}\left\{-\frac{\epsilon}{2}\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\}\\
&+ 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\times 0 + \frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\times K\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\\
&+ K\frac{2\Theta}{3g}\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\hat{\Phi}'_c(\tilde{u})\times\epsilon\tilde{u}^2\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\times\epsilon\tilde{u}^2\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\\
&+ K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\times\frac{1}{2}\tilde{u}_i\left\{\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\right\}\\
&+ K\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{2}{3g}\left\{-\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\Theta\frac{\partial}{\partial r_i}\left(\ln n + \frac{3}{2}\ln\Theta\right)\right\}
\end{aligned}$$

or

$$\begin{aligned}
\tilde{\mathscr{D}}_\epsilon \Phi_K = K\epsilon\frac{2\Theta}{3g}\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}&\left[2\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} + 2\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\hat{\Phi}'_v(\tilde{u})\tilde{u}^2\right.\\
&+ 2\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} - \hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\underbrace{+\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\tilde{u}^2}\\
&+ \left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\hat{\Phi}'_c(\tilde{u})\tilde{u}^2 + \hat{\Phi}_c(\tilde{u})\tilde{u}^2\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\underbrace{+\frac{1}{2}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\tilde{u}^2}\\
&\left.- \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln n}{\partial r_i}\underbrace{-\frac{3}{2}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}}\right],
\end{aligned}$$

where the underbraces denote the terms cancelling each other. Therefore

$$\tilde{\mathscr{D}}_\epsilon \Phi_K = K\epsilon \frac{2\Theta}{3g}\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\left[2\left\{\hat{\Phi}_v'(\tilde{u})\tilde{u}^2 + \frac{3}{2}\hat{\Phi}_v(\tilde{u})\right\}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} + \left\{\hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right) + \hat{\Phi}_c(\tilde{u})\right\}\tilde{u}^2\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} - \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln n}{\partial r_i}\right]. \quad (4.20)$$

Now let us evaluate the last term, $K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\Phi_\epsilon$ on the left-hand side of eq. (4.8).

$$K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\Phi_\epsilon = K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\left\{\epsilon\hat{\Phi}_e(\tilde{u})\right\} = K\epsilon\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\frac{\partial}{\partial v_i}\hat{\Phi}_e(\tilde{u})$$
$$= K\epsilon\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\hat{\Phi}_e'(\tilde{u})\frac{\partial\tilde{u}^2}{\partial v_i},$$

where prime denotes the differentiation with respect to $\tilde{u}^2$ and

$$\frac{\partial\tilde{u}^2}{\partial v_i} = \frac{\partial}{\partial v_i}\left(\frac{3}{2\Theta}(\boldsymbol{v}-\boldsymbol{V})^2\right) = \frac{3}{2\Theta}\frac{\partial}{\partial v_i}(\boldsymbol{v}-\boldsymbol{V})^2 = \frac{3}{2\Theta}2(v_j - V_j)\frac{\partial}{\partial v_i}(v_j - V_j)$$
$$= \frac{3}{2\Theta}2(v_j - V_j)\frac{\partial v_j}{\partial v_i} = \frac{3}{2\Theta}2(v_j - V_j)\delta_{ji} = 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(v_i - V_i)\right\} = 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i.$$

Hence

$$K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\Phi_\epsilon = 2K\epsilon\hat{\Phi}_e'(\tilde{u})\tilde{g}_i\tilde{u}_i. \quad (4.21)$$

On substituting the above values along with the values of Φ_K and Φ_ϵ from eqs. (3.18) and (3.36) respectively, the left-hand side of eq. (4.8) changes to following:

$$\left\{2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\right\}\left(\tilde{u}^2 - \frac{3}{2}\right)\left\{-\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\right\}$$
$$+ \epsilon\hat{\Phi}_e(\tilde{u})\left\{2K\frac{2\Theta}{3g}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} + K\frac{2\Theta}{3g}\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\right\}$$
$$+ K\epsilon\frac{2\Theta}{3g}\hat{\Phi}_e'(\tilde{u})\left\{\tilde{u}_i\frac{\partial\ln n}{\partial r_i} - 2\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} - (\tilde{u}^2 - 1)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\right\} - 2K\epsilon\hat{\Phi}_e'(\tilde{u})\tilde{g}_i\tilde{u}_i$$
$$+ K\epsilon\frac{2\Theta}{3g}\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\left[2\left\{\hat{\Phi}_v'(\tilde{u})\tilde{u}^2 + \frac{3}{2}\hat{\Phi}_v(\tilde{u})\right\}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\right.$$
$$\left. + \left\{\hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right) + \hat{\Phi}_c(\tilde{u})\right\}\tilde{u}^2\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i} - \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln n}{\partial r_i}\right] + 2K\epsilon\hat{\Phi}_e'(\tilde{u})\tilde{g}_i\tilde{u}_i.$$

Thus simplying the left-hand side of eq. (4.8) further, it reduces to

$$
\begin{aligned}
&K\epsilon\frac{2\Theta}{3g}\left[\frac{4}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left\{-\hat{\Phi}_v(\tilde{u})\left(\tilde{u}^2-\frac{3}{2}\right)+\hat{\Phi}_v'(\tilde{u})\tilde{u}^2+\frac{3}{2}\hat{\Phi}_v(\tilde{u})\right\}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\right.\\
&+\left\{2\hat{\Phi}_e(\tilde{u})-2\hat{\Phi}_e'(\tilde{u})\right\}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}+\left\{-\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{3}{2}\right)\right.\\
&\left.+\hat{\Phi}_e(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)-\hat{\Phi}_e'(\tilde{u})(\tilde{u}^2-1)+\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left\{\hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)+\hat{\Phi}_c(\tilde{u})\right\}\tilde{u}^2\right\}\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\\
&\left.+\left\{\hat{\Phi}_e'(\tilde{u})-\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\right\}\tilde{u}_i\frac{\partial\ln n}{\partial r_i}\right]\\
&=K\epsilon\frac{2\Theta}{3g}\left[\frac{4}{3}\left\{\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\hat{\Phi}_v'(\tilde{u})\tilde{u}^2-\hat{\Phi}_v(\tilde{u})(\tilde{u}^2-3)\right)+\frac{3}{2}\left\{\hat{\Phi}_e(\tilde{u})-\hat{\Phi}_e'(\tilde{u})\right\}\right\}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\right.\\
&\quad+\left\{\hat{\Phi}_e(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)-\hat{\Phi}_e'(\tilde{u})(\tilde{u}^2-1)+\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\hat{\Phi}_c'(\tilde{u})\tilde{u}^2\left(\tilde{u}^2-\frac{5}{2}\right)\right.\right.\\
&\quad\left.\left.\left.-\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^4-5\tilde{u}^2+\frac{15}{4}\right)\right)\right\}\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}+\left\{\hat{\Phi}_e'(\tilde{u})-\frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\right\}\tilde{u}_i\frac{\partial\ln n}{\partial r_i}\right]\\
&\equiv\epsilon\,\tilde{S}_K.\quad\text{(let)}
\end{aligned}
\tag{4.22}
$$

Hence eq. (4.8) can be written as

$$
\epsilon\,\tilde{S}_K=\epsilon\,\tilde{\mathscr{L}}(\varphi_K^{(1)})+\epsilon\tilde{\Xi}(\Phi_K)+\epsilon\tilde{\Lambda}(\Phi_K)+\epsilon\tilde{\Omega}(\Phi_K,\varphi_1^{(1)})
$$

or

$$
\tilde{\mathscr{L}}(\varphi_K^{(1)})=\tilde{S}_K-\tilde{\Xi}(\Phi_K)-\tilde{\Lambda}(\Phi_K)-\tilde{\Omega}(\Phi_K,\varphi_1^{(1)}).
\tag{4.23}
$$

Eq. (4.23) is the final (simplified) form of the expanded Boltzmann equation and will be used to derive the constitutive relations at $O(K\epsilon)$.

4.2.2 Constitutive Relations

Heat Flux

From eq. (2.30), the contribution of $\Phi_{K\epsilon}$ to the heat flux is

$$
Q_i^{K\epsilon}=\frac{n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int\mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}^2\tilde{u}_i\,\mathrm{e}^{-\tilde{u}^2}\,\Phi_{K\epsilon}=\frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\varphi_K^{(1)}.
$$

The second equality in the above equation results from the orthogonality of $\Phi_{K\epsilon}$ to $\tilde{u}_i$, which is, $\int\mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{f}_0(\tilde{u})\tilde{u}_i\Phi_{K\epsilon}=0$ (cf. eq. (2.18)). Now, using eq. (3.21), one can write

$$
Q_i^{K\epsilon}=\frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\tilde{\mathscr{L}}\left[\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\right]\varphi_K^{(1)}
$$

and using the fact that $\tilde{\mathscr{L}}$ is self-adjoint with $\mathrm{e}^{-\tilde{u}^2}$ serving as a weight function (see §4.1), the above equation changes to

$$Q_i^{K\epsilon} = \frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \tilde{\mathscr{L}}(\varphi_K^{(1)}), \tag{4.24}$$

Substituting the value of $\tilde{\mathscr{L}}(\varphi_K^{(1)})$ from eq. (4.23) in the above equation,

$$\begin{aligned} Q_i^{K\epsilon} &= \frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2} \left\{\tilde{S}_K - \tilde{\varXi}(\varPhi_K) - \tilde{\varLambda}(\varPhi_K) - \tilde{\varOmega}(\varPhi_K, \varphi_1^{(1)})\right\} \\ &\equiv Q_{i_1}^{K\epsilon} + Q_{i_2}^{K\epsilon} + Q_{i_3}^{K\epsilon}, \quad \text{(let)} \end{aligned} \tag{4.25}$$

where

$$Q_{i_1}^{K\epsilon} = \frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{S}_K, \tag{4.26}$$

$$Q_{i_2}^{K\epsilon} = -\frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2} \left\{\tilde{\varXi}(\varPhi_K) + \tilde{\varLambda}(\varPhi_K)\right\}, \tag{4.27}$$

$$Q_{i_3}^{K\epsilon} = -\frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{\varOmega}(\varPhi_K, \varphi_1^{(1)}). \tag{4.28}$$

Now let us evaluate the quantities $Q_{i_1}^{K\epsilon}$, $Q_{i_2}^{K\epsilon}$ and $Q_{i_3}^{K\epsilon}$ as following. Substituting the explicit expression of $\tilde{S}_K$ from eq. (4.22) into eq. (4.26), we see that the term containing velocity gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}$. Hence eq. (4.26) reduces to

$$\begin{aligned} Q_{i_1}^{K\epsilon} = \frac{\epsilon n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} K\frac{2\Theta}{3g} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2} \Bigg[\Bigg\{\hat{\varPhi}_e(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) - \hat{\varPhi}_e'(\tilde{u})(\tilde{u}^2 - 1) \\ + \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left\{\hat{\varPhi}_c'(\tilde{u})\tilde{u}^2 \left(\tilde{u}^2 - \frac{5}{2}\right) - \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^4 - 5\tilde{u}^2 + \frac{15}{4}\right)\right\}\Bigg\} \tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} \\ + \left\{\hat{\varPhi}_e'(\tilde{u}) - \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right)\right\} \tilde{u}_j \frac{\partial \ln n}{\partial r_j}\Bigg] \end{aligned}$$

or

$$\begin{aligned} Q_{i_1}^{K\epsilon} = \epsilon\ell \frac{n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \frac{\partial \ln \Theta}{\partial r_j} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i \tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2} \Bigg[\hat{\varPhi}_e(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \\ - \hat{\varPhi}_e'(\tilde{u})(\tilde{u}^2 - 1) + \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left\{\hat{\varPhi}_c'(\tilde{u})\tilde{u}^2 \left(\tilde{u}^2 - \frac{5}{2}\right) - \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^4 - 5\tilde{u}^2 + \frac{15}{4}\right)\right\}\Bigg] \\ + \epsilon\ell \frac{n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \frac{\partial \ln n}{\partial r_j} \\ \times \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i \tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2} \left\{\hat{\varPhi}_e'(\tilde{u}) - \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right)\right\}. \end{aligned}$$

Using eq. (F.9b),

$$Q_{i_1}^{K\epsilon} = \epsilon\ell \frac{n}{2\pi^{3/2}} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \frac{4\pi}{3} \frac{\partial \ln \Theta}{\partial r_i} \int_0^\infty \mathrm{d}\tilde{u}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}^4\, \mathrm{e}^{-\tilde{u}^2} \left[\hat{\Phi}_e(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) - \hat{\Phi}_e'(\tilde{u})(\tilde{u}^2 - 1) + \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left\{\hat{\Phi}_c'(\tilde{u})\tilde{u}^2 \left(\tilde{u}^2 - \frac{5}{2}\right) - \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^4 - 5\tilde{u}^2 + \frac{15}{4}\right)\right\}\right]$$
$$+ \epsilon\ell \frac{n}{2\pi^{3/2}} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \frac{4\pi}{3} \frac{\partial \ln n}{\partial r_i} \times \int_0^\infty \mathrm{d}\tilde{u}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}^4\, \mathrm{e}^{-\tilde{u}^2} \left\{\hat{\Phi}_e'(\tilde{u}) - \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right)\right\}$$

or

$$Q_{i_1}^{K\epsilon} = \alpha_1 \epsilon n \ell \Theta^{1/2} \frac{\partial \Theta}{\partial r_i} + \beta_1 \epsilon \ell \Theta^{3/2} \frac{\partial n}{\partial r_i}, \tag{4.29}$$

where

$$\alpha_1 = \frac{4}{9}\left(\frac{2}{3\pi}\right)^{\frac{1}{2}} \int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^4 \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \mathrm{e}^{-\tilde{u}^2} \left[\hat{\Phi}_e(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) - \hat{\Phi}_e'(\tilde{u})(\tilde{u}^2 - 1) + \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left\{\hat{\Phi}_c'(\tilde{u})\tilde{u}^2 \left(\tilde{u}^2 - \frac{5}{2}\right) - \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^4 - 5\tilde{u}^2 + \frac{15}{4}\right)\right\}\right] \tag{4.30}$$

and

$$\beta_1 = \frac{4}{9}\left(\frac{2}{3\pi}\right)^{\frac{1}{2}} \int_0^\infty \mathrm{d}\tilde{u}\, \tilde{u}^4 \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \mathrm{e}^{-\tilde{u}^2} \left\{\hat{\Phi}_e'(\tilde{u}) - \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right)\right\}. \tag{4.31}$$

The integrals in eqs. (4.30) and (4.31) have been evaluated numerically. The result is $\alpha_1 \approx -0.3627$ and $\beta_1 \approx -0.2110$. The second term contributing to $Q_i^{K\epsilon}$ is given by eq. (4.27). The reason behind considering $\tilde{\Xi}$ and $\tilde{\Lambda}$ together in eq. (4.27) is that this way one obtains a cancelation of terms. Substituting the values of $\tilde{\Xi}$ and $\tilde{\Lambda}$ from eqs. (2.39) and (2.40) respectively in eq. (4.27), one obtains

$$Q_{i_2}^{K\epsilon} = -\frac{\epsilon n}{2\pi^4} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \times \Big(\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2')\Big) \hat{\Phi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i}$$
$$- \frac{\epsilon n}{2\pi^4} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0} \frac{\partial}{\partial \epsilon} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \times \Big(\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2')\Big) \hat{\Phi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i}.$$

Note that in the above equation, the velocity transformation is characterized by elastic collisons in the first integral while that is characterized by inelastic collisons in the second integral. Hence, to change the integration variables from postcollisional velocities to precollisional velocities—in the first integral, relations $\mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 = \mathrm{d}\tilde{\boldsymbol{u}}_1' \mathrm{d}\tilde{\boldsymbol{u}}_2'$, $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12} = -\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'$ and $\tilde{u}_1^2 + \tilde{u}_2^2 = \tilde{u}_1'^2 + \tilde{u}_2'^2$ are employed, and in the second integral, relations $\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2 = e\,\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'$, $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12} = -e(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')$ and $\tilde{u}_1^2+\tilde{u}_2^2 = \tilde{u}_1'^2+\tilde{u}_2'^2 - \frac{1}{2}\epsilon(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2$ are employed. In this way, one obtains

$$\begin{aligned}
Q_{i_2}^{K\epsilon} &= -\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int_{-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0}\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\left(1-\frac{1}{2}(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2\right)\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\\
&\quad\times\left(\Phi_K(\tilde{\boldsymbol{u}}_1')+\Phi_K(\tilde{\boldsymbol{u}}_2')\right)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\\
&\quad-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{-e\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0}e\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(-e\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\,\mathrm{e}^{-\left(\tilde{u}_1'^2+\tilde{u}_2'^2-\frac{1}{2}\epsilon(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2\right)}\\
&\quad\times\left(\Phi_K(\tilde{\boldsymbol{u}}_1')+\Phi_K(\tilde{\boldsymbol{u}}_2')\right)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\\
&= -\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0}\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\left(1-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2\right)\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\\
&\quad\times\left(\Phi_K(\tilde{\boldsymbol{u}}_1')+\Phi_K(\tilde{\boldsymbol{u}}_2')\right)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\\
&\quad-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0}\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\left\{(1-\epsilon)\,\mathrm{e}^{\frac{1}{2}\epsilon(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2}\right\}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\,\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\\
&\quad\times\left(\Phi_K(\tilde{\boldsymbol{u}}_1')+\Phi_K(\tilde{\boldsymbol{u}}_2')\right)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}.\quad\text{(using eq. (F.8) and } \epsilon = 1-e^2)
\end{aligned}$$

Since,

$$\begin{aligned}
(1-\epsilon)\,\mathrm{e}^{\frac{1}{2}\epsilon(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2} &= (1-\epsilon)\left(1+\frac{1}{2}\epsilon(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2+\frac{1}{2!4}\epsilon^2(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^4+\dots\right)\\
&= 1+\epsilon\left(\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2-1\right)+\epsilon^2\left(\frac{1}{2!4}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^4-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2\right)+\dots,
\end{aligned}$$

therefore

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0}\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\left\{(1-\epsilon)\,\mathrm{e}^{\frac{1}{2}\epsilon(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2}\right\}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\,\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\left(\Phi_K(\tilde{\boldsymbol{u}}_1')+\Phi_K(\tilde{\boldsymbol{u}}_2')\right)\\
&\quad\times\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\\
&=\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0}\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\,\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\left(\Phi_K(\tilde{\boldsymbol{u}}_1')+\Phi_K(\tilde{\boldsymbol{u}}_2')\right)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\\
&\quad+\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\Bigg\{\epsilon\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0}\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\left(\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2-1\right)(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\,\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\\
&\quad\times\left(\Phi_K(\tilde{\boldsymbol{u}}_1')+\Phi_K(\tilde{\boldsymbol{u}}_2')\right)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\Bigg\}\\
&\quad+\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\Bigg\{\epsilon^2\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0}\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\left(\frac{1}{2!4}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^4-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2\right)(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\,\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\\
&\quad\times\left(\Phi_K(\tilde{\boldsymbol{u}}_1')+\Phi_K(\tilde{\boldsymbol{u}}_2')\right)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\Bigg\}+\dots
\end{aligned}$$

or

$$\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}'_1\mathrm{d}\tilde{\boldsymbol{u}}'_2\mathrm{d}\hat{\boldsymbol{k}}\left\{(1-\epsilon)\,\mathrm{e}^{\frac{1}{2}\epsilon(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})^2}\right\}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})\,\mathrm{e}^{-(\tilde{u}'^2_1+\tilde{u}'^2_2)}\Big(\Phi_K(\tilde{\boldsymbol{u}}'_1)+\Phi_K(\tilde{\boldsymbol{u}}'_2)\Big)$$
$$\times\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}$$
$$=\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}'_1\mathrm{d}\tilde{\boldsymbol{u}}'_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})\,\mathrm{e}^{-(\tilde{u}'^2_1+\tilde{u}'^2_2)}\Big(\Phi_K(\tilde{\boldsymbol{u}}'_1)+\Phi_K(\tilde{\boldsymbol{u}}'_2)\Big)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}$$
$$+\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}'_1\mathrm{d}\tilde{\boldsymbol{u}}'_2\mathrm{d}\hat{\boldsymbol{k}}\left(\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})^2-1\right)(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})\,\mathrm{e}^{-(\tilde{u}'^2_1+\tilde{u}'^2_2)}\Big(\Phi_K(\tilde{\boldsymbol{u}}'_1)+\Phi_K(\tilde{\boldsymbol{u}}'_2)\Big)$$
$$\times\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}.$$

In the above equation other terms have vanished due to limit. Substituting this value in the expression of $Q_{i_2}^{K\epsilon}$, we get

$$Q_{i_2}^{K\epsilon}=-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}'_1\mathrm{d}\tilde{\boldsymbol{u}}'_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})\left(1-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})^2\right)\mathrm{e}^{-(\tilde{u}'^2_1+\tilde{u}'^2_2)}$$
$$\times\Big(\Phi_K(\tilde{\boldsymbol{u}}'_1)+\Phi_K(\tilde{\boldsymbol{u}}'_2)\Big)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}$$
$$-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}'_1\mathrm{d}\tilde{\boldsymbol{u}}'_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})\,\mathrm{e}^{-(\tilde{u}'^2_1+\tilde{u}'^2_2)}$$
$$\times\Big(\Phi_K(\tilde{\boldsymbol{u}}'_1)+\Phi_K(\tilde{\boldsymbol{u}}'_2)\Big)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}$$
$$-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}'_1\mathrm{d}\tilde{\boldsymbol{u}}'_2\mathrm{d}\hat{\boldsymbol{k}}\left(\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})^2-1\right)(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})\,\mathrm{e}^{-(\tilde{u}'^2_1+\tilde{u}'^2_2)}$$
$$\times\Big(\Phi_K(\tilde{\boldsymbol{u}}'_1)+\Phi_K(\tilde{\boldsymbol{u}}'_2)\Big)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}.$$

In the above equation, the first integral uses elastic velocity transformation and the limit $\epsilon\to 0$ in the third integral makes the velocity transformation elastic. Hence the two integrals are equal in magnitude and therefore the corresponding terms will cancel each other. Thus, one obtains

$$Q_{i_2}^{K\epsilon}=-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}'_1\mathrm{d}\tilde{\boldsymbol{u}}'_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})\,\mathrm{e}^{-(\tilde{u}'^2_1+\tilde{u}'^2_2)}$$
$$\times\Big(\Phi_K(\tilde{\boldsymbol{u}}'_1)+\Phi_K(\tilde{\boldsymbol{u}}'_2)\Big)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}.$$

Now, by renaming the variables $\tilde{\boldsymbol{u}}'_1$ and $\tilde{\boldsymbol{u}}'_2$ to $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ respectively and vice-versa, the above equation can be written as

$$Q_{i_2}^{K\epsilon}=-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}^2_1+\tilde{u}^2_2)}$$
$$\times\Big(\Phi_K(\tilde{\boldsymbol{u}}_1)+\Phi_K(\tilde{\boldsymbol{u}}_2)\Big)\hat{\Phi}_c(\tilde{u}'_1)\left(\tilde{u}'^2_1-\frac{5}{2}\right)\tilde{u}'_{1i}. \tag{4.32}$$

Remember that the variables are just renamed in eq. (4.32); the non-primed velocities are now precollisional velocities and the primed velocities are now postcollisional velocities. To simplify the integral in eq. (4.32) further, we shall make use of delta function and write the primed quantities in terms of corresponding non-primed quantities as below.

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,\delta(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1')\,\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i=\hat{\varPhi}_c(\tilde{u}_1')\left(\tilde{u}_1'^2-\frac{5}{2}\right)\tilde{u}_{1i}'.$$

Since $\tilde{\boldsymbol{u}}_1'$ is the postcollisional velocity here, the left-hand side of the above equation can be written in terms of precollisional velocity (cf. eq. (2.1a) for relation between postcollisional and precollisional velocity) as

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,\delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1+q(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big)\,\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i=\hat{\varPhi}_c(\tilde{u}_1')\left(\tilde{u}_1'^2-\frac{5}{2}\right)\tilde{u}_{1i}',$$

where $q=\frac{1+e}{2}$. Using the above equation, eq. (4.32) can be written as

$$\begin{aligned}
Q_{i_2}^{K\epsilon}&=-\frac{\epsilon n}{2\pi^4}\left(\frac{2\varTheta}{3}\right)^{\frac{3}{2}}\lim_{\epsilon\to0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\Big(\varPhi_K(\tilde{\boldsymbol{u}}_1)+\varPhi_K(\tilde{\boldsymbol{u}}_2)\Big)\\
&\quad\times\int \mathrm{d}\tilde{\boldsymbol{u}}\,\delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1+q(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big)\,\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\\
&=-\frac{\epsilon n}{2\pi^4}\left(\frac{2\varTheta}{3}\right)^{\frac{3}{2}}\lim_{\epsilon\to0}\frac{\partial}{\partial\epsilon}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\Big(\varPhi_K(\tilde{\boldsymbol{u}}_1)+\varPhi_K(\tilde{\boldsymbol{u}}_2)\Big)\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_iI_\delta,
\end{aligned}\tag{4.33}$$

where

$$I_\delta=\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1+q(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big)\,.\tag{4.34}$$

The integral in eq. (4.33) is then split into two parts. The first part is

$$\begin{aligned}
(I)&=-\frac{\epsilon n}{2\pi^4}\left(\frac{2\varTheta}{3}\right)^{\frac{3}{2}}\lim_{\epsilon\to0}\frac{\partial}{\partial\epsilon}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\varPhi_K(\tilde{\boldsymbol{u}}_1)\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_iI_\delta\\
&=-\frac{\epsilon n}{2\pi^4}\left(\frac{2\varTheta}{3}\right)^{\frac{3}{2}}\lim_{\epsilon\to0}\frac{\partial}{\partial\epsilon}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_iI_\delta\\
&\quad\times\left[2K\frac{2\varTheta}{3g}\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1j}\tilde{u}_{1k}}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_k}+K\frac{2\varTheta}{3g}\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1j}\frac{\partial\ln\varTheta}{\partial r_j}\right].
\end{aligned}\tag{4.35}$$

Let us define a new variable $\tilde{\boldsymbol{s}}\equiv\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1$. We shall replace the variable $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}$ in the above equation. Note that, $\mathrm{d}\tilde{\boldsymbol{u}}_1=-\mathrm{d}\tilde{\boldsymbol{s}}$ and the limits of integration for each component of $\tilde{\boldsymbol{s}}$ will be ∞ to $-\infty$. Hence, making the limits of each component of $\tilde{\boldsymbol{s}}$ as $-\infty$ to ∞ by using the property: $\int_a^b f(x)\,\mathrm{d}x=-\int_b^a f(x)\,\mathrm{d}x$ and eq. (F.3), one can write

$$
\begin{aligned}
(I) = &-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\lim_{\epsilon\to 0}\frac{\partial}{\partial \epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta \\
&\times \hat{\varPhi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_j-\tilde{s}_j)(\tilde{u}_k-\tilde{s}_k) \\
&-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial \epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta \\
&\times \hat{\varPhi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_j-\tilde{s}_j).
\end{aligned}
$$

In both the integrals, we shall first integrate over $\tilde{\boldsymbol{u}}_2$ as following. Let the spherical coordinates of $\tilde{\boldsymbol{s}}$ in the original coordinate system be $(\tilde{s},\theta_{\tilde{s}},\phi_{\tilde{s}})$ and the integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2,\theta_2',\phi_2')$, which results from two rotations: (i) rotation of xy-plane in positive direction (x towards y) around z-axis by an angle $\phi_{\tilde{s}}$ and (ii) rotation of new zx-plane in positive direction (z towards x) around new y-axis by an angle $\theta_{\tilde{s}}$, so that $\tilde{\boldsymbol{s}}$ coincides with the new z-axis and $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_2=\tilde{s}\tilde{u}_2\cos\theta_2'$ (see Appendix H). Hence

$$
\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}I_\delta = \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-\tilde{u}_2^2}I_\delta\,\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2 = 2\pi\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^2\,\mathrm{e}^{-\tilde{u}_2^2}\left(\int_{\theta_2'=0}^{\pi}\sin\theta_2'\,I_\delta\right)\mathrm{d}\tilde{u}_2.
$$

From eq. (C.12),

$$
\int_{\theta_2'=0}^{\pi}\sin\theta_2'\,I_\delta = \frac{1}{q^2\tilde{s}\tilde{u}_2}H\left(\tilde{u}_2-\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|\right).
$$

$$
\begin{aligned}
\therefore\quad \int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,I_\delta &= 2\pi\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^2\,\mathrm{e}^{-\tilde{u}_2^2}\,\frac{1}{q^2\tilde{s}\tilde{u}_2}H\left(\tilde{u}_2-\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|\right)\mathrm{d}\tilde{u}_2 \\
&= \frac{2\pi}{q^2\tilde{s}}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2 = \frac{\pi}{q^2\tilde{s}}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(I) = &-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\lim_{\epsilon\to 0}\frac{\partial}{\partial \epsilon}\frac{\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2} \\
&\times \hat{\varPhi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i(\tilde{u}_j-\tilde{s}_j)(\tilde{u}_k-\tilde{s}_k) \\
&-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial \epsilon}\frac{\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2} \\
&\times \hat{\varPhi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\varPhi}_c(\tilde{u})\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i(\tilde{u}_j-\tilde{s}_j).
\end{aligned}
$$

Let the spherical coordinates of $\tilde{\boldsymbol{u}}$ in the original coordinate system be $(\tilde{u},\theta_{\tilde{u}},\phi_{\tilde{u}})$ and the integration over $\tilde{\boldsymbol{s}}$ is performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$, which results from two rotations: (i) rotation of xy-plane in positive direction (x towards y) around z-axis by an angle $\phi_{\tilde{u}}$ and (ii) rotation of new zx-plane in positive direction (z towards x) around new y-axis by an angle $\theta_{\tilde{u}}$, so that $\tilde{\boldsymbol{u}}$ coincides with the new z-axis and $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}=\tilde{s}\tilde{u}\cos\theta'$ (see Appendix H). Hence

$$
\begin{aligned}
(I) = &-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\times\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\\
&\times\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}\hat{\Phi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i(\tilde{u}_j-\tilde{s}_j)(\tilde{u}_k-\tilde{s}_k)\\
&-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\\
&\times\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}\hat{\Phi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big)\hat{\Phi}_c(\tilde{u})\\
&\times\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i(\tilde{u}_j-\tilde{s}_j).
\end{aligned}
$$

Note that the components of $\tilde{\boldsymbol{s}}$ are the only functions of ϕ' (see Appendix H). The values of integrations over ϕ' are given in eqs. (H.19) and (H.6) respectively. Substituting these values in the above equation, we get

$$
\begin{aligned}
(I) = &-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\times\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\\
&\times\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}\hat{\Phi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\\
&\times 2\pi\frac{1}{\tilde{u}^2}\overline{\frac{\partial V_j}{\partial r_k}}\tilde{u}_j\tilde{u}_k\left\{\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\frac{1}{2}\tilde{s}^2(3\cos^2\theta'-1)\right\}\\
&-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\\
&\times\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}\hat{\Phi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big)\\
&\times\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i(2\pi\tilde{u}_j-2\pi\frac{\tilde{s}}{\tilde{u}}\tilde{u}_j\cos\theta').
\end{aligned}
$$

Clearly, the term containing velocity gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}$ and let $\cos\theta' = y$. Therefore

$$
\begin{aligned}
(I) = &-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{2\pi^2}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}\\
&\times\hat{\Phi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\left(1-\frac{\tilde{s}}{\tilde{u}}y\right).
\end{aligned}
$$

Using eq. (F.9b), the integration over $\tilde{\boldsymbol{u}}$ results into

$$
\begin{aligned}
(I) = &-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{2\pi^2}{q^2}\frac{4\pi}{3}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{y=-1}^{1}\mathrm{d}y\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\,\tilde{s}\\
&\times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}\hat{\Phi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\\
&\times\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^4\left(1-\frac{\tilde{s}}{\tilde{u}}y\right)
\end{aligned}
$$

or

$$(I) = -\frac{4}{3\pi}\left(\frac{2}{3}\right)^{\frac{3}{2}} \epsilon n \ell \Theta^{1/2} \frac{\partial \Theta}{\partial r_i} \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \frac{1}{q^2} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{y=-1}^{1} \mathrm{d}y \, \tilde{s} \, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)} \, \mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2}$$
$$\times \; \hat{\Phi}_c\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\right) \times \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right)\left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}^3 (\tilde{u} - \tilde{s}y). \quad (4.36)$$

The differentiation with respect to ϵ at $\epsilon = 0$ is carried out next. Note that $q \to 1$ as $\epsilon \to 0$.

$$\because \quad q = \frac{1+e}{2}, \; \epsilon = 1 - e^2 \quad \Rightarrow \quad q = \frac{1+\sqrt{1-\epsilon}}{2} \quad \Rightarrow \quad \lim_{\epsilon \to 0} \frac{\partial q}{\partial \epsilon} = -\lim_{\epsilon \to 0} \frac{1}{4\sqrt{1-\epsilon}} = -\frac{1}{4},$$

therefore

$$\lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon}\left[\frac{1}{q^2} \mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2}\right] = \lim_{\epsilon \to 0}\left[\frac{2\tilde{s}\left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right) \mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2} - 2q\, \mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2}}{q^4} \times \frac{\partial q}{\partial \epsilon}\right]$$
$$= \left\{2\tilde{s}\tilde{u}y \, \mathrm{e}^{-\tilde{u}^2 y^2} - 2\, \mathrm{e}^{-\tilde{u}^2 y^2}\right\}\left(-\frac{1}{4}\right) = \frac{1}{2}(1 - \tilde{s}\tilde{u}y)\, \mathrm{e}^{-\tilde{u}^2 y^2}. \quad (4.37)$$

Using the above equation, (I) can be written as

$$(I) = -\frac{2}{3\pi}\left(\frac{2}{3}\right)^{\frac{3}{2}} \epsilon n \ell \Theta^{1/2} \frac{\partial \Theta}{\partial r_i} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{y=-1}^{1} \mathrm{d}y \, \tilde{s}\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)} (1 - \tilde{s}\tilde{u}y)\, \mathrm{e}^{-\tilde{u}^2 y^2}$$
$$\times \hat{\Phi}_c\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\right) \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right)\left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}^3 (\tilde{u} - \tilde{s}y)$$
$$= \alpha_2 \epsilon n \ell \Theta^{1/2} \frac{\partial \Theta}{\partial r_i}, \quad (4.38)$$

where

$$\alpha_2 = -\frac{4\sqrt{2}}{9\pi\sqrt{3}} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{y=-1}^{1} \mathrm{d}y \, \tilde{s}\tilde{u}^3 (\tilde{u} - \tilde{s}y)(1 - \tilde{s}\tilde{u}y) \hat{\Phi}_c\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\right)$$
$$\times \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right)\left(\tilde{u}^2 - \frac{5}{2}\right) \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \mathrm{e}^{-\tilde{u}^2 y^2}. \quad (4.39)$$

The integrals in eq. (4.39) have been evaluated numerically. The result is $\alpha_2 \approx -0.0282$. The second part of eq. (4.33) is

$$(II) = -\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)}\, \Phi_K(\tilde{\boldsymbol{u}}_2) \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i I_\delta$$
$$= -\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i I_\delta$$
$$\times \left[2K \frac{2\Theta}{3g} \hat{\Phi}_v(\tilde{u}_2) \overline{\tilde{u}_{2j}\tilde{u}_{2k}} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_k} + K \frac{2\Theta}{3g} \hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_{2j} \frac{\partial \ln \Theta}{\partial r_j}\right]. \quad (4.40)$$

We follow the same procedure as above. Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}$ in the above equation and using eq. (F.3), we get

$$\begin{aligned}(II) = &-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2j}\tilde{u}_{2k}I_\delta \\ &-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i \\ &\times\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2j}I_\delta. \end{aligned} \tag{4.41}$$

The integrations over $\tilde{\boldsymbol{u}}_2$ are performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_2 = \tilde{s}\tilde{u}_2\cos\theta_2'$, i.e.,

$$\begin{aligned}(II) = &-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i \\ &\times\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2j}\tilde{u}_{2k}I_\delta \\ &-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i \\ &\times\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2j}I_\delta. \end{aligned}$$

The integrations over ϕ_2' result into (cf. eqs. (H.18) and (H.6) respectively)

$$\begin{aligned}(II) = &-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i \\ &\times\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{\pi\frac{\tilde{u}_2^2}{\tilde{s}^2}\tilde{s}_j\tilde{s}_k(3\cos^2\theta_2'-1)\right\}I_\delta \\ &-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i \\ &\times\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(2\pi\tilde{u}_2\frac{\tilde{s}_j}{\tilde{s}}\cos\theta_2'\right)I_\delta. \end{aligned}$$

Using eq. (C.12),

$$\begin{aligned}&\int_{\theta_2'=0}^{\pi}\sin\theta_2'(3\cos^2\theta_2'-1)I_\delta\,\mathrm{d}\theta_2' \\ &\qquad=\frac{1}{q^2\tilde{s}\tilde{u}_2}\left\{3\left(\frac{(1-q)}{q}\frac{\tilde{s}}{\tilde{u}_2}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right)^2-1\right\}H\left(\tilde{u}_2-\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|\right)\end{aligned}$$

and

$$\int_0^{\pi}\sin\theta_2'\,\cos\theta_2' I_\delta\,\mathrm{d}\theta_2' = \frac{1}{q^2\tilde{s}\tilde{u}_2}\left(\frac{(1-q)}{q}\frac{\tilde{s}}{\tilde{u}_2}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right)H\left(\tilde{u}_2-\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|\right).$$

Substituting these values in the above expression of (II), we get

$$
\begin{aligned}
(II) = & -\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i \\
& \times\frac{\tilde{s}_j\tilde{s}_k}{\tilde{s}^3}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^3\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\frac{\tilde{s}}{\tilde{u}_2}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right)^2-1\right\} \\
& -\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{2\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i \\
& \times\frac{\tilde{s}_j}{\tilde{s}^2}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{(1-q)}{q}\frac{\tilde{s}}{\tilde{u}_2}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right). \qquad (4.42)
\end{aligned}
$$

The integration over $\tilde{\boldsymbol{s}}$ is performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}=\tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$
\begin{aligned}
(II) = & -\frac{K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
& \times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{s}_j\tilde{s}_k\frac{1}{\tilde{s}^3} \\
& \times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2-\tilde{u}_2^2\right\} \\
& -\frac{K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
& \times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{s}_j\frac{1}{\tilde{s}^2} \\
& \times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right).
\end{aligned}
$$

Note that the components of $\tilde{\boldsymbol{s}}$ are the only functions of ϕ' (see Appendix H). Hence, using eqs. (H.18) and (H.6) respectively, we have

$$
\begin{aligned}
(II) = & -\frac{K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\frac{2\Theta}{3}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)} \\
& \times\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\left\{\pi\frac{\tilde{s}^2}{\tilde{u}^2}\overline{\frac{\partial V_j}{\partial r_k}}\tilde{u}_j\tilde{u}_k(3\cos^2\theta'-1)\right\}\frac{1}{\tilde{s}^3} \\
& \times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2-\tilde{u}_2^2\right\} \\
& -\frac{K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
& \times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\left(2\pi\frac{\tilde{s}}{\tilde{u}}\tilde{u}_j\cos\theta'\right)\frac{1}{\tilde{s}^2} \\
& \times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right).
\end{aligned}
$$

Clearly, the term containing velocity gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}$ and let $\cos\theta' = y$. Therefore

$$(II) = -\frac{2K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)$$
$$\times\,\tilde{u}_i\tilde{u}_j\frac{\tilde{s}}{\tilde{u}}y\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right).$$

Let us replace $\tilde{u}_2^2$ by $\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2$. This shift results into

$$(II) = -\frac{2K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_j}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}$$
$$\times\,\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\tilde{s}}{\tilde{u}}y\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}}$$
$$\times\,\hat{\varPhi}_c\left(\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}^{1/2}\right)\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\frac{5}{2}\right\}.$$

Using eq. (F.9b), the integration over $\tilde{\boldsymbol{u}}$ results into

$$(II) = -\frac{2K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\frac{4\pi}{3}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}$$
$$\times\,\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^4\frac{\tilde{s}}{\tilde{u}}y\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}}$$
$$\times\,\hat{\varPhi}_c\left(\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}^{1/2}\right)\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\frac{5}{2}\right\}$$

or

$$(II) = -\frac{8K\epsilon n}{3\pi}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\tilde{u}^3y$$
$$\times\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\hat{\varPhi}_c\left(\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}^{1/2}\right)\hat{\varPhi}_c(\tilde{u})$$
$$\times\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\frac{5}{2}\right\}\left(\tilde{u}^2-\frac{5}{2}\right)\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}}. \quad (4.43)$$

The differentiation with respect to ϵ at $\epsilon = 0$ is carried out next. Note that $q \to 1$ as $\epsilon \to 0$. Let

$$\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left[\frac{1}{q^2}\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\hat{\varPhi}_c\left(\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}^{1/2}\right)\right.$$
$$\left.\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\frac{5}{2}\right\}\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}\right] = A$$

$$\begin{aligned}\Rightarrow \quad A = &\lim_{\epsilon\to 0}\left(-\frac{2}{q^3}\frac{\partial q}{\partial \epsilon}\right)\tilde{u}y\,\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\mathrm{e}^{-\tilde{u}^2y^2}\\ &+\lim_{\epsilon\to 0}\left(-\frac{\tilde{s}}{q^2}\frac{\partial q}{\partial \epsilon}\right)\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\mathrm{e}^{-\tilde{u}^2y^2}\\ &+\lim_{\epsilon\to 0}\left[\hat{\Phi}_c'\left(\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}^{1/2}\right)2\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\left(-\frac{\tilde{s}}{q^2}\frac{\partial q}{\partial \epsilon}\right)\right]\\ &\times\tilde{u}y\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\mathrm{e}^{-\tilde{u}^2y^2}\\ &+\lim_{\epsilon\to 0}\left[2\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\left(-\frac{\tilde{s}}{q^2}\frac{\partial q}{\partial \epsilon}\right)\right]\tilde{u}y\,\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\mathrm{e}^{-\tilde{u}^2y^2}\\ &+\lim_{\epsilon\to 0}\left[-\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}2\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\left(-\frac{\tilde{s}}{q^2}\frac{\partial q}{\partial \epsilon}\right)\right]\\ &\times\tilde{u}y\,\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right).\end{aligned}$$

Here prime on $\hat{\Phi}_c$ denotes the differentiation with respect to square of its argument. Noting that $\lim_{\epsilon\to 0}\frac{\partial q}{\partial \epsilon} = -\frac{1}{4}$ (as above),

$$\begin{aligned}A = &\frac{1}{4}\,\mathrm{e}^{-\tilde{u}^2y^2}\Bigg[2\tilde{u}y\,\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\\ &+\tilde{s}\,\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)+2\tilde{s}\tilde{u}^2y^2\,\hat{\Phi}_c'\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\\ &+2\tilde{s}\tilde{u}^2y^2\,\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)-2\tilde{s}\tilde{u}^2y^2\,\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\Bigg]\\ =&\frac{1}{2}\,\mathrm{e}^{-\tilde{u}^2y^2}\left[\tilde{u}y+\tilde{s}\left\{\frac{1}{2}-\tilde{u}^2y^2\left(1-\frac{\hat{\Phi}_c'\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)}{\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)}-\frac{1}{\big(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\big)}\right)\right\}\right]\\ &\times\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right).\end{aligned}$$

Hence

$$\begin{aligned}(II) = &-\frac{4}{3\pi}\epsilon n\left(K\frac{2\Theta}{3g}\right)\left(\frac{2}{3}\right)^{\frac{3}{2}}\Theta^{1/2}\frac{\partial\Theta}{\partial r_i}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\tilde{u}^3y\\ &\times\left[\tilde{u}y+\tilde{s}\left\{\frac{1}{2}-\tilde{u}^2y^2\left(1-\frac{\hat{\Phi}_c'\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)}{\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)}-\frac{1}{\big(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\big)}\right)\right\}\right]\\ &\times\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\hat{\Phi}_c\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\hat{\Phi}_c(\tilde{u})\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}.\end{aligned}$$

Now, one can integrate over $\tilde{s}$ to get

$$\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\,\tilde{s}\,\mathrm{e}^{2\tilde{u}\tilde{s}y-\tilde{s}^2} = \frac{1}{2}\left\{1+\sqrt{\pi}\tilde{u}y\,\mathrm{e}^{\tilde{u}^2y^2}\left(1+\mathrm{erf}(\tilde{u}y)\right)\right\}, \tag{4.44a}$$

$$\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\,\tilde{s}^2\,\mathrm{e}^{2\tilde{u}\tilde{s}y-\tilde{s}^2} = \frac{1}{2}\left\{\tilde{u}y+\frac{\sqrt{\pi}}{2}(1+2\tilde{u}^2y^2)\,\mathrm{e}^{\tilde{u}^2y^2}\left(1+\mathrm{erf}(\tilde{u}y)\right)\right\}. \tag{4.44b}$$

Substituting these values in the expression of (II), we get

$$
\begin{aligned}
(II) = &-\frac{4\sqrt{2}}{9\pi\sqrt{3}}\epsilon n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{u}_2\tilde{u}^3 y\left[\tilde{u}y\left\{1+\sqrt{\pi}\tilde{u}y\,\mathrm{e}^{\tilde{u}^2y^2}\left(1+\mathrm{erf}(\tilde{u}y)\right)\right\}\right.\\
&+\left\{\frac{1}{2}-\tilde{u}^2y^2\left(1-\frac{\hat{\varPhi}_c'\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)}{\hat{\varPhi}_c\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)}-\frac{1}{(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2})}\right)\right\}\\
&\left.\times\left\{\tilde{u}y+\frac{\sqrt{\pi}}{2}(1+2\tilde{u}^2y^2)\,\mathrm{e}^{\tilde{u}^2y^2}\left(1+\mathrm{erf}(\tilde{u}y)\right)\right\}\right]\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\\
&\times\hat{\varPhi}_c\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\,\hat{\varPhi}_c(\tilde{u})\,\mathrm{e}^{-\tilde{u}^2}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\\
= &\,\alpha_3\epsilon n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i},
\end{aligned}
\tag{4.45}
$$

where

$$
\begin{aligned}
\alpha_3 = &-\frac{4\sqrt{2}}{9\pi\sqrt{3}}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{u}_2\tilde{u}^3 y\left[\left\{\tilde{u}y+\frac{\sqrt{\pi}}{2}(1+2\tilde{u}^2y^2)\,\mathrm{e}^{\tilde{u}^2y^2}\left(1+\mathrm{erf}(\tilde{u}y)\right)\right\}\right.\\
&\times\left\{\frac{1}{2}-\tilde{u}^2y^2\left(1-\frac{\hat{\varPhi}_c'\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)}{\hat{\varPhi}_c\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)}-\frac{1}{(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2})}\right)\right\}\\
&\left.+\tilde{u}y+\sqrt{\pi}\,\tilde{u}^2y^2\,\mathrm{e}^{\tilde{u}^2y^2}\left(1+\mathrm{erf}(\tilde{u}y)\right)\right]\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\\
&\times\hat{\varPhi}_c\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\,\hat{\varPhi}_c(\tilde{u})\,\mathrm{e}^{-\tilde{u}^2}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}.
\end{aligned}
\tag{4.46}
$$

The integrals in eq. (4.46) have been evaluated numerically. The result is $\alpha_3 \approx 0.2849$. The third contribution to $Q_i^{K\epsilon}$ is given by eq. (4.28). Substituting the value of $\tilde{\varOmega}$ from eq. (2.38) in eq. (4.28), one obtains

$$
\begin{aligned}
Q_{i_3}^{K\epsilon} = &-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\\
&\times\{\varPhi_K(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')+\varPhi_K(\tilde{\boldsymbol{u}}_2')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1')-\varPhi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)-\varPhi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\}.
\end{aligned}
\tag{4.47}
$$

Note that eq. (4.47) uses elastic velocity transformation. The part of integral containing primed velocities can be simplified as follows. Let

$$
\begin{aligned}
\dot{I} = &\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\\
&\times\{\varPhi_K(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')+\varPhi_K(\tilde{\boldsymbol{u}}_2')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1')\}.
\end{aligned}
$$

The integration variables in the above equation can be changed by using the relations: $\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2 = \mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'$, $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12} = -\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'$ and $\tilde{u}_1^2+\tilde{u}_2^2 = \tilde{u}_1'^2+\tilde{u}_2'^2$. Thus

$$
\begin{aligned}
\dot{I} &= \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}<0} \mathrm{d}\tilde{\boldsymbol{u}}'_1 \mathrm{d}\tilde{\boldsymbol{u}}'_2 \mathrm{d}\hat{\boldsymbol{k}}\, (-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})\, \mathrm{e}^{-(\tilde{u}'^2_1+\tilde{u}'^2_2)}\, \hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i} \\
&\quad \times \{\Phi_K(\tilde{\boldsymbol{u}}'_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}'_2)+\Phi_K(\tilde{\boldsymbol{u}}'_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}'_1)\} \\
&= \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}'_1 \mathrm{d}\tilde{\boldsymbol{u}}'_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}'_{12})\, \mathrm{e}^{-(\tilde{u}'^2_1+\tilde{u}'^2_2)}\, \hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i} \\
&\quad \times \{\Phi_K(\tilde{\boldsymbol{u}}'_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}'_2)+\Phi_K(\tilde{\boldsymbol{u}}'_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}'_1)\}. \qquad \text{(using eqn. (F.8))}
\end{aligned}
$$

Now, changing the integration variables $\tilde{\boldsymbol{u}}'_1$ and $\tilde{\boldsymbol{u}}'_2$ to $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ respectively, we get

$$
\begin{aligned}
\dot{I} &= \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}^2_1+\tilde{u}^2_2)}\, \hat{\Phi}_c(\tilde{u}'_1)\left(\tilde{u}'^2_1-\frac{5}{2}\right)\tilde{u}'_{1i} \\
&\quad \times \{\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\}.
\end{aligned}
$$

Next, using the relation (cf. see text and equations below eq. (4.32))

$$
\hat{\Phi}_c(\tilde{u}'_1)\left(\tilde{u}'^2_1-\frac{5}{2}\right)\tilde{u}'_{1i} = \int \mathrm{d}\tilde{\boldsymbol{u}}\, \delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1+q(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big)\, \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i,
$$

one can write

$$
\begin{aligned}
\dot{I} &= \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}} \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}^2_1+\tilde{u}^2_2)}\, \delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1+(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big)\, \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i \\
&\quad \times \{\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\} \\
&= \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}^2_1+\tilde{u}^2_2)} \{\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\}\, \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta^{(0)},
\end{aligned}
$$

where

$$
I_\delta^{(0)} = I_\delta(q=1) = \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1+(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big). \tag{4.48}
$$

Hence $Q_{i_3}^{K\epsilon}$ can be written as

$$
\begin{aligned}
Q_{i_3}^{K\epsilon} &= -\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}^2_1+\tilde{u}^2_2)} \{\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\} \\
&\quad \times \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta^{(0)} \\
&\quad + \frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}^2_1+\tilde{u}^2_2)} \\
&\quad \times \{\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\}\, \hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}.
\end{aligned} \tag{4.49}
$$

The term $Q_{i_3}^{K\epsilon}$ is split into four parts. The first part is

$$
Q_{i_3}^{K\epsilon} = -\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}^2_1+\tilde{u}^2_2)}\, \Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta^{(0)}.
$$

Substituting the explicit forms of Φ_K and $\varphi_1^{(1)}$,

$$(I) = -\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta^{(0)}$$
$$\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1j}\tilde{u}_{1k}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_k} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1j}\frac{\partial \ln \Theta}{\partial r_j}\right]. \tag{4.50}$$

Note that except for the extra term $\hat{\Phi}_e(\tilde{u}_2)$ and the above definition of $I_\delta^{(0)}$, the integrand in eq. (4.50) is similar to that in eq. (4.35). Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}$ in the above equation and using eq. (F.3), we get

$$(I) = -\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta^{(0)}$$
$$\times\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_j-\tilde{s}_j)(\tilde{u}_k-\tilde{s}_k)$$
$$-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta^{(0)}$$
$$\times\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_j-\tilde{s}_j).$$

First we integrate over $\tilde{\boldsymbol{u}}_2$ (as above) to get

$$\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_e(\tilde{u}_2)I_\delta^{(0)} = \frac{2\pi}{\tilde{s}}\int_{\tilde{u}_2=\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.$$

Let us replace $\tilde{u}_2^2$ by $\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2$. This shift implies that

$$\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_e(\tilde{u}_2)I_\delta^{(0)} = \frac{2\pi}{\tilde{s}}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2\,\hat{\Phi}_e\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\,\mathrm{d}\tilde{u}_2. \tag{4.51}$$

Therefore

$$(I) = -\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\frac{2\pi}{\tilde{s}}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\hat{\Phi}_e\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}$$
$$\times\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i(\tilde{u}_j-\tilde{s}_j)(\tilde{u}_k-\tilde{s}_k)$$
$$-\frac{K\epsilon n}{2\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\frac{2\pi}{\tilde{s}}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\hat{\Phi}_e\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)$$
$$\times\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u})\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i(\tilde{u}_j-\tilde{s}_j).$$

The integrations over $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{u}}$ are performed by following a similar procedure as performed following eq. (4.35). Finally, we get (cf. eq. (4.36))

$$
\begin{aligned}
(I) = &-\frac{8}{3\pi}\epsilon n\ell\left(\frac{2}{3}\right)^{\frac{3}{2}}\Theta^{1/2}\frac{\partial\Theta}{\partial r_i}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{u}_2\tilde{u}^3\tilde{s}\\
&\times\ \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\,\hat{\Phi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\Phi}_e\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\\
&\times\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)(\tilde{u}-\tilde{s}y)\\
= &\,\alpha_4\epsilon n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i},
\end{aligned}
\tag{4.52}
$$

where

$$
\begin{aligned}
\alpha_4 = &-\frac{16\sqrt{2}}{9\pi\sqrt{3}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\tilde{u}^3(\tilde{u}-\tilde{s}y)\hat{\Phi}_c(\tilde{u})\hat{\Phi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\\
&\times\hat{\Phi}_e\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}.
\end{aligned}
\tag{4.53}
$$

The integrals in eq. (4.53) have been evaluated numerically. The result is $\alpha_4 \approx -0.0018$. The second part of eq. (4.49) is

$$
\begin{aligned}
(II) = &-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_iI_\delta^{(0)}\\
= &-\frac{\epsilon n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_iI_\delta^{(0)}\\
&\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{2j}\tilde{u}_{2k}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_k}+K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2j}\frac{\partial\ln\Theta}{\partial r_j}\right].
\end{aligned}
\tag{4.54}
$$

Note that except for the extra term $\hat{\Phi}_e(\tilde{u}_1)$ and the above definition of $I_\delta^{(0)}$, the integrand in eq. (4.54) is similar to the one in eq. (4.40). Therefore, the integrations are carried out by following a similar procedure as performed following eq. (4.40). Finally, we get (cf. eq. (4.43))

$$
\begin{aligned}
(II) = &-\frac{8K\epsilon n}{3\pi}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_i}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\tilde{u}^3y\\
&\times\tilde{u}y\,\hat{\Phi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\Phi}_c\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\,\hat{\Phi}_c(\tilde{u})\\
&\times\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\left(\tilde{u}_2^2+\tilde{u}^2y^2\right)}\\
= &\,\alpha_5\epsilon n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i},
\end{aligned}
\tag{4.55}
$$

where

$$
\begin{aligned}
\alpha_5 = &-\frac{16\sqrt{2}}{9\pi\sqrt{3}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\tilde{u}^4y^2\hat{\Phi}_c(\tilde{u})\hat{\Phi}_c\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\\
&\times\hat{\Phi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{5}{2}\right)\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}.
\end{aligned}
\tag{4.56}
$$

The integrals in eq. (4.56) have been evaluated numerically. The result is $\alpha_5 \approx -0.0018$. The computation of the third and fourth parts in eq. (4.49) is much simpler than that of first and second parts because the corresponding integrands do not contain a mixture of precollisional and postcollisional velocities. Therefore, the integration over $\hat{\boldsymbol{k}}$ is trivial in third and fourth parts in eq. (4.49). Hence, using eq. (G.1b), the third part of eq. (4.49) reads

$$(III) = \frac{\epsilon n}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \varPhi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2 - \frac{5}{2}\right)\tilde{u}_{1i}.$$

Substituting the explicit forms of $\varPhi_K$ and $\varphi_1^{(1)}$ and using eq. (F.3),

$$\begin{aligned}(III) = {}& \frac{\epsilon n}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_2)\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2 - \frac{5}{2}\right)\tilde{u}_{1i} \\ & \times \left[2K\frac{2\Theta}{3g}\hat{\varPhi}_v(\tilde{u}_1)\tilde{u}_{1j}\tilde{u}_{1k}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_j}{\partial r_k}} + K\frac{2\Theta}{3g}\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2 - \frac{5}{2}\right)\tilde{u}_{1j}\frac{\partial \ln\Theta}{\partial r_j}\right].\end{aligned}$$

Let the spherical coordinates of $\tilde{\boldsymbol{u}}_1$ in the original coordinate system be $(\tilde{u}_1, \theta_1, \phi_1)$ and the integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$, which results from two rotations: (i) rotation of xy-plane in positive direction (x towards y) around z-axis by an angle ϕ_1 and (ii) rotation of new zx-plane in positive direction (z towards x) around new y-axis by an angle θ_1, so that $\tilde{\boldsymbol{u}}_1$ coincides with the new z-axis and $\tilde{\boldsymbol{u}}_1 \cdot \tilde{\boldsymbol{u}}_2 = \tilde{u}_1\tilde{u}_2\cos\theta_2'$ (cf. Appendix H). Hence

$$\begin{aligned}(III) = {}& \frac{K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\, \tilde{u}_2^2\sin\theta_2' \\ & \times \left(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2\right)^{1/2}\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_e(\tilde{u}_2)\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1)\left(\tilde{u}_1^2 - \frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{1k} \\ & + \frac{K\epsilon n}{2\pi^3}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\, \tilde{u}_2^2\sin\theta_2' \\ & \times \left(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2\right)^{1/2}\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_e(\tilde{u}_2)\hat{\varPhi}_c^2(\tilde{u}_1)\left(\tilde{u}_1^2 - \frac{5}{2}\right)^2\tilde{u}_{1i}\tilde{u}_{1j}.\end{aligned}$$

Clearly, the term containing velocity gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}_1$ and the integration over ϕ_2' is just 2π in the other term. Hence

$$\begin{aligned}(III) = {}& \frac{K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_2^2 \\ & \times\left\{\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\,\sin\theta_2'\left(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2\right)^{1/2}\right\}\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_e(\tilde{u}_2)\hat{\varPhi}_c^2(\tilde{u}_1)\left(\tilde{u}_1^2 - \frac{5}{2}\right)^2 \\ = {}& \frac{K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_2^2 R_0(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_e(\tilde{u}_2) \\ & \times\hat{\varPhi}_c^2(\tilde{u}_1)\left(\tilde{u}_1^2 - \frac{5}{2}\right)^2,\end{aligned}$$

where the function $R_n(\tilde{u}_1, \tilde{u}_2)$ is defined as

$$
\begin{aligned}
R_n(\tilde{u}_1, \tilde{u}_2) &\equiv \int_0^{\pi} \mathrm{d}\theta_2' \, \sin\theta_2' \, P_n(\cos\theta_2') \, (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \\
&= \int_{-1}^{1} \mathrm{d}y \, P_n(y) \, (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 y + \tilde{u}_2^2)^{1/2},
\end{aligned} \tag{4.57}
$$

and $P_n(x)$ is the n^{th} order Legendre polynomial. The value of $R_n(\tilde{u}_1, \tilde{u}_2)$ is (Pekeris 1955):

$$
R_n(\tilde{u}_1, \tilde{u}_2) = \begin{cases} \dfrac{2\,\tilde{u}_2}{(2n+1)} \left[\dfrac{1}{(2n+3)} \left(\dfrac{\tilde{u}_2}{\tilde{u}_1}\right)^{n+1} - \dfrac{1}{(2n-1)} \left(\dfrac{\tilde{u}_2}{\tilde{u}_1}\right)^{n-1} \right], & \text{if} \quad \tilde{u}_1 > \tilde{u}_2, \\ \dfrac{2\,\tilde{u}_1}{(2n+1)} \left[\dfrac{1}{(2n+3)} \left(\dfrac{\tilde{u}_1}{\tilde{u}_2}\right)^{n+1} - \dfrac{1}{(2n-1)} \left(\dfrac{\tilde{u}_1}{\tilde{u}_2}\right)^{n-1} \right], & \text{if} \quad \tilde{u}_2 > \tilde{u}_1. \end{cases} \tag{4.58}
$$

Now, using eq. (F.9b), the expression for (III) simplifies to

$$
\begin{aligned}
(III) &= \frac{K\epsilon n}{\pi^2} \frac{2\Theta}{3g} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \frac{\partial \ln\Theta}{\partial r_i} \frac{4\pi}{3} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \tilde{u}_1^4 \tilde{u}_2^2 \\
&\quad \times R_0(\tilde{u}_1, \tilde{u}_2) \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \hat{\varPhi}_e(\tilde{u}_2) \hat{\varPhi}_c^2(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right)^2 \\
&= \frac{4}{3\pi} \left(\frac{2}{3}\right)^{\frac{3}{2}} \epsilon n \ell \Theta^{1/2} \frac{\partial\Theta}{\partial r_i} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \tilde{u}_1^4 \tilde{u}_2^2 \\
&\quad \times R_0(\tilde{u}_1, \tilde{u}_2) \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \hat{\varPhi}_e(\tilde{u}_2) \hat{\varPhi}_c^2(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right)^2 \\
&= \alpha_6 \epsilon n \ell \Theta^{1/2} \frac{\partial\Theta}{\partial r_i},
\end{aligned} \tag{4.59}
$$

where

$$
\alpha_6 = \frac{8\sqrt{2}}{9\pi\sqrt{3}} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \tilde{u}_1^4 \tilde{u}_2^2 \, R_0(\tilde{u}_1, \tilde{u}_2) \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \hat{\varPhi}_e(\tilde{u}_2) \hat{\varPhi}_c^2(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right)^2. \tag{4.60}
$$

From eq. (4.58), the value of $R_0(\tilde{u}_1, \tilde{u}_2)$ is given by

$$
R_0(\tilde{u}_1, \tilde{u}_2) = \begin{cases} \dfrac{2\,\tilde{u}_2^2}{3\tilde{u}_1} + 2\tilde{u}_1, & \text{if} \quad \tilde{u}_1 > \tilde{u}_2, \\ \dfrac{2\,\tilde{u}_1^2}{3\tilde{u}_2} + 2\tilde{u}_2, & \text{if} \quad \tilde{u}_2 > \tilde{u}_1. \end{cases} \tag{4.61}
$$

The integrals in eq. (4.60) have been evaluated numerically. The result is $\alpha_6 \approx 0.0025$. Using eq. (G.1b), the fourth part of eq. (4.49) reads

$$
(IV) = \frac{\epsilon n}{2\pi^3} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \tilde{u}_{12} \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \varPhi_K(\tilde{\boldsymbol{u}}_2) \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) \hat{\varPhi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i}.
$$

Substituting the explicit forms of $\varPhi_K$ and $\varphi_1^{(1)}$ and using eq. (F.3),

$$(IV) = \frac{\epsilon n}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}$$
$$\times \left[2K\frac{2\Theta}{3g}\hat{\varPhi}_v(\tilde{u}_2)\tilde{u}_{2j}\tilde{u}_{2k}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_j}{\partial r_k}} + K\frac{2\Theta}{3g}\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2j}\frac{\partial \ln \Theta}{\partial r_j}\right].$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2 = \tilde{u}_1\tilde{u}_2\cos\theta_2'$, i.e.,

$$(IV) = \frac{K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\, \tilde{u}_2^2\sin\theta_2'$$
$$\times (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{\frac{1}{2}}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{2j}\tilde{u}_{2k}$$
$$+ \frac{K\epsilon n}{2\pi^3}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\, \tilde{u}_2^2\sin\theta_2'$$
$$\times (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{\frac{1}{2}}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{2j}.$$

Note that the components of $\tilde{\boldsymbol{u}}_2$ are the only functions of ϕ_2' (see Appendix H). Hence, the integrations over ϕ_2' result into (cf. eqs. (H.18) and (H.6) respectively)

$$(IV) = \frac{K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\frac{2\Theta}{3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\, \tilde{u}_2^2\sin\theta_2'(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{\frac{1}{2}}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}$$
$$\times \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\times\pi\frac{\tilde{u}_2^2}{\tilde{u}_1^2}\overline{\frac{\partial V_j}{\partial r_k}}\tilde{u}_{1j}\tilde{u}_{1k}(3\cos^2\theta_2' - 1)$$
$$+ \frac{K\epsilon n}{2\pi^3}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\, \tilde{u}_2^2\sin\theta_2'(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{\frac{1}{2}}$$
$$\times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1i}\times 2\pi\frac{\tilde{u}_2}{\tilde{u}_1}\tilde{u}_{1j}\cos\theta_2'.$$

Clearly, the term containing velocity gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}_1$. Hence

$$(IV) = \frac{K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\,\frac{1}{\tilde{u}_1}\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_2^3$$
$$\times\left\{\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\,\sin\theta_2'\cos\theta_2'(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{\frac{1}{2}}\right\}$$
$$\times\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)$$
$$= \frac{K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\,\frac{1}{\tilde{u}_1}\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_2^3$$
$$\times R_1(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right).$$

Using eq. (F.9b),

$$(IV) = \frac{K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial \ln\Theta}{\partial r_i}\frac{4\pi}{3}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\frac{1}{\tilde{u}_1}\tilde{u}_1^4\tilde{u}_2^3R_1(\tilde{u}_1,\tilde{u}_2)$$
$$\times\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)$$
$$=\frac{4}{3\pi}\left(\frac{2}{3}\right)^{\frac{3}{2}}\epsilon n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_1^3\tilde{u}_2^3\,R_1(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}$$
$$\times\,\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)$$
$$=\alpha_7\,\epsilon n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i}, \tag{4.62}$$

where

$$\alpha_7=\frac{8\sqrt{2}}{9\pi\sqrt{3}}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_1^3\tilde{u}_2^3\,R_1(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}$$
$$\times\,\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right). \tag{4.63}$$

From eq. (4.58), the value of $R_1(\tilde{u}_1,\tilde{u}_2)$ is given by

$$R_1(\tilde{u}_1,\tilde{u}_2)=\begin{cases}\dfrac{2\,\tilde{u}_2^3}{15\tilde{u}_1^2}-\dfrac{2}{3}\tilde{u}_2, & \text{if}\quad \tilde{u}_1>\tilde{u}_2,\\[2mm] \dfrac{2\,\tilde{u}_1^3}{15\tilde{u}_2^2}-\dfrac{2}{3}\tilde{u}_1, & \text{if}\quad \tilde{u}_2>\tilde{u}_1.\end{cases} \tag{4.64}$$

The integrals in eq. (4.63) have been evaluated numerically. The result is $\alpha_7\approx -0.0006$. Summing all the contributions to $Q_i^{K\epsilon}$, one obtains

$$\boxed{Q_i^{K\epsilon}=-\epsilon\tilde{\kappa}_1 n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i}-\epsilon\tilde{\tau}_1\ell\Theta^{3/2}\frac{\partial n}{\partial r_i}} \tag{4.65}$$

where $\tilde{\kappa}_1\approx 0.1078$ and $\tilde{\tau}_1\approx 0.2110$. Eq. (4.65) can be written in another form to get

$$Q_i^{K\epsilon}=-\kappa_1\frac{\partial\Theta}{\partial r_i}-\lambda_1\frac{\partial n}{\partial r_i}, \tag{4.66}$$

where $\kappa_1\approx 0.1078\epsilon n\ell\,\Theta^{1/2}$ and $\lambda_1\approx 0.2110\epsilon\ell\,\Theta^{3/2}$. Heat flux has a non Fourier term proportional to density gradient at this order and this term is zero in case of elastic particles.

Pressure Tensor

From eq. (2.28), the contribution of $\Phi_{K\epsilon}$ to the pressure tensor is given by

$$P_{ij}^{K\epsilon}=\frac{2n\Theta}{3\pi^{3/2}}\int\mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}_i\tilde{u}_j\,\mathrm{e}^{-\tilde{u}^2}\,\Phi_{K\epsilon}=\frac{2\epsilon n\Theta}{3\pi^{3/2}}\int\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\left(\frac{\tilde{u}_i\tilde{u}_j+\tilde{u}_j\tilde{u}_i}{2}-\frac{1}{3}\tilde{u}^2\delta_{ij}\right)\varphi_K^{(1)}.$$

The second equality in the above equation results from the orthogonality of $\Phi_{K\epsilon}$ to $\tilde{u}^2$, which is $\int\mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{f}_0(\tilde{u})\tilde{u}^2\Phi_{K\epsilon}=0$ (cf. eq. (2.18)). Noting that the quantity in brackets in the above integral is $\overline{\tilde{u}_i\tilde{u}_j}$ and using eq. (3.20), one can write

$$P_{ij}^{K\epsilon} = \frac{2\epsilon n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{\mathscr{L}}\left[\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\right] \varphi_K^{(1)},$$

and using the fact that $\tilde{\mathscr{L}}$ is self-adjoint with $\mathrm{e}^{-\tilde{u}^2}$ serving as a weight function (see §4.1), the above equation changes to

$$P_{ij}^{K\epsilon} = \frac{2\epsilon n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\, \tilde{\mathscr{L}}(\varphi_K^{(1)}). \tag{4.67}$$

Substituting the value of $\tilde{\mathscr{L}}(\varphi_K^{(1)})$ from eq. (4.23), the above equation changes to

$$\begin{aligned} P_{ij}^{K\epsilon} &= \frac{2\epsilon n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\, \mathrm{e}^{-\tilde{u}^2} \left\{\tilde{S}_K - \tilde{\Xi}(\Phi_K) - \tilde{\Lambda}(\Phi_K) - \tilde{\Omega}(\Phi_K, \varphi_1^{(1)})\right\} \\ &\equiv P_{ij_1}^{K\epsilon} + P_{ij_2}^{K\epsilon} + P_{ij_3}^{K\epsilon}, \quad \text{(let)} \end{aligned} \tag{4.68}$$

where

$$P_{ij_1}^{K\epsilon} = \frac{2\epsilon n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{S}_K, \tag{4.69}$$

$$P_{ij_2}^{K\epsilon} = -\frac{2\epsilon n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\, \mathrm{e}^{-\tilde{u}^2} \left\{\tilde{\Xi}(\Phi_K) + \tilde{\Lambda}(\Phi_K)\right\}, \tag{4.70}$$

$$P_{ij_3}^{K\epsilon} = -\frac{2\epsilon n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{\Omega}(\Phi_K, \varphi_1^{(1)}). \tag{4.71}$$

Substituting the explicit form of $\tilde{S}_K$ from eq. (4.22) into eq. (4.69), we have

$$\begin{aligned} P_{ij_1}^{K\epsilon} &= \frac{2\epsilon n\Theta}{3\pi^{3/2}} \left(K\frac{2\Theta}{3g}\right) \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\, \mathrm{e}^{-\tilde{u}^2} \\ &\times \left[\frac{4}{3}\left\{\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\hat{\Phi}_v'(\tilde{u})\tilde{u}^2 - \hat{\Phi}_v(\tilde{u})(\tilde{u}^2-3)\right) + \frac{3}{2}\left\{\hat{\Phi}_e(\tilde{u}) - \hat{\Phi}_e'(\tilde{u})\right\}\right\} \overline{\tilde{u}_k\tilde{u}_l} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_l}\right. \\ &+ \left\{\hat{\Phi}_e(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right) - \hat{\Phi}_e'(\tilde{u})(\tilde{u}^2-1) + \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\hat{\Phi}_c'(\tilde{u})\tilde{u}^2\left(\tilde{u}^2 - \frac{5}{2}\right) - \hat{\Phi}_c(\tilde{u})\right.\right. \\ &\times \left.\left.\left(\tilde{u}^4 - 5\tilde{u}^2 + \frac{15}{4}\right)\right)\right\} \tilde{u}_k \frac{\partial \ln\Theta}{\partial r_k} + \left.\left\{\hat{\Phi}_e'(\tilde{u}) - \frac{2}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\right\} \tilde{u}_k \frac{\partial \ln n}{\partial r_k}\right]. \end{aligned}$$

Clearly, the terms containing number density gradient and temperature gradient vanish upon integration beacuse the corresponding integrands are odd functions in components of $\tilde{\boldsymbol{u}}$. Using eq. (F.3) in the term containing velocity gradient, above equation reduces to

$$\begin{aligned} P_{ij_1}^{K\epsilon} &= \frac{4\epsilon n}{3\pi^{3/2}} \left(K\frac{2\Theta}{3g}\right) \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l\, \mathrm{e}^{-\tilde{u}^2} \\ &\times \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\hat{\Phi}_v'(\tilde{u})\tilde{u}^2 - \hat{\Phi}_v(\tilde{u})(\tilde{u}^2-3)\right) + \frac{3}{2}\left\{\hat{\Phi}_e(\tilde{u}) - \hat{\Phi}_e'(\tilde{u})\right\}\right]. \end{aligned}$$

Using eq. (F.12), we get

$$\begin{aligned} P_{ij_1}^{K\epsilon} &= \frac{4\epsilon n}{3\pi^{3/2}}\left(K\frac{2\Theta}{3g}\right)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}}\int_0^\infty \mathrm{d}\tilde{u}\,\hat{\Phi}_v(\tilde{u})\tilde{u}^6\,\mathrm{e}^{-\tilde{u}^2} \\ &\quad\times\left[\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\hat{\Phi}_v'(\tilde{u})\tilde{u}^2-\hat{\Phi}_v(\tilde{u})(\tilde{u}^2-3)\right)+\frac{3}{2}\left\{\hat{\Phi}_e(\tilde{u})-\hat{\Phi}_e'(\tilde{u})\right\}\right] \\ &= \zeta_1\epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}, \end{aligned} \tag{4.72}$$

where

$$\begin{aligned} \zeta_1 &= \frac{32}{45}\left(\frac{2}{3\pi}\right)^{\frac{1}{2}}\int_0^\infty \mathrm{d}\tilde{u}\,\tilde{u}^6\hat{\Phi}_v(\tilde{u})\,\mathrm{e}^{-\tilde{u}^2} \\ &\quad\times\left[\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\hat{\Phi}_v'(\tilde{u})\tilde{u}^2-\hat{\Phi}_v(\tilde{u})(\tilde{u}^2-3)\right)+\frac{3}{2}\left\{\hat{\Phi}_e(\tilde{u})-\hat{\Phi}_e'(\tilde{u})\right\}\right]. \end{aligned} \tag{4.73}$$

The integral in eq. (4.73) has been evaluated numerically. The result is $\zeta_1 \approx -0.0942$. The second term contributing to $P_{ij}^{K\epsilon}$ is given by eq. (4.70). Substituting the values of $\tilde{\Xi}$ and $\tilde{\Lambda}$ from eqs. (2.39) and (2.40) respectively, in eq. (4.70), we get

$$\begin{aligned} P_{ij_2}^{K\epsilon} &= -\frac{2\epsilon n\Theta}{3\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(1-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right)\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\ &\quad\times\Big(\Phi_K(\tilde{\boldsymbol{u}}_1')+\Phi_K(\tilde{\boldsymbol{u}}_2')\Big)\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}} \\ &\quad-\frac{2\epsilon n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\ &\quad\times\Big(\Phi_K(\tilde{\boldsymbol{u}}_1')+\Phi_K(\tilde{\boldsymbol{u}}_2')\Big)\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}. \end{aligned}$$

Following a similar procedure as in the derivation of $Q_{i_2}^{K\epsilon}$, we get

$$\begin{aligned} P_{ij_2}^{K\epsilon} &= -\frac{2\epsilon n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\ &\quad\times\Big(\Phi_K(\tilde{\boldsymbol{u}}_1)+\Phi_K(\tilde{\boldsymbol{u}}_2)\Big)\hat{\Phi}_v(\tilde{u}_1')\overline{\tilde{u}_{1i}'\tilde{u}_{1j}'}. \end{aligned} \tag{4.74}$$

Note that, in eq. (4.74), the non-primed velocities are now precollisional velocities and the primed velocities are now postcollisional velocities. To simplify the integral in eq. (4.74) further, we use the property of delta function, i.e.,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,\delta(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1')\hat{\Phi}_v(\tilde{u})\,\overline{\tilde{u}_i\tilde{u}_j} = \hat{\Phi}_v(\tilde{u}_1')\,\overline{\tilde{u}_{1i}'\tilde{u}_{1j}'}.$$

Since $\tilde{\boldsymbol{u}}_1'$ is the postcollisional velocity here, hence using eq. (2.1a), the above equation can be written as

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,\delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1+q(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big)\,\hat{\Phi}_v(\tilde{u})\,\overline{\tilde{u}_i\tilde{u}_j} = \hat{\Phi}_v(\tilde{u}_1')\,\overline{\tilde{u}_{1i}'\tilde{u}_{1j}'}\,,$$

where $q = \frac{1+e}{2}$. Substituting this in eq. (4.74), we get

$$
\begin{aligned}
P_{ij_2}^{K\epsilon} &= -\frac{2\epsilon n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\Big(\varPhi_K(\tilde{\boldsymbol{u}}_1)+\varPhi_K(\tilde{\boldsymbol{u}}_2)\Big)\\
&\quad\times\int \mathrm{d}\tilde{\boldsymbol{u}}\,\delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1+q(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big)\,\hat{\varPhi}_v(\tilde{u})\,\overline{\tilde{u}_i\tilde{u}_j}\\
&= -\frac{2\epsilon n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\Big(\varPhi_K(\tilde{\boldsymbol{u}}_1)+\varPhi_K(\tilde{\boldsymbol{u}}_2)\Big)\hat{\varPhi}_v(\tilde{u})\,\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta,
\end{aligned}
\tag{4.75}
$$

where I_δ is given in eq. (4.34). The integral in eq. (4.75) is then split into two parts. The first part is

$$
\begin{aligned}
(I) &= -\frac{2\epsilon n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\varPhi_K(\tilde{\boldsymbol{u}}_1)\hat{\varPhi}_v(\tilde{u})\,\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta,\\
&= -\frac{\epsilon n}{\pi^4}\frac{2\Theta}{3}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_v(\tilde{u})\,\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta,\\
&\quad\times\left[2K\frac{2\Theta}{3g}\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1k}\tilde{u}_{1l}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_l}+K\frac{2\Theta}{3g}\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1k}\frac{\partial\ln\Theta}{\partial r_k}\right].
\end{aligned}
\tag{4.76}
$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}$ in the above equation and using eq. (F.3), we get

$$
\begin{aligned}
(I) &= -\frac{2K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\varPhi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta\\
&\quad\times\hat{\varPhi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)\\
&\quad-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\frac{\partial\ln\Theta}{\partial r_k}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\varPhi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta\\
&\quad\times\hat{\varPhi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_k-\tilde{s}_k).
\end{aligned}
$$

Following exactly similar procedure as in the derivation of $Q_{i_2}^{K\epsilon}$, we have

$$
\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,I_\delta = \frac{2\pi}{q^2\tilde{s}}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2 = \frac{\pi}{q^2\tilde{s}}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}. \tag{4.77}
$$

Therefore

$$
\begin{aligned}
(I) &= -\frac{2K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\varPhi}_v(\tilde{u})\hat{\varPhi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\\
&\quad\times\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}\\
&\quad-\frac{K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\frac{2\Theta}{3}\frac{\partial\ln\Theta}{\partial r_k}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\varPhi}_v(\tilde{u})\hat{\varPhi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\\
&\quad\times\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}.
\end{aligned}
$$

The integrations over $\tilde{\boldsymbol{s}}$ are performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}=\tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$
\begin{aligned}
(I) = &-\frac{2K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\overline{\partial V_k}}{\partial r_l}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
&\times\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\,\hat{\Phi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big) \\
&\times\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2} \\
&-\frac{K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\frac{2\Theta}{3}\frac{\partial\ln\Theta}{\partial r_k}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
&\times\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\hat{\Phi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big)\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2-\frac{5}{2}\right) \\
&\times\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}.
\end{aligned}
$$

Note that the components of $\tilde{\boldsymbol{s}}$ are the only functions of ϕ' (see Appendix H). Hence, using eqs. (H.19) and (H.6) respectively, we get

$$
\begin{aligned}
(I) = &-\frac{2K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}\sin\theta' \\
&\times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\,\hat{\Phi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big) \\
&\times\overline{\tilde{u}_i\tilde{u}_j}\left[2\pi\frac{1}{\tilde{u}^2}\frac{\overline{\partial V_k}}{\partial r_l}\tilde{u}_k\tilde{u}_l\left\{\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\frac{1}{2}\tilde{s}^2(3\cos^2\theta'-1)\right\}\right]\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2} \\
&-\frac{K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\frac{2\Theta}{3}\frac{\partial\ln\Theta}{\partial r_k}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}\sin\theta' \\
&\times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\hat{\Phi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big)\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2-\frac{5}{2}\right) \\
&\times\overline{\tilde{u}_i\tilde{u}_j}\left(2\pi\tilde{u}_k-2\pi\frac{\tilde{s}}{\tilde{u}}\tilde{u}_k\cos\theta'\right)\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}.
\end{aligned}
$$

Clearly, the term containing temperature gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}$ and let $\cos\theta' = y$. Therefore

$$
\begin{aligned}
(I) = &-\frac{4K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\overline{\partial V_k}}{\partial r_l}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\frac{\tilde{s}}{\tilde{u}^2}\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l \\
&\times\left\{\tilde{u}^2-2\tilde{u}\tilde{s}y+\frac{1}{2}\tilde{s}^2(3y^2-1)\right\}\hat{\Phi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\Phi}_v(\tilde{u}) \\
&\times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}.
\end{aligned}
$$

Using eq. (F.12), the integration over $\tilde{\boldsymbol{u}}$ results into

$$
\begin{aligned}
(I) = &-\frac{4K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\overline{\partial V_i}}{\partial r_j}\frac{8\pi}{15}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\frac{\tilde{s}}{\tilde{u}^2}\tilde{u}^6 \\
&\times\left\{\tilde{u}^2-2\tilde{u}\tilde{s}y+\frac{1}{2}\tilde{s}^2(3y^2-1)\right\}\hat{\Phi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\Phi}_v(\tilde{u}) \\
&\times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}
\end{aligned}
$$

or

$$
\begin{aligned}
(I) = &-\frac{32}{15\pi}\left(\frac{2}{3}\right)^{\frac{1}{2}} \epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}^4 \\
&\times\left\{\tilde{u}^2-2\tilde{u}\tilde{s}y+\frac{1}{2}\tilde{s}^2(3y^2-1)\right\}\hat{\Phi}_v\left(\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2\right)^{1/2}\right)\hat{\Phi}_v(\tilde{u}) \\
&\times \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}.
\end{aligned}
\tag{4.78}
$$

Using eq. (4.37),

$$
\begin{aligned}
(I) &= -\frac{16\sqrt{2}}{15\pi\sqrt{3}}\epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}^4\left\{\tilde{u}^2-2\tilde{s}\tilde{u}y+\frac{1}{2}\tilde{s}^2(3y^2-1)\right\} \\
&\quad\times(1-\tilde{s}\tilde{u}y)\hat{\Phi}_v\left(\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2\right)^{1/2}\right)\hat{\Phi}_v(\tilde{u})\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\tilde{u}^2y^2} \\
&= \zeta_2\,\epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}},
\end{aligned}
\tag{4.79}
$$

where

$$
\begin{aligned}
\zeta_2 = &-\frac{16\sqrt{2}}{15\pi\sqrt{3}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}^4\left\{\tilde{u}^2-2\tilde{s}\tilde{u}y+\frac{1}{2}\tilde{s}^2(3y^2-1)\right\} \\
&\times(1-\tilde{s}\tilde{u}y)\hat{\Phi}_v\left(\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2\right)^{1/2}\right)\hat{\Phi}_v(\tilde{u})\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\tilde{u}^2y^2}.
\end{aligned}
\tag{4.80}
$$

The integrals in eq. (4.80) have been evaluated numerically. The result is $\zeta_2 \approx -0.1349$. The second part of $P_{ij_2}^{K\epsilon}$ is

$$
\begin{aligned}
(II) &= -\frac{2\epsilon n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\Phi_K(\tilde{\boldsymbol{u}}_2)\hat{\Phi}_v(\tilde{u})\,\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta, \\
&= -\frac{2\epsilon n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u})\,\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta, \\
&\quad\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{2k}\tilde{u}_{2l}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_l}+K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2k}\frac{\partial\ln\Theta}{\partial r_k}\right].
\end{aligned}
\tag{4.81}
$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}$ in the above equation and using eq. (F.3), we get

$$
\begin{aligned}
(II) = &-\frac{2K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2k}\tilde{u}_{2l}I_\delta \\
&-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\frac{\partial\ln\Theta}{\partial r_k}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2k}I_\delta.
\end{aligned}
\tag{4.82}
$$

The functions depending on $\tilde{u}_2$ in the integrands of eq. (4.82) are exactly similar to the functions depending on $\tilde{u}_2$ in the integrands of eq. (4.41) (except for the indices names). Therefore the integrations over $\tilde{\boldsymbol{u}}_2$ are carried out by following a similar procedure as given below eq. (4.41).

Finally, we get (cf. eq. (4.42))

$$
\begin{aligned}
(II) = &-\frac{2K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j} \\
&\times\frac{\tilde{s}_k\tilde{s}_l}{\tilde{s}^3}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^3\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\frac{\tilde{s}}{\tilde{u}_2}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right)^2-1\right\} \\
&-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\frac{\partial\ln\Theta}{\partial r_k}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{2\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j} \\
&\times\frac{\tilde{s}_k}{\tilde{s}^2}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{(1-q)}{q}\frac{\tilde{s}}{\tilde{u}_2}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right). \qquad (4.83)
\end{aligned}
$$

The integrations over $\tilde{\boldsymbol{s}}$ are performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}=\tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$
\begin{aligned}
(II) = &-\frac{2K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
&\times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{s}_k\tilde{s}_l\frac{1}{\tilde{s}^3} \\
&\times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2-\tilde{u}_2^2\right\} \\
&-\frac{2K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\frac{2\Theta}{3}\frac{\partial\ln\Theta}{\partial r_k}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
&\times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{s}_k\frac{1}{\tilde{s}^2} \\
&\times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right).
\end{aligned}
$$

Note that the components of $\tilde{\boldsymbol{s}}$ are the only functions of ϕ' (see Appendix H). Hence, using eqs. (H.18) and (H.6) respectively, we get

$$
\begin{aligned}
(II) = &-\frac{2K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
&\times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left\{\pi\frac{\tilde{s}^2}{\tilde{u}^2}\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_k\tilde{u}_l(3\cos^2\theta'-1)\right\}\frac{1}{\tilde{s}^3} \\
&\times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2-\tilde{u}_2^2\right\} \\
&-\frac{2K\epsilon n}{\pi^3}\frac{2\Theta}{3g}\frac{2\Theta}{3}\frac{\partial\ln\Theta}{\partial r_k}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
&\times\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(2\pi\frac{\tilde{s}}{\tilde{u}}\tilde{u}_k\cos\theta'\right)\frac{1}{\tilde{s}^2} \\
&\times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right).
\end{aligned}
$$

Clearly, the term containing temperature gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}$ and let $\cos\theta' = y$. Therefore

$$(II) = -\frac{2K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\frac{\tilde{s}}{\tilde{u}^2}\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l(3y^2-1)\hat{\Phi}_v(\tilde{u})$$
$$\times\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\tilde{u}_2^2\right\}.$$

Using eq. (F.12), the integration over $\tilde{\boldsymbol{u}}$ results into

$$(II) = -\frac{2K\epsilon n}{\pi^2}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\frac{\tilde{s}}{\tilde{u}^2}\tilde{u}^6(3y^2-1)\hat{\Phi}_v(\tilde{u})$$
$$\times\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\tilde{u}_2^2\right\}.$$

Let us replace $\tilde{u}_2^2$ by $\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2$. This shift results into

$$(II) = -\frac{16K\epsilon n}{15\pi}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\tilde{s}\tilde{u}^4(3y^2-1)\hat{\Phi}_v(\tilde{u})$$
$$\times\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}}$$
$$\times\,\hat{\Phi}_v\left(\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}^{1/2}\right)\left\{2\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\tilde{u}_2^2\right\}.$$

or

$$(II) = -\frac{16K\epsilon n}{15\pi}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\tilde{u}^4$$
$$\times\,(3y^2-1)\hat{\Phi}_v(\tilde{u})\,\hat{\Phi}_v\left(\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}^{1/2}\right)$$
$$\times\left\{2\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\tilde{u}_2^2\right\}\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}}. \tag{4.84}$$

The differentiation with respect to ϵ at $\epsilon=0$ is carried out next. Note that $q\to 1$ as $\epsilon\to 0$. Let

$$\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left[\frac{1}{q^2}\hat{\Phi}_v\left(\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}^{\frac{1}{2}}\right)\left\{2\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\tilde{u}_2^2\right\}\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}\right] = B.$$

$$
\begin{aligned}
B = & \lim_{\epsilon\to 0}\left(-\frac{2}{q^3}\frac{\partial q}{\partial \epsilon}\right)\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)(2\tilde{u}^2y^2-\tilde{u}_2^2)\,\mathrm{e}^{-\tilde{u}^2y^2} \\
& + \lim_{\epsilon\to 0}\left[\hat{\Phi}_v'\left(\left\{\tilde{u}_2^2+\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2\right\}^{1/2}\right)2\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\left(-\frac{\tilde{s}}{q^2}\frac{\partial q}{\partial \epsilon}\right)\right] \\
& \times(2\tilde{u}^2y^2-\tilde{u}_2^2)\,\mathrm{e}^{-\tilde{u}^2y^2} \\
& + \lim_{\epsilon\to 0}\left[4\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\left(-\frac{\tilde{s}}{q^2}\frac{\partial q}{\partial \epsilon}\right)\right]\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\,\mathrm{e}^{-\tilde{u}^2y^2} \\
& + \lim_{\epsilon\to 0}\left[-\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}2\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)\left(-\frac{\tilde{s}}{q^2}\frac{\partial q}{\partial \epsilon}\right)\right]\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)(2\tilde{u}^2y^2-\tilde{u}_2^2).
\end{aligned}
$$

Here prime on $\hat{\Phi}_v$ denotes the differentiation with respect to square of its argument. Noting that $\lim_{\epsilon\to 0}\frac{\partial q}{\partial \epsilon}=-\frac{1}{4}$ (as above),

$$
\begin{aligned}
B = & \frac{1}{2}\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)(2\tilde{u}^2y^2-\tilde{u}_2^2)\,\mathrm{e}^{-\tilde{u}^2y^2}+\frac{1}{2}\hat{\Phi}_v'\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\tilde{u}y\tilde{s}(2\tilde{u}^2y^2-\tilde{u}_2^2)\,\mathrm{e}^{-\tilde{u}^2y^2} \\
& + \tilde{u}y\tilde{s}\,\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\,\mathrm{e}^{-\tilde{u}^2y^2}-\frac{1}{2}\tilde{u}y\tilde{s}\,\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)(2\tilde{u}^2y^2-\tilde{u}_2^2)\,\mathrm{e}^{-\tilde{u}^2y^2} \\
= & \frac{1}{2}\,\mathrm{e}^{-\tilde{u}^2y^2}\Big[\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)(2\tilde{u}^2y^2-\tilde{u}_2^2)+\hat{\Phi}_v'\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\tilde{u}y\tilde{s}(2\tilde{u}^2y^2-\tilde{u}_2^2) \\
& +2\tilde{u}y\tilde{s}\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)-\tilde{u}y\tilde{s}\,\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)(2\tilde{u}^2y^2-\tilde{u}_2^2)\Big] \\
= & \frac{1}{2}\,\mathrm{e}^{-\tilde{u}^2y^2}\left[1-\tilde{u}y\tilde{s}\left\{1-\frac{\hat{\Phi}_v'\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)}{\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)}-\frac{2}{(2\tilde{u}^2y^2-\tilde{u}_2^2)}\right\}\right]\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big) \\
& \times(2\tilde{u}^2y^2-\tilde{u}_2^2).
\end{aligned}
$$

Hence

$$
\begin{aligned}
(II) = & -\frac{8}{15\pi}\epsilon n\left(K\frac{2\Theta}{3g}\right)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\tilde{u}^4(3y^2-1) \\
& \times\hat{\Phi}_v(\tilde{u})\left[1-\tilde{u}y\tilde{s}\left\{1-\frac{\hat{\Phi}_v'\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)}{\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)}-\frac{2}{(2\tilde{u}^2y^2-\tilde{u}_2^2)}\right\}\right] \\
& \times\hat{\Phi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)(2\tilde{u}^2y^2-\tilde{u}_2^2)\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}.
\end{aligned}
$$

Substituting the values of integrations over $\tilde{s}$ from eqs. (4.44a) and (4.44b) in the expression of (II), we get

$$\begin{aligned}(II) = &-\frac{8\sqrt{2}}{15\pi\sqrt{3}}\epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{u}_2\tilde{u}^4(3y^2-1)\hat{\Phi}_v(\tilde{u})\\
&\times\frac{1}{2}\Bigg[\left\{1+\sqrt{\pi}\tilde{u}y\,\mathrm{e}^{\tilde{u}^2y^2}\left(1+\mathrm{erf}(\tilde{u}y)\right)\right\}-\tilde{u}y\left\{1-\frac{\hat{\Phi}_v'\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)}{\hat{\Phi}_v\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)}-\frac{2}{(2\tilde{u}^2y^2-\tilde{u}_2^2)}\right\}\\
&\times\left\{\tilde{u}y+\frac{\sqrt{\pi}}{2}(1+2\tilde{u}^2y^2)\,\mathrm{e}^{\tilde{u}^2y^2}\left(1+\mathrm{erf}(\tilde{u}y)\right)\right\}\Bigg]\\
&\times\hat{\Phi}_v\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)(2\tilde{u}^2y^2-\tilde{u}_2^2)\,\mathrm{e}^{-\tilde{u}^2}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\\
=&\ \zeta_3\,\epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}},\end{aligned}\tag{4.85}$$

where

$$\begin{aligned}\zeta_3 = &-\frac{4\sqrt{2}}{15\pi\sqrt{3}}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{u}_2\tilde{u}^4(3y^2-1)(2\tilde{u}^2y^2-\tilde{u}_2^2)\\
&\times\Bigg[-\tilde{u}y\left\{\tilde{u}y+\frac{\sqrt{\pi}}{2}(1+2\tilde{u}^2y^2)\,\mathrm{e}^{\tilde{u}^2y^2}\left(1+\mathrm{erf}(\tilde{u}y)\right)\right\}\\
&\times\left\{1-\frac{\hat{\Phi}_v'\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)}{\hat{\Phi}_v\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)}-\frac{2}{(2\tilde{u}^2y^2-\tilde{u}_2^2)}\right\}+1+\sqrt{\pi}\tilde{u}y\,\mathrm{e}^{\tilde{u}^2y^2}\left(1+\mathrm{erf}(\tilde{u}y)\right)\Bigg]\\
&\times\hat{\Phi}_v\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)\hat{\Phi}_v(\tilde{u})\,\mathrm{e}^{-\tilde{u}^2}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}.\end{aligned}\tag{4.86}$$

The integrals in eq. (4.86) have been evaluated numerically. The result is $\zeta_3 \approx 0.1094$. The third contribution to $P_{ij}^{K\epsilon}$ is given by eq. (4.71). Substituting the value of $\tilde{\Omega}$ from eq. (2.38) in eq. (4.71), one obtains

$$\begin{aligned}P_{ij_3}^{K\epsilon} = &-\frac{2\epsilon n\Theta}{3\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\\
&\times\{\Phi_K(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')+\Phi_K(\tilde{\boldsymbol{u}}_2')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1')-\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)-\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\}.\end{aligned}\tag{4.87}$$

In eq. (4.87), the velocity transformation corresponds to the elastic limit. Following a similar procedure as in the derivation of $Q_{i_3}^{K\epsilon}$, one obtains

$$\begin{aligned}P_{ij_3}^{K\epsilon} = &-\frac{2\epsilon n\Theta}{3\pi^4}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)}\\
&+\frac{2\epsilon n\Theta}{3\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\\
&\times\{\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}},\end{aligned}\tag{4.88}$$

where $I_\delta^{(0)}$ is given in eq. (4.48). The term $P_{ij_3}^{K\epsilon}$ is split into four parts. The first part is

$$(I) = -\frac{2\epsilon n\Theta}{3\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)}$$
$$= -\frac{2\epsilon n\Theta}{3\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)}$$
$$\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1k}\tilde{u}_{1l}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_l}+K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1k}\frac{\partial\ln\Theta}{\partial r_k}\right]. \tag{4.89}$$

Note that except for the extra term $\hat{\Phi}_e(\tilde{u}_2)$ and the above definition of $I_\delta^{(0)}$, the integrand in eq. (4.89) is similar to that in eq. (4.76). Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}$ in the above equation and using eq. (F.3), one can write

$$(I) = -\frac{2K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta^{(0)}$$
$$\times\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)$$
$$-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\frac{\partial\ln\Theta}{\partial r_k}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta^{(0)}$$
$$\times\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_k-\tilde{s}_k).$$

The integral over $\tilde{\boldsymbol{u}}_2$ is given in eq. (4.51). Hence using eq. (4.51), we get

$$(I) = -\frac{2K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\frac{2\pi}{\tilde{s}}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\hat{\Phi}_e\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)$$
$$\times\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)$$
$$-\frac{K\epsilon n}{\pi^4}\frac{2\Theta}{3g}\frac{2\Theta}{3}\frac{\partial\ln\Theta}{\partial r_k}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\frac{2\pi}{\tilde{s}}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\hat{\Phi}_e\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)$$
$$\times\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_k-\tilde{s}_k).$$

The integrations over $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{u}}$ are performed by following a similar procedure as performed following eq. (4.76). Finally, we get (cf. eq. (4.78))

$$(I) = -\frac{64}{15\pi}\epsilon n\ell\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{u}_2\tilde{s}\tilde{u}^4\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})$$
$$\times\hat{\Phi}_v\left((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\right)\hat{\Phi}_e\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)\left\{\tilde{u}^2-2\tilde{s}\tilde{u}y+\frac{1}{2}\tilde{s}^2(3y^2-1)\right\}\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}$$
$$=\zeta_4\,\epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}, \tag{4.90}$$

where

$$\zeta_4 = -\frac{64\sqrt{2}}{15\pi\sqrt{3}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{u}_2\tilde{s}\tilde{u}^4\left\{\tilde{u}^2-2\tilde{s}\tilde{u}y+\frac{1}{2}\tilde{s}^2(3y^2-1)\right\}$$
$$\times\hat{\Phi}_v\left((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\right)\hat{\Phi}_e\left((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\right)\hat{\Phi}_v(\tilde{u})\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}. \tag{4.91}$$

The integrals in eq. (4.91) have been evaluated numerically. The result is $\zeta_4 \approx 0.0014$. The second part of eq. (4.88) is

$$\begin{aligned}(II) &= -\frac{2\epsilon n\Theta}{3\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\varPhi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\,\hat{\varPhi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)} \\ &= -\frac{2\epsilon n\Theta}{3\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)} \\ &\quad\times\left[2K\frac{2\Theta}{3g}\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2k}\tilde{u}_{2l}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_l}+K\frac{2\Theta}{3g}\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2k}\frac{\partial\ln\Theta}{\partial r_k}\right]. \end{aligned} \tag{4.92}$$

Note that except for the extra term $\hat{\varPhi}_e(\tilde{u}_1)$ and the above definition of $I_\delta^{(0)}$, the integrand in eq. (4.92) is similar to the one in eq. (4.81). Therefore, the above equation is simplified by following a similar procedure as performed following eq. (4.81). Finally, we get (cf. eq. (4.84))

$$\begin{aligned}(II) &= -\frac{16K\epsilon n}{15\pi}\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\tilde{u}^4(3y^2-1)\,\hat{\varPhi}_v(\tilde{u}) \\ &\quad\times\hat{\varPhi}_v\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\,\hat{\varPhi}_e\Big(\big(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2\big)^{1/2}\Big)\,(2\tilde{u}^2y^2-\tilde{u}_2^2)\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)} \\ &= \zeta_5\,\epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}, \end{aligned} \tag{4.93}$$

where

$$\begin{aligned}\zeta_5 = &-\frac{16\sqrt{2}}{15\pi\sqrt{3}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\tilde{u}^4(3y^2-1)(2\tilde{u}^2y^2-\tilde{u}_2^2) \\ &\times\hat{\varPhi}_e\Big(\big(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2\big)^{1/2}\Big)\,\hat{\varPhi}_v\Big(\big(\tilde{u}_2^2+\tilde{u}^2y^2\big)^{1/2}\Big)\,\hat{\varPhi}_v(\tilde{u})\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}. \end{aligned} \tag{4.94}$$

The integrals in eq. (4.94) have been evaluated numerically. The result is $\zeta_5 \approx 0.0014$. In the third and fourth part of eq. (4.88), the integration over $\hat{\boldsymbol{k}}$ is trivial. Hence, using eq. (G.1b), the third part of eq. (4.88) reads

$$(III) = \frac{2\epsilon n\Theta}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\varPhi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\,.$$

Substituting the explicit forms of $\varPhi_K$ and $\varphi_1^{(1)}$ and using eq. (F.3),

$$\begin{aligned}(III) &= \frac{2\epsilon n\Theta}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_e(\tilde{u}_2)\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}} \\ &\quad\times\left[2K\frac{2\Theta}{3g}\hat{\varPhi}_v(\tilde{u}_1)\tilde{u}_{1k}\tilde{u}_{1l}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}+K\frac{2\Theta}{3g}\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1k}\frac{\partial\ln\Theta}{\partial r_k}\right]. \end{aligned}$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1 \cdot \tilde{\boldsymbol{u}}_2 = \tilde{u}_1 \tilde{u}_2 \cos\theta_2'$, i.e.,

$$
\begin{aligned}
(III) = {} & \frac{2}{\pi^3}\epsilon n \left(K\frac{2\Theta}{3g}\right)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2' \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2' \\
& \times \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v^2(\tilde{u}_1) \\
& + \frac{1}{\pi^3}\epsilon n \left(K\frac{2\Theta}{3g}\right)\frac{2\Theta}{3}\frac{\partial \ln\Theta}{\partial r_k} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2' \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2' \\
& \times \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2 - \frac{5}{2}\right).
\end{aligned}
$$

Clearly, the term containing temperature gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}_1$ and the integration over ϕ_2' is just 2π in the other term. Hence

$$
\begin{aligned}
(III) = {} & \frac{4}{\pi^2}\epsilon n \left(K\frac{2\Theta}{3g}\right)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_2^2 \\
& \times \left\{\int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2' \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}\right\} \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v^2(\tilde{u}_1) \\
= {} & \frac{4}{\pi^2}\left(\frac{2}{3}\right)^{\frac{1}{2}} \epsilon n \ell \Theta^{1/2} \overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_2^2 R_0(\tilde{u}_1,\tilde{u}_2)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v^2(\tilde{u}_1),
\end{aligned}
$$

where $R_n(\tilde{u}_1,\tilde{u}_2)$ is defined in eq. (4.57) and the value of $R_0(\tilde{u}_1,\tilde{u}_2)$ is given in eq. (4.61). Now, using eq. (F.12), the expression for (III) simplifies to

$$
\begin{aligned}
(III) &= \frac{4\sqrt{2}}{\pi^2\sqrt{3}}\epsilon n \ell \Theta^{1/2}\frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_1^6\tilde{u}_2^2 R_0(\tilde{u}_1,\tilde{u}_2)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v^2(\tilde{u}_1) \\
&= \zeta_6\, \epsilon n \ell \Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}},
\end{aligned}
\tag{4.95}
$$

where

$$
\zeta_6 = \frac{32\sqrt{2}}{15\pi\sqrt{3}} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_1^6\tilde{u}_2^2 R_0(\tilde{u}_1,\tilde{u}_2)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v^2(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_2). \tag{4.96}
$$

The integrals in eq. (4.96) have been evaluated numerically. The result is $\zeta_6 \approx 0.0016$. Using eq. (G.1b), the fourth part of eq. (4.88) reads

$$
(IV) = \frac{2\epsilon n\Theta}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}.
$$

Substituting the explicit forms of Φ_K and $\varphi_1^{(1)}$ and using eq. (F.3),

$$
\begin{aligned}
(IV) = {} & \frac{2\epsilon n\Theta}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}} \\
& \times \left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2k}\tilde{u}_{2l}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2 - \frac{5}{2}\right)\tilde{u}_{2k}\frac{\partial\ln\Theta}{\partial r_k}\right].
\end{aligned}
$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta'_2, \phi'_2)$ described in part (III) of $Q^{K\epsilon}_{i3}$ such that $\tilde{\boldsymbol{u}}_1 \cdot \tilde{\boldsymbol{u}}_2 = \tilde{u}_1 \tilde{u}_2 \cos\theta'_2$, i.e.,

$$\begin{aligned}(IV) = & \frac{2}{\pi^3}\epsilon n \left(K\frac{2\Theta}{3g}\right)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta'_2=0}^{\pi} \int_{\phi'_2=0}^{2\pi} \mathrm{d}\phi'_2 \mathrm{d}\theta'_2 \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta'_2 \\ & \times \left(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta'_2 + \tilde{u}_2^2\right)^{1/2} \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{2k}\tilde{u}_{2l} \\ & + \frac{1}{\pi^3}\epsilon n \left(K\frac{2\Theta}{3g}\right)\frac{2\Theta}{3}\frac{\partial \ln\Theta}{\partial r_k} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta'_2=0}^{\pi} \int_{\phi'_2=0}^{2\pi} \mathrm{d}\phi'_2 \mathrm{d}\theta'_2 \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta'_2 \\ & \times \left(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta'_2 + \tilde{u}_2^2\right)^{1/2} \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_2^2 - \frac{5}{2}\right)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{2k}.\end{aligned}$$

Note that the components of $\tilde{\boldsymbol{u}}_2$ are the only functions of ϕ'_2 (see Appendix H). Hence, the integrations over ϕ'_2 result into (cf. eqs. (H.18) and (H.6) respectively)

$$\begin{aligned}(IV) = & \frac{2}{\pi^3}\epsilon n \left(K\frac{2\Theta}{3g}\right)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta'_2=0}^{\pi} \mathrm{d}\theta'_2 \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta'_2 \left(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta'_2 + \tilde{u}_2^2\right)^{1/2} \\ & \times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{1i}\tilde{u}_{1j}} \times \pi\frac{\tilde{u}_2^2}{\tilde{u}_1^2}\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_{1k}\tilde{u}_{1l}(3\cos^2\theta'_2 - 1) \\ & + \frac{1}{\pi^3}\epsilon n \left(K\frac{2\Theta}{3g}\right)\frac{2\Theta}{3}\frac{\partial \ln\Theta}{\partial r_k} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta'_2=0}^{\pi} \mathrm{d}\theta'_2 \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta'_2 \left(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta'_2 + \tilde{u}_2^2\right)^{1/2} \\ & \times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_2^2 - \frac{5}{2}\right)\overline{\tilde{u}_{1i}\tilde{u}_{1j}} \times 2\pi\frac{\tilde{u}_2}{\tilde{u}_1}\tilde{u}_{1k}\cos\theta'_2.\end{aligned}$$

Clearly, the term containing temperature gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}_1$. Thus

$$\begin{aligned}(IV) = & \frac{4\sqrt{2}}{\pi^2\sqrt{3}}\epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}\frac{\tilde{u}_2^4}{\tilde{u}_1^2} \\ & \times \left[\int_{\theta'_2=0}^{\pi} \mathrm{d}\theta'_2\, \sin\theta'_2 \left\{\frac{1}{2}(3\cos^2\theta'_2 - 1)\right\} \left(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta'_2 + \tilde{u}_2^2\right)^{1/2}\right] \\ & \times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2) \\ = & \frac{4\sqrt{2}}{\pi^2\sqrt{3}}\epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}\frac{\tilde{u}_2^4}{\tilde{u}_1^2} \\ & \times R_2(\tilde{u}_1, \tilde{u}_2)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2),\end{aligned}$$

where $R_n(\tilde{u}_1, \tilde{u}_2)$ is defined in eq. (4.57). Now, using eq. (F.12), the above equation changes to

$$\begin{aligned}(IV) & = \frac{4\sqrt{2}}{\pi^2\sqrt{3}}\epsilon n\ell\Theta^{1/2}\frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_1^6\frac{\tilde{u}_2^4}{\tilde{u}_1^2}R_2(\tilde{u}_1, \tilde{u}_2)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2) \\ & = \zeta_7\, \epsilon n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}, \end{aligned} \tag{4.97}$$

where

$$\zeta_7 = \frac{32\sqrt{2}}{15\pi\sqrt{3}} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_1^4\tilde{u}_2^4 R_2(\tilde{u}_1, \tilde{u}_2)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2)\hat{\varPhi}_e(\tilde{u}_1). \tag{4.98}$$

From eq. (4.58), the value of $R_2(\tilde{u}_1, \tilde{u}_2)$ is given by

$$R_2(\tilde{u}_1,\tilde{u}_2) = \begin{cases} \dfrac{2\tilde{u}_2^4}{35\tilde{u}_1^3} - \dfrac{2\tilde{u}_2^2}{15\tilde{u}_1}, & \text{if} \quad \tilde{u}_1 > \tilde{u}_2 \\ \dfrac{2\tilde{u}_1^4}{35\tilde{u}_2^3} - \dfrac{2\tilde{u}_1^2}{15\tilde{u}_2}, & \text{if} \quad \tilde{u}_2 > \tilde{u}_1. \end{cases} \tag{4.99}$$

The integrals in eq. (4.98) have been evaluated numerically. The result is $\zeta_7 \approx -0.0004$. Adding all the contributions to $P_{ij}^{K\epsilon}$, we get

$$\boxed{P_{ij}^{K\epsilon} = -2\epsilon\tilde{\mu}_1 n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}} \tag{4.100}$$

where $\tilde{\mu}_1 \approx 0.0578$. Clearly, pressure tensor is Newton's law of viscosity at this order also.

Collisional Dissipation

Eq. (2.13) implies that the contribution of $\Phi_{K\epsilon}$ to collisional dissipation Γ is of $O(K\epsilon^2)$. From eq. (2.34), the contribution of $\Phi_{K\epsilon}$ to collisional dissipation Γ is given by

$$\begin{aligned}\Gamma_{K\epsilon\epsilon} = \frac{\epsilon\Theta}{12\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\ \times \{\Phi_{K\epsilon}(\tilde{\boldsymbol{u}}_1) + \Phi_{K\epsilon}(\tilde{\boldsymbol{u}}_2) + \Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2) + \Phi_K(\tilde{\boldsymbol{u}}_2)\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)\}.\end{aligned}$$

Again, (similar as above) on interchanging $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$,

$$\begin{aligned}&\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \{\Phi_{K\epsilon}(\tilde{\boldsymbol{u}}_1) + \Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)\} \\ &= \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \{\Phi_{K\epsilon}(\tilde{\boldsymbol{u}}_2) + \Phi_K(\tilde{\boldsymbol{u}}_2)\Phi_\epsilon(\tilde{\boldsymbol{u}}_1)\}\,.\end{aligned}$$

$$\therefore \quad \Gamma_{K\epsilon\epsilon} = \frac{\epsilon\Theta}{6\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}(I_1 + I_2), \tag{4.101}$$

where

$$I_1 = \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_{K\epsilon}(\tilde{\boldsymbol{u}}_1) \tag{4.102}$$

and

$$I_2 = \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2). \tag{4.103}$$

First consider eq. (4.102). Using eq. (3.26), I_1 can be written as

$$I_1 = \int \mathrm{d}\tilde{\boldsymbol{u}}_1\, \mathrm{e}^{-\tilde{u}_1^2}\, \chi(\tilde{u}_1)\Phi_{K\epsilon}(\tilde{\boldsymbol{u}}_1) = \int \mathrm{d}\tilde{\boldsymbol{u}}\; \mathrm{e}^{-\tilde{u}^2}\chi(\tilde{u})\Phi_{K\epsilon}(\tilde{\boldsymbol{u}}). \tag{4.104}$$

Since we do not know the explicit form of $\Phi_{K\epsilon}$, we shall use the self adjoint property of $\tilde{\mathscr{L}}$ as following. The function χ is orthogonal to summational invariant $\tilde{\boldsymbol{u}}$, with $\tilde{f}_0(\tilde{u}) = \pi^{-3/2}\,\mathrm{e}^{-\tilde{u}^2}$ serving as a weight function, because $\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\chi(\tilde{u})\tilde{u}_i = 0$ (since the integrand is an odd function in components of $\tilde{\boldsymbol{u}}$); but χ is not orthogonal to summational invariants 1 and $\tilde{u}^2$ because

$$\left.\begin{aligned}\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\chi(\tilde{u}) = 8\pi\sqrt{2},\\ \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\chi(\tilde{u})\tilde{u}^2 = 18\pi\sqrt{2},\end{aligned}\right\} \tag{4.105}$$

but one can take advantage of the orthogonality of $\Phi_{K\epsilon}$ to the summational invariants of $\tilde{\mathscr{L}}$ and replace χ in eq. (4.104) by $\bar{\chi} \equiv \chi + \alpha + \beta\tilde{u}^2$ (where α and β need to be evaluated) so that $\bar{\chi}$ would be orthogonal to all the summational invariants of $\tilde{\mathscr{L}}$, i.e.,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\bar{\chi}(\tilde{u}) = \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\bar{\chi}(\tilde{u})\tilde{u}^2 = 0. \tag{4.106}$$

Note that the ortogonality condition with respect to summational invariant $\tilde{\boldsymbol{u}}$ will be identically satisfied because of the same reason as mentioned above. We can evalute α and β by substituting the value of $\bar{\chi}$ in eq. (4.106) and making use of eq. (4.105) which implies that

$$\begin{aligned} 8\pi\sqrt{2} + \alpha \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u}) + \beta \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\tilde{u}^2 = 0,\\ 18\pi\sqrt{2} + \alpha \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\tilde{u}^2 + \beta \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}_0(\tilde{u})\tilde{u}^4 = 0. \end{aligned}$$

The integrals above can be evaluated by changing them to spherical coordinate system. The result is

$$\left.\begin{aligned} 8\pi\sqrt{2} + \alpha + \frac{3}{2}\beta = 0\\ 18\pi\sqrt{2} + \frac{3}{2}\alpha + \frac{15}{4}\beta = 0 \end{aligned}\right\} \quad \Rightarrow \quad \alpha = -2\pi\sqrt{2} \quad \text{and} \quad \beta = -4\pi\sqrt{2}.$$

Hence $\bar{\chi}(\tilde{u}) = \chi(\tilde{u}) - 2\pi\sqrt{2}(1+2\tilde{u}^2)$, which is orthogonal to all the summational invariants. Now, define $\bar{\eta}$ as the *unique* solution of the equation $\tilde{\mathscr{L}}(\bar{\eta}) = \bar{\chi}$. By Fredholm second (uniqueness) theorem (Harris 2004, Section 6-1), $\bar{\eta}(\tilde{u})$ is orthogonal to 1 and $\tilde{u}^2$, with $\tilde{f}_0(\tilde{u}) = \pi^{-3/2}\,\mathrm{e}^{-\tilde{u}^2}$ serving as a weight function. The function $\bar{\eta}(\tilde{u})$ depends on speed $\tilde{u}$ alone and has been evaluated in a similar way as $\hat{\Phi}_e$. Hence eq. (4.104) can be written as

$$I_1 = \int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\bar{\chi}(\tilde{u})\Phi_{K\epsilon}(\tilde{\boldsymbol{u}}) = \int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\,\tilde{\mathscr{L}}(\bar{\eta}(\tilde{u}))\,\Phi_{K\epsilon}(\tilde{\boldsymbol{u}}) = \epsilon \int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\,\tilde{\mathscr{L}}(\bar{\eta}(\tilde{u}))\,\varphi_K^{(1)}$$

and using the fact that $\tilde{\mathscr{L}}$ is self-adjoint with $\mathrm{e}^{-\tilde{u}^2}$ serving as a weight function, the above expression can be written as

$$I_1 = \epsilon \int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\,\bar{\eta}(\tilde{u})\,\tilde{\mathscr{L}}(\varphi_K^{(1)}).$$

Substituting the value of $\tilde{\mathscr{L}}(\varphi_K^{(1)})$ from eq. (4.23), the above equation changes to

$$\begin{aligned} I_1 &= \epsilon \int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\,\bar{\eta}(\tilde{u}) \left\{ \tilde{S}_K - \tilde{\Xi}(\Phi_K) - \tilde{\Lambda}(\Phi_K) - \tilde{\Omega}(\Phi_K, \varphi_1^{(1)}) \right\}\\ &\equiv H_1^{K\epsilon} + H_2^{K\epsilon} + H_3^{K\epsilon}, \quad \text{(let)} \end{aligned} \tag{4.107}$$

where

$$H_1^{K\epsilon} = \epsilon \int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\ \overline{\eta}(\tilde{u})\tilde{S}_K \tag{4.108}$$

$$H_2^{K\epsilon} = -\epsilon \int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\ \overline{\eta}(\tilde{u}) \left\{\tilde{\Xi}(\Phi_K) + \tilde{\Lambda}(\Phi_K)\right\} \tag{4.109}$$

$$H_3^{K\epsilon} = -\epsilon \int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\ \overline{\eta}(\tilde{u})\, \tilde{\Omega}(\Phi_K, \varphi_1^{(1)}) \tag{4.110}$$

Now we shall simplify $H_1^{K\epsilon}$, $H_2^{K\epsilon}$ and $H_3^{K\epsilon}$ as following. Substituting the explicit form of $\tilde{S}_K$ from eq. (4.22) into eq. (4.108), we see that the terms containing number density gradient and temperature gradient vanish upon integration beacuse the corresponding integrands are odd functions in components of $\tilde{\boldsymbol{u}}$. Hence

$$\begin{aligned} H_1^{K\epsilon} = \frac{4}{3}K\epsilon\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\ \overline{\eta}(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j} \\ \times\left[\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\hat{\Phi}_v'(\tilde{u})\tilde{u}^2 - \hat{\Phi}_v(\tilde{u})(\tilde{u}^2-3)\right) + \frac{3}{2}\left\{\hat{\Phi}_e(\tilde{u}) - \hat{\Phi}_e'(\tilde{u})\right\}\right] \end{aligned}$$

and using eq. (F.10),

$$H_1^{K\epsilon} = 0. \tag{4.111}$$

Substituting the values of $\tilde{\Xi}$ and $\tilde{\Lambda}$ from eqs. (2.39) and (2.40) respectively in eq. (4.109), one obtains

$$\begin{aligned} H_2^{K\epsilon} = &-\frac{\epsilon}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(1-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right)\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2')\}\overline{\eta}(\tilde{u}_1) \\ &-\frac{\epsilon}{\pi^{5/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2')\}\,\overline{\eta}(\tilde{u}_1). \end{aligned}$$

Following a similar procedure as in the derivation of $Q_{i_2}^{K\epsilon}$, one obtains

$$H_2^{K\epsilon} = -\frac{\epsilon}{\pi^{5/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\Phi_K(\tilde{\boldsymbol{u}}_1) + \Phi_K(\tilde{\boldsymbol{u}}_2)\}\,\overline{\eta}(\tilde{u}_1'). \tag{4.112}$$

Note that, in eq. (4.112), the non-primed velocities are now precollisional velocities and the primed velocities are now postcollisional velocities. To simplify the integral in eq. (4.112) further, we use the property of delta function, i.e.,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,\delta(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1')\,\overline{\eta}(\tilde{u}) = \overline{\eta}(\tilde{u}_1').$$

Since $\tilde{\boldsymbol{u}}_1'$ is the postcollisional velocity here, hence using eq. (2.1a), the above equation can be written as

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,\delta\Big(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1 + q(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big)\,\overline{\eta}(\tilde{u}) = \overline{\eta}(\tilde{u}_1'),$$

where $q = \frac{1+e}{2}$. Substituting this in eq. (4.112), we get

$$H_2^{K\epsilon} = -\frac{\epsilon}{\pi^{5/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\Phi_K(\tilde{\boldsymbol{u}}_1) + \Phi_K(\tilde{\boldsymbol{u}}_2)\}\,\overline{\eta}(\tilde{u})I_\delta, \tag{4.113}$$

where I_δ is given in eq. (4.34). The integral in eq. (4.113) is then split into two parts. The first part is

$$\begin{aligned}(I) &= -\frac{\epsilon}{\pi^{5/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\varPhi_K(\tilde{\boldsymbol{u}}_1)\bar{\eta}(\tilde{u})I_\delta \\ &= -\frac{\epsilon}{\pi^{5/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u})I_\delta \\ &\times\left[2K\frac{2\varTheta}{3g}\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}+K\frac{2\varTheta}{3g}\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\frac{\partial\ln\varTheta}{\partial r_i}\right].\end{aligned} \tag{4.114}$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}$ in the above equation and using eq. (F.3), we get

$$\begin{aligned}(I) = &-\frac{2K\epsilon}{\pi^{5/2}}\frac{2\varTheta}{3g}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\bar{\eta}(\tilde{u})I_\delta \\ &\times\hat{\varPhi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j) \\ &-\frac{K\epsilon}{\pi^{5/2}}\frac{2\varTheta}{3g}\frac{\partial\ln\varTheta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\bar{\eta}(\tilde{u})I_\delta \\ &\times\hat{\varPhi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i).\end{aligned}$$

The integral over $\tilde{\boldsymbol{u}}_2$ is given in eq. (4.77). Hence using eq. (4.77), we get

$$\begin{aligned}(I) = &-\frac{2K\epsilon}{\pi^{3/2}}\frac{2\varTheta}{3g}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\bar{\eta}(\tilde{u})\hat{\varPhi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|) \\ &\times(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2} \\ &-\frac{K\epsilon}{\pi^{3/2}}\frac{2\varTheta}{3g}\frac{\partial\ln\varTheta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\bar{\eta}(\tilde{u})\hat{\varPhi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|) \\ &\times\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i)\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}.\end{aligned}$$

The integrations over $\tilde{\boldsymbol{s}}$ are performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}=\tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$\begin{aligned}(I) = &-\frac{2K\epsilon}{\pi^{3/2}}\frac{2\varTheta}{3g}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)} \\ &\times\bar{\eta}(\tilde{u})\hat{\varPhi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2} \\ &-\frac{K\epsilon}{\pi^{3/2}}\frac{2\varTheta}{3g}\frac{\partial\ln\varTheta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\ &\times\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\bar{\eta}(\tilde{u})\hat{\varPhi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big) \\ &\times\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i)\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}.\end{aligned}$$

Note that the components of $\tilde{\boldsymbol{s}}$ are the only functions of ϕ' (see Appendix H). Hence, using eqs. (H.19) and (H.6) respectively, we get

$$
\begin{aligned}
(I) = &-\frac{2K\epsilon}{\pi^{3/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
&\times\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\bar{\eta}(\tilde{u})\hat{\varPhi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big) \\
&\times\left[2\pi\frac{1}{\tilde{u}^2}\overline{\frac{\partial V_i}{\partial r_j}}\tilde{u}_i\tilde{u}_j\left\{\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\frac{1}{2}\tilde{s}^2(3\cos^2\theta'-1)\right\}\right]\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2} \\
&-\frac{K\epsilon}{\pi^{3/2}}\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta' \\
&\times\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\bar{\eta}(\tilde{u})\hat{\varPhi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big) \\
&\times\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2-\frac{5}{2}\right)\left(2\pi\tilde{u}_i-2\pi\frac{\tilde{s}}{\tilde{u}}\tilde{u}_i\cos\theta'\right)\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}.
\end{aligned}
$$

Clearly, the term containing temperature gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}$ and let $\cos\theta' = y$. Therefore

$$
\begin{aligned}
(I) = &-\frac{4K\epsilon}{\pi^{1/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\tilde{u}_i\tilde{u}_j\frac{\tilde{s}}{\tilde{u}^2}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)} \\
&\times\bar{\eta}(\tilde{u})\hat{\varPhi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\left\{\tilde{u}^2-2\tilde{u}\tilde{s}y+\frac{1}{2}\tilde{s}^2(3y^2-1)\right\}\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}.
\end{aligned}
$$

Using eq. (F.9c), the integration over $\tilde{\boldsymbol{u}}$ results into

$$
\begin{aligned}
(I) = &-\frac{4K\epsilon}{\pi^{1/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{4\pi}{3}\overline{\frac{\partial V_i}{\partial r_i}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{y=-1}^{1}\mathrm{d}y\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\,\tilde{u}^4\frac{\tilde{s}}{\tilde{u}^2}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)} \\
&\times\bar{\eta}(\tilde{u})\hat{\varPhi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\left\{\tilde{u}^2-2\tilde{u}\tilde{s}y+\frac{1}{2}\tilde{s}^2(3y^2-1)\right\}\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}. \qquad (4.115)
\end{aligned}
$$

but since $\overline{\dfrac{\partial V_i}{\partial r_i}} = 0$, we have

$$
(I) = 0.
$$

The second part of eq. (4.113) is

$$
\begin{aligned}
(II) &= -\frac{\epsilon}{\pi^{5/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\varPhi_K(\tilde{\boldsymbol{u}}_2)\bar{\eta}(\tilde{u})I_\delta \\
&= -\frac{\epsilon}{\pi^{5/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u})I_\delta \\
&\quad\times\left[2K\frac{2\Theta}{3g}\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}+K\frac{2\Theta}{3g}\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\frac{\partial\ln\Theta}{\partial r_i}\right]. \qquad (4.116)
\end{aligned}
$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}$ in the above equation and using eq. (F.3), we get

$$
\begin{aligned}
(II) = &-\frac{2K\epsilon}{\pi^{5/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\bar{\eta}(\tilde{u})\hat{\varPhi}_v(\tilde{u}_2)\tilde{u}_{2i}\tilde{u}_{2j}I_\delta \\
&-\frac{K\epsilon}{\pi^{5/2}}\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\bar{\eta}(\tilde{u})\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}I_\delta. \qquad (4.117)
\end{aligned}
$$

The functions depending on $\tilde{u}_2$ in the integrands of eq. (4.117) are exactly similar to the functions depending on $\tilde{u}_2$ in the integrands of eq. (4.41) (except for the indices names). Therefore the integrations over $\tilde{\boldsymbol{u}}_2$ are carried out by following a similar procedure as given below eq. (4.41). Finally, we get (cf. eq. (4.42))

$$
\begin{aligned}
(II) = &-\frac{2K\epsilon}{\pi^{5/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\bar{\eta}(\tilde{u})\\
&\times\frac{\tilde{s}_i\tilde{s}_j}{\tilde{s}^3}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^3\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\frac{\tilde{s}}{\tilde{u}_2}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right)^2-1\right\}\\
&-\frac{K\epsilon}{\pi^{5/2}}\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{2\pi}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\bar{\eta}(\tilde{u})\\
&\times\frac{\tilde{s}_i}{\tilde{s}^2}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{(1-q)}{q}\frac{\tilde{s}}{\tilde{u}_2}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right).
\end{aligned}
\tag{4.118}
$$

The integrations over $\tilde{\boldsymbol{s}}$ are performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q^{K\epsilon}_{i_2}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}=\tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$
\begin{aligned}
(II) = &-\frac{2K\epsilon}{\pi^{3/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\\
&\times\bar{\eta}(\tilde{u})\frac{\tilde{s}_i\tilde{s}_j}{\tilde{s}^3}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2-\tilde{u}_2^2\right\}\\
&-\frac{2K\epsilon}{\pi^{3/2}}\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\\
&\times\bar{\eta}(\tilde{u})\frac{\tilde{s}_i}{\tilde{s}^2}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right).
\end{aligned}
$$

Note that the components of $\tilde{\boldsymbol{s}}$ are the only functions of ϕ' (see Appendix H). Hence, using eqs. (H.18) and (H.6) respectively, we get

$$
\begin{aligned}
(II) = &-\frac{2K\epsilon}{\pi^{3/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\\
&\times\bar{\eta}(\tilde{u})\left\{\pi\frac{\tilde{s}^2}{\tilde{u}^2}\overline{\frac{\partial V_i}{\partial r_j}}\tilde{u}_i\tilde{u}_j(3\cos^2\theta'-1)\right\}\frac{1}{\tilde{s}^3}\\
&\times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2-\tilde{u}_2^2\right\}\\
&-\frac{2K\epsilon}{\pi^{3/2}}\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\bar{\eta}(\tilde{u})\frac{1}{\tilde{s}^2}\\
&\times 2\pi\frac{\tilde{s}}{\tilde{u}}\tilde{u}_i\cos\theta'\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right).
\end{aligned}
$$

Clearly, the term containing temperature gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}$ and let $\cos\theta' = y$. Therefore

$$(II) = -\frac{2K\epsilon}{\pi^{1/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{y=-1}^{1}\mathrm{d}y\mathrm{d}\tilde{s}\,\tilde{u}_i\tilde{u}_j\frac{\tilde{s}}{\tilde{u}^2}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}$$
$$\times\,\bar{\eta}(\tilde{u})(3y^2-1)\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\tilde{u}_2^2\right\}.$$

Using eq. (F.9c), the integration over $\tilde{\boldsymbol{u}}$ results into

$$(II) = -\frac{2K\epsilon}{\pi^{1/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{4\pi}{3}\overline{\frac{\partial V_i}{\partial r_i}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{y=-1}^{1}\mathrm{d}y\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\,\tilde{u}^4\frac{\tilde{s}}{\tilde{u}^2}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}$$
$$\times\,\bar{\eta}(\tilde{u})(3y^2-1)\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2-\tilde{u}_2^2\right\} \quad (4.119)$$

but since $\overline{\dfrac{\partial V_i}{\partial r_i}} = 0$, we have

$$(II) = 0.$$

Therefore

$$H_2^{K\epsilon} = 0. \quad (4.120)$$

Substituting the value of $\tilde{\Omega}$ from eq. (2.38) in eq. (4.110), one obtains

$$H_3^{K\epsilon} = -\frac{\epsilon}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)$$
$$\times\,\{\Phi_K(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')+\Phi_K(\tilde{\boldsymbol{u}}_2')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1')-\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)-\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\}. \quad (4.121)$$

In eq. (4.121), the velocity transformation corresponds to the elastic limit. Following a similar procedure as in the derivation of $Q_{i3}^{K\epsilon}$, one obtains

$$H_3^{K\epsilon} = -\frac{\epsilon}{\pi^{5/2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\}\bar{\eta}(\tilde{u})I_\delta^{(0)}$$
$$-\frac{\epsilon}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\{\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)+\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\}, \quad (4.122)$$

where $I_\delta^{(0)}$ is given in eq. (4.48). The term $H_3^{K\epsilon}$ is split into four parts. The first part is

$$(I) = -\frac{\epsilon}{\pi^{5/2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\bar{\eta}(\tilde{u})I_\delta^{(0)}$$
$$-\frac{\epsilon}{\pi^{5/2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\bar{\eta}(\tilde{u})I_\delta^{(0)}$$
$$\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}+K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\frac{\partial\ln\Theta}{\partial r_i}\right]. \quad (4.123)$$

Note that except for the extra term $\hat{\Phi}_e(\tilde{u}_2)$ and the above definition of $I_\delta^{(0)}$, the integrand in eq. (4.123) is similar to that in eq. (4.114). Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}$ in the above equation and using eq. (F.3), we get

$$\begin{aligned}(I) = &-\frac{2K\epsilon}{\pi^{5/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_e(\tilde{u}_2)\bar{\eta}(\tilde{u})I_\delta^{(0)}\\ &\times \hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\\ &-\frac{K\epsilon}{\pi^{5/2}}\frac{2\Theta}{3g}\frac{\partial \ln\Theta}{\partial r_i}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_e(\tilde{u}_2)\bar{\eta}(\tilde{u})I_\delta^{(0)}\\ &\times \hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i).\end{aligned}$$

The integral over $\tilde{\boldsymbol{u}}_2$ is given in eq. (4.51). Hence using eq. (4.51), we get

$$\begin{aligned}(I) = &-\frac{2K\epsilon}{\pi^{5/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\frac{2\pi}{\tilde{s}}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\hat{\Phi}_e\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)\\ &\times \mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\bar{\eta}(\tilde{u})\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\\ &-\frac{K\epsilon}{\pi^{5/2}}\frac{2\Theta}{3g}\frac{\partial \ln\Theta}{\partial r_i}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\frac{2\pi}{\tilde{s}}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\hat{\Phi}_e\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)\\ &\times \mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\bar{\eta}(\tilde{u})\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i).\end{aligned}$$

The integrations over $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{u}}$ are performed by following a similar procedure as performed following eq. (4.114). Finally, we get (cf. eq. (4.115))

$$\begin{aligned}(I) = &-\frac{8K\epsilon}{\pi^{1/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{4\pi}{3}\overline{\frac{\partial V_i}{\partial r_i}}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{y=-1}^{1}\mathrm{d}y\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\,\tilde{u}_2\tilde{u}^4\frac{\tilde{s}}{\tilde{u}^2}\\ &\times \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\bar{\eta}(\tilde{u})\hat{\Phi}_v\left((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\right)\hat{\Phi}_e((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2})\\ &\times\left\{\tilde{u}^2-2\tilde{u}\tilde{s}y+\frac{1}{2}\tilde{s}^2(3y^2-1)\right\}\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\end{aligned}$$

but since $\overline{\dfrac{\partial V_i}{\partial r_i}} = 0$, we have

$$(I) = 0.$$

The second part of eq. (4.122) is

$$\begin{aligned}(II) &= -\frac{\epsilon}{\pi^{5/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\bar{\eta}(\tilde{u})I_\delta^{(0)}\\ &= -\frac{\epsilon}{\pi^{5/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_1)\bar{\eta}(\tilde{u})I_\delta^{(0)}\\ &\quad\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}+K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{2i}\frac{\partial \ln\Theta}{\partial r_i}\right].\end{aligned}\tag{4.124}$$

Note that except for the extra term $\hat{\varPhi}_e(\tilde{u}_1)$ and the above definition of $I_\delta^{(0)}$, the integrand in eq. (4.124) is similar to the one in eq. (4.116). Therefore, the integrations are carried out by following a similar procedure as performed following eq. (4.116). Finally, we get (cf. eq. (4.119))

$$(II) = -\frac{2K\epsilon}{\pi^{1/2}}\frac{2\varTheta}{3g}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\frac{4\pi}{3}\overline{\frac{\partial V_i}{\partial r_i}}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{y=-1}^{1}\mathrm{d}y\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\,\tilde{u}^4\frac{\tilde{s}}{\tilde{u}^2}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\bar{\eta}(\tilde{u})$$
$$\times(3y^2-1)\hat{\varPhi}_e\left((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\right)\int_{\tilde{u}_2=|\tilde{u}y|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\varPhi}_v(\tilde{u}_2)(3\tilde{u}^2y^2-\tilde{u}_2^2)$$

but since $\overline{\dfrac{\partial V_i}{\partial r_i}}=0$, we have

$$(II)=0.$$

In the third and fourth part of eq. (4.122), the integration over $\hat{\boldsymbol{k}}$ is trivial. Hence, using eq. (G.1b), the third part of eq. (4.122) reads

$$(III) = -\frac{\epsilon}{\pi^{3/2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\varPhi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2).$$

Substituing the explicit forms of $\varPhi_K$ and $\varphi_1^{(1)}$ and using eq. (F.3),

$$(III) = -\frac{\epsilon}{\pi^{3/2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_e(\tilde{u}_2)\bar{\eta}(\tilde{u}_1)$$
$$\times\left[2K\frac{2\varTheta}{3g}\hat{\varPhi}_v(\tilde{u}_1)\tilde{u}_{1i}\tilde{u}_{1j}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}+K\frac{2\varTheta}{3g}\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\frac{\partial\ln\varTheta}{\partial r_i}\right].$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2,\theta_2',\phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2=\tilde{u}_1\tilde{u}_2\cos\theta_2'$, i.e.,

$$(III) = -\frac{2K\epsilon}{\pi^{3/2}}\frac{2\varTheta}{3g}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'$$
$$\times(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2)^{1/2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_e(\tilde{u}_2)\bar{\eta}(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1)\tilde{u}_{1i}\tilde{u}_{1j}$$
$$-\frac{K\epsilon}{\pi^{3/2}}\frac{2\varTheta}{3g}\frac{\partial\ln\varTheta}{\partial r_i}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'$$
$$\times(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2)^{1/2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_e(\tilde{u}_2)\bar{\eta}(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}.$$

Clearly, the term containing temperature gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}_1$ and the integration over ϕ_2' is just 2π in the other term. Hence

$$(III) = -\frac{4K\epsilon}{\pi^{1/2}}\frac{2\varTheta}{3g}\left(\frac{3}{2\varTheta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_2^2\sin\theta_2'$$
$$\times(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2)^{1/2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_e(\tilde{u}_2)\bar{\eta}(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1).$$

Using eq. (F.9c), the expression for (III) simplifies to

$$(III) = -\frac{4K\epsilon}{\pi^{1/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{4\pi}{3}\overline{\frac{\partial V_i}{\partial r_i}}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_1^4\tilde{u}_2^2\sin\theta_2'$$
$$\times\left(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2\right)^{1/2}\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\bar{\eta}(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_1).$$

but since $\overline{\frac{\partial V_i}{\partial r_i}} = 0$, we have

$$(III) = 0.$$

Using eq. (G.1b), the fourth part of eq. (4.122) reads

$$(IV) = -\frac{\epsilon}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1).$$

Substituting the explicit forms of Φ_K and $\varphi_1^{(1)}$ and using eq. (F.3),

$$(IV) = -\frac{\epsilon}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1)$$
$$\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2i}\tilde{u}_{2j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}+K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\frac{\partial\ln\Theta}{\partial r_i}\right].$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2 = \tilde{u}_1\tilde{u}_2\cos\theta_2'$, i.e.,

$$(IV) = -\frac{2K\epsilon}{\pi^{3/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'$$
$$\times\left(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2\right)^{1/2}\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2i}\tilde{u}_{2j}$$
$$-\frac{K\epsilon}{\pi^{3/2}}\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'$$
$$\times\left(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2\right)^{1/2}\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}.$$

Note that the components of $\tilde{\boldsymbol{u}}_2$ are the only functions of ϕ_2' (see Appendix H). Hence, the integrations over ϕ_2' result into (cf. eqs. (H.18) and (H.6) respectively)

$$(IV) = -\frac{2K\epsilon}{\pi^{3/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\left(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2\right)^{1/2}$$
$$\times\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\left\{\pi\frac{\tilde{u}_2^2}{\tilde{u}_1^2}\overline{\frac{\partial V_i}{\partial r_j}}\tilde{u}_{1i}\tilde{u}_{1j}(3\cos^2\theta_2'-1)\right\}$$
$$-\frac{K\epsilon}{\pi^{3/2}}\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\left(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2\right)^{1/2}$$
$$\times\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(2\pi\frac{\tilde{u}_2}{\tilde{u}_1}\tilde{u}_{1i}\cos\theta_2'\right).$$

Clearly, the term containing temperature gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}_1$. Thus

$$(IV) = -\frac{2K\epsilon}{\pi^{1/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_{1i}\tilde{u}_{1j}\frac{\tilde{u}_2^4}{\tilde{u}_1^2}\sin\theta_2'(3\cos^2\theta_2'-1)$$
$$\times(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\overline{\eta}(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2).$$

Using eq. (F.9c), the expression for (IV) simplifies to

$$(IV) = -\frac{2K\epsilon}{\pi^{1/2}}\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{4\pi}{3}\overline{\frac{\partial V_i}{\partial r_i}}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_1^4\frac{\tilde{u}_2^4}{\tilde{u}_1^2}\sin\theta_2'(3\cos^2\theta_2'-1)$$
$$\times(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\overline{\eta}(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)$$

but since $\overline{\frac{\partial V_i}{\partial r_i}} = 0$, we have

$$(IV) = 0.$$

Hence

$$H_3^{K\epsilon} = 0. \tag{4.125}$$

From eqs. (4.111), (4.120), (4.125) and (4.107), we conclude that

$$I_1 = 0. \tag{4.126}$$

Next we shall evaluate I_2 (given in eq. (4.103)). Substituting the explicit forms of Φ_K and Φ_ϵ and using eq. (F.3), eq. (4.103) can be written as

$$I_2 = \epsilon\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)$$
$$\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_1)\tilde{u}_{1i}\tilde{u}_{1j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\frac{\partial\ln\Theta}{\partial r_i}\right]$$
$$= 2K\epsilon\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u}_1)\tilde{u}_{1i}\tilde{u}_{1j}$$
$$+ K\epsilon\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}.$$

The integrations over $\tilde{\boldsymbol{u}}_2$ are performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2 = \tilde{u}_1\tilde{u}_2\cos\theta_2'$, i.e.,

$$I_2 = 2K\epsilon\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'$$
$$\times(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{3/2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u}_1)\tilde{u}_{1i}\tilde{u}_{1j}$$
$$+ K\epsilon\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'$$
$$\times(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{3/2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}.$$

Clearly, the term containing temperature gradient vanishes because the corresponding integrand is an odd function in components of $\tilde{\boldsymbol{u}}_1$ and the integration over ϕ_2' is just 2π in the other term.

$$\therefore \quad I_2 = 4\pi K\epsilon\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_2^2\sin\theta_2' \\ \times\left(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2\right)^{3/2}\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u}_1).$$

Using eq. (F.9c), the expression for I_2 simplifies to

$$I_2 = 4\pi K\epsilon\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{4\pi}{3}\overline{\frac{\partial V_i}{\partial r_i}}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_1^4\tilde{u}_2^2\sin\theta_2' \\ \times\left(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2\right)^{3/2}\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u}_1)$$

but since $\overline{\dfrac{\partial V_i}{\partial r_i}} = 0$, we have

$$I_2 = 0. \tag{4.127}$$

Hence from eqs. (4.126), (4.127) and (4.101),

$$\boxed{\Gamma_{K\epsilon\epsilon} = 0} \tag{4.128}$$

i.e., the collisional dissipation is zero at this order also.

4.3 Constitutive Relations at $O(KK)$

In the following, we shall first simplify the expanded Boltzmann equation at this order.

4.3.1 Simplified form of eq. (2.36)

Collecting $O(K^2)$ terms in eq. (2.36), we have

$$\tilde{\mathscr{D}}_{KK}\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_{KK}V_i + \left(\tilde{u}^2-\frac{3}{2}\right)\tilde{\mathscr{D}}_{KK}\ln\Theta \\ + \Phi_K\left\{\tilde{\mathscr{D}}_K\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_K V_i + \left(\tilde{u}^2-\frac{3}{2}\right)\tilde{\mathscr{D}}_K\ln\Theta - 2K\tilde{g}_i\tilde{u}_i\right\} \\ + \tilde{\mathscr{D}}_K\Phi_K + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\Phi_K = \tilde{\mathscr{L}}(\Phi_{KK}) + \frac{1}{2}\tilde{\Omega}(\Phi_K,\Phi_K)$$

$$\Rightarrow \quad \tilde{\mathscr{L}}(\Phi_{KK}) = \tilde{\mathscr{D}}_{KK}\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_{KK}V_i + \left(\tilde{u}^2-\frac{3}{2}\right)\tilde{\mathscr{D}}_{KK}\ln\Theta \\ + \Phi_K\left\{\tilde{\mathscr{D}}_K\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_K V_i + \left(\tilde{u}^2-\frac{3}{2}\right)\tilde{\mathscr{D}}_K\ln\Theta - 2K\tilde{g}_i\tilde{u}_i\right\} \\ + \tilde{\mathscr{D}}_K\Phi_K + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\Phi_K - \frac{1}{2}\tilde{\Omega}(\Phi_K,\Phi_K). \tag{4.129}$$

Now, the operation of $\tilde{\mathscr{D}}_{KK}$ on the hydrodynamic fields can be computed as following: From eq. (2.20),

$$\tilde{\mathscr{D}}_{KK} \ln n = 0. \tag{4.130}$$

From eqs. (2.21) and (3.22),

$$\begin{aligned}\tilde{\mathscr{D}}_{KK} V_i &= -K\frac{2\Theta}{3g}\frac{1}{n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial P^K_{ij}}{\partial r_j} = -K\frac{2\Theta}{3g}\frac{1}{n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_j}\left(-2\tilde{\mu}_0 n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}\right)\\
&= 2\tilde{\mu}_0 K\frac{2\Theta}{3g}\frac{1}{n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_j}\left(n\times\frac{1}{\pi n d^2}\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}\right)\\
&= 2\tilde{\mu}_0 K\frac{2\Theta}{3g}\frac{1}{\pi n d^2}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_j}\left(\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}\right)\\
&= 2\tilde{\mu}_0 K\frac{2\Theta}{3g}\left(K\frac{2\Theta}{3g}\right)\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(\frac{1}{2\Theta^{1/2}}\frac{\partial\Theta}{\partial r_j}\overline{\frac{\partial V_i}{\partial r_j}} + \Theta^{1/2}\frac{\partial}{\partial r_j}\overline{\frac{\partial V_i}{\partial r_j}}\right)\end{aligned}$$

or

$$\tilde{\mathscr{D}}_{KK} V_i = \left(K\frac{2\Theta}{3g}\right)^2\tilde{\mu}_0\left(\frac{3}{2}\right)^{\frac{1}{2}}\left(\frac{\partial\ln\Theta}{\partial r_j}\overline{\frac{\partial V_i}{\partial r_j}} + 2\frac{\partial}{\partial r_j}\overline{\frac{\partial V_i}{\partial r_j}}\right). \tag{4.131}$$

From eqs. (2.22), (3.22) and (3.24),

$$\begin{aligned}\tilde{\mathscr{D}}_{KK}\ln\Theta &= K\frac{2\Theta}{3g}\left\{-\frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}P^K_{ij} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial Q^K_j}{\partial r_j}\right\}\\
&= -K\frac{2\Theta}{3g}\frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\frac{\partial V_i}{\partial r_j}\left(-2\tilde{\mu}_0 n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}\right) + \frac{\partial}{\partial r_j}\left(-\tilde{\kappa}_0 n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_j}\right)\right\}\\
&= K\frac{2\Theta}{3g}\frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\frac{\partial V_i}{\partial r_j}\left(2\tilde{\mu}_0 n\frac{1}{\pi n d^2}\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}\right) + \frac{\partial}{\partial r_j}\left(\tilde{\kappa}_0 n\frac{1}{\pi n d^2}\Theta^{1/2}\frac{\partial\Theta}{\partial r_j}\right)\right\}\\
&= K\frac{2\Theta}{3g}\frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{1}{\pi d^2}\left\{2\tilde{\mu}_0\Theta^{1/2}\frac{\partial V_i}{\partial r_j}\overline{\frac{\partial V_i}{\partial r_j}} + \tilde{\kappa}_0\frac{\partial}{\partial r_j}\left(\Theta^{1/2}\frac{\partial\Theta}{\partial r_j}\right)\right\}\\
&= K\frac{2\Theta}{3g}\frac{1}{\pi n d^2}\left(\frac{3}{2}\right)^{\frac{1}{2}}\frac{2}{\Theta^{3/2}}\left\{2\tilde{\mu}_0\Theta^{1/2}\frac{\partial V_i}{\partial r_j}\overline{\frac{\partial V_i}{\partial r_j}} + \tilde{\kappa}_0\left(\Theta^{1/2}\frac{\partial^2\Theta}{\partial r_j^2} + \frac{1}{2\Theta^{1/2}}\frac{\partial\Theta}{\partial r_j}\frac{\partial\Theta}{\partial r_j}\right)\right\}\\
&= \left(K\frac{2\Theta}{3g}\right)^2\left(\frac{3}{2}\right)^{\frac{1}{2}}\left[4\tilde{\mu}_0\frac{1}{\Theta}\frac{\partial V_i}{\partial r_j}\overline{\frac{\partial V_i}{\partial r_j}} + \tilde{\kappa}_0\left\{\frac{2}{\Theta}\frac{\partial^2\Theta}{\partial r_j^2} + \left(\frac{1}{\Theta}\frac{\partial\Theta}{\partial r_j}\right)\left(\frac{1}{\Theta}\frac{\partial\Theta}{\partial r_j}\right)\right\}\right],\end{aligned}$$

but,

$$\begin{aligned}\because\ \frac{\partial^2\ln\Theta}{\partial r_j\partial r_j} &= \frac{\partial}{\partial r_j}\left(\frac{\partial\ln\Theta}{\partial r_j}\right) = \frac{\partial}{\partial r_j}\left(\frac{1}{\Theta}\frac{\partial\Theta}{\partial r_j}\right) = -\frac{1}{\Theta^2}\frac{\partial\Theta}{\partial r_j}\frac{\partial\Theta}{\partial r_j} + \frac{1}{\Theta}\frac{\partial^2\Theta}{\partial r_j^2}\\
\Rightarrow\ \frac{1}{\Theta}\frac{\partial^2\Theta}{\partial r_j^2} &= \frac{\partial^2\ln\Theta}{\partial r_j\partial r_j} + \left(\frac{1}{\Theta}\frac{\partial\Theta}{\partial r_j}\right)\left(\frac{1}{\Theta}\frac{\partial\Theta}{\partial r_j}\right) = \frac{\partial^2\ln\Theta}{\partial r_j\partial r_j} + \frac{\partial\ln\Theta}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_j}.\end{aligned}$$

Hence

$$\tilde{\mathscr{D}}_{KK}\ln\Theta = \left(K\frac{2\Theta}{3g}\right)^2\left(\frac{3}{2}\right)^{\frac{1}{2}}\left[\frac{8}{3}\tilde{\mu}_0\frac{3}{2\Theta}\frac{\partial V_i}{\partial r_j}\overline{\frac{\partial V_i}{\partial r_j}} + \tilde{\kappa}_0\left\{2\frac{\partial^2\ln\Theta}{\partial r_j\partial r_j} + 3\frac{\partial\ln\Theta}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_j}\right\}\right]$$

or

$$\tilde{\mathscr{D}}_{KK} \ln \Theta = \left(K\frac{2\Theta}{3g}\right)^2 \left(\frac{3}{2}\right)^{\frac{1}{2}} \left(\frac{8}{3}\tilde{\mu}_0 \frac{3}{2\Theta}\frac{\partial V_i}{\partial r_j}\overline{\frac{\partial V_i}{\partial r_j}} + 2\tilde{\kappa}_0 \frac{\partial^2 \ln \Theta}{\partial r_j \partial r_j} + 3\tilde{\kappa}_0 \frac{\partial \ln \Theta}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_j}\right). \quad (4.132)$$

Note that the quantity in the curly brackets in eq. (4.129) is the left-hand side of eq. (3.1), whose simplified value is the right-hand side of eq. (3.5). Hence, with the help of eqs. (3.5) and (3.18), the term containing this curly bracketed quantity in eq. (4.129) can be simplified as follows:

$$\begin{aligned}
&\Phi_K \left\{\tilde{\mathscr{D}}_K \ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{u}_i \tilde{\mathscr{D}}_K V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathscr{D}}_K \ln \Theta - 2K\tilde{g}_i \tilde{u}_i\right\} \\
&= \left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln \Theta}{\partial r_i}\right] \\
&\quad \times \left[2K\frac{2\Theta}{3g}\overline{\tilde{u}_k\tilde{u}_l}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_l} + K\frac{2\Theta}{3g}\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_k\frac{\partial \ln \Theta}{\partial r_k}\right].
\end{aligned}$$

Using, eq. (F.3) and the definition $\overline{\tilde{u}_k\tilde{u}_l} = \frac{\tilde{u}_k\tilde{u}_l + \tilde{u}_l\tilde{u}_k}{2} - \frac{1}{3}\tilde{u}^2\delta_{kl}$, we get

$$\begin{aligned}
&\Phi_K \left\{\tilde{\mathscr{D}}_K \ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{u}_i \tilde{\mathscr{D}}_K V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathscr{D}}_K \ln \Theta - 2K\tilde{g}_i \tilde{u}_i\right\} \\
&= \left(K\frac{2\Theta}{3g}\right)^2 \left[2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}} + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln \Theta}{\partial r_i}\right] \\
&\quad \times \left[2\tilde{u}_k\tilde{u}_l\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_l} - \frac{2}{3}\tilde{u}^2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k} + \left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_k\frac{\partial \ln \Theta}{\partial r_k}\right] \\
&= \left(K\frac{2\Theta}{3g}\right)^2 \left[4\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_l} - \frac{4}{3}\hat{\Phi}_v(\tilde{u})\tilde{u}^2\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_k}\right. \\
&\quad + 2\hat{\Phi}_v(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial \ln \Theta}{\partial r_k} + 2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\tilde{u}_k\tilde{u}_l\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_l}\frac{\partial \ln \Theta}{\partial r_i} \\
&\quad \left. - \frac{2}{3}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}^2\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\frac{\partial \ln \Theta}{\partial r_i} + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)^2\tilde{u}_i\tilde{u}_k\frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_k}\right] \\
&= \left(K\frac{2\Theta}{3g}\right)^2 \left[4\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_l} - \frac{4}{3}\hat{\Phi}_v(\tilde{u})\tilde{u}^2\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_k}\right. \\
&\quad + 2\hat{\Phi}_v(\tilde{u})\left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial \ln \Theta}{\partial r_k} - 2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial \ln \Theta}{\partial r_k} \\
&\quad + 2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\tilde{u}_k\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i} - \frac{2}{3}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}^2\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i} \\
&\quad \left. + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j} - \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j}\right]
\end{aligned}$$

or

$$\varPhi_K\left\{\tilde{\mathscr{D}}_K\ln n+2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_K V_i+\left(\tilde{u}^2-\frac{3}{2}\right)\tilde{\mathscr{D}}_K\ln\Theta-2K\tilde{g}_i\tilde{u}_i\right\}$$
$$=\left(K\frac{2\Theta}{3g}\right)^2\left[4\hat{\varPhi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_l}-\frac{4}{3}\hat{\varPhi}_v(\tilde{u})\tilde{u}^2\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_k}\right.$$
$$+2\hat{\varPhi}_v(\tilde{u})\left(\tilde{u}^2-\frac{3}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial\ln\Theta}{\partial r_k}-2\hat{\varPhi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial\ln(n\Theta)}{\partial r_k}$$
$$+2\hat{\varPhi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial\ln n}{\partial r_k}+2\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_i}$$
$$-\frac{2}{3}\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^2\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_i}+\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{3}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}$$
$$\left.-\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln(n\Theta)}{\partial r_j}+\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln n}{\partial r_j}\right]$$

or

$$\varPhi_K\left\{\tilde{\mathscr{D}}_K\ln n+2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_K V_i+\left(\tilde{u}^2-\frac{3}{2}\right)\tilde{\mathscr{D}}_K\ln\Theta-2K\tilde{g}_i\tilde{u}_i\right\}$$
$$=\left(K\frac{2\Theta}{3g}\right)^2\left[4\hat{\varPhi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_l}-\frac{4}{3}\hat{\varPhi}_v(\tilde{u})\tilde{u}^2\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_k}\right.$$
$$+2\hat{\varPhi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial\ln n}{\partial r_k}+2\hat{\varPhi}_v(\tilde{u})\left(\tilde{u}^2-\frac{3}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial\ln\Theta}{\partial r_k}$$
$$-2\hat{\varPhi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial\ln(n\Theta)}{\partial r_k}+2\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_i}$$
$$-\frac{2}{3}\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^2\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_i}+\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln n}{\partial r_j}$$
$$\left.+\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{3}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}-\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln(n\Theta)}{\partial r_j}\right] \tag{4.133}$$

Next, let us simplify the term $K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\varPhi_K$ in the right-hand side of eq. (4.129) as following. First, let us evaluate $\boldsymbol{\nabla}_v\,\varPhi_K$.

$$\boldsymbol{\nabla}_v\,\varPhi_K=\boldsymbol{\nabla}_v\left\{2K\frac{2\Theta}{3g}\hat{\varPhi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}+K\frac{2\Theta}{3g}\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\right\}$$
$$=2K\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\left[\overline{\tilde{u}_i\tilde{u}_j}\left\{\boldsymbol{\nabla}_v\,\hat{\varPhi}_v(\tilde{u})\right\}+\hat{\varPhi}_v(\tilde{u})\boldsymbol{\nabla}_v\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\}\right]$$
$$+K\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\left[\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\left\{\boldsymbol{\nabla}_v\,\hat{\varPhi}_c(\tilde{u})\right\}+\hat{\varPhi}_c(\tilde{u})\boldsymbol{\nabla}_v\left\{\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\right\}\right]. \tag{4.134}$$

Let us evaluate the following terms in eq. (4.134) separately,

$$\boldsymbol{\nabla}_v\,\hat{\varPhi}_v(\tilde{u})=\hat{\varPhi}_v'(\tilde{u})\boldsymbol{\nabla}_v(\tilde{u}^2),$$

where the prime denotes the integration with respect to $\tilde{u}^2$ and the value of $\boldsymbol{\nabla}_v(\tilde{u}^2)$ is:

$$\begin{aligned}\boldsymbol{\nabla}_v(\tilde{u}^2) &= \boldsymbol{\nabla}_v\left\{\frac{3}{2\Theta}(\boldsymbol{v}-\boldsymbol{V})^2\right\} = \frac{3}{2\Theta}\boldsymbol{\nabla}_v\left\{(\boldsymbol{v}-\boldsymbol{V})^2\right\} = \frac{3}{2\Theta}\sum_i \tilde{\boldsymbol{\delta}}_i\frac{\partial}{\partial v_i}\left\{\sum_j (v_j-V_j)^2\right\}\\ &= \frac{3}{2\Theta}\sum_i \tilde{\boldsymbol{\delta}}_i\left\{\sum_j 2(v_j-V_j)\frac{\partial v_j}{\partial v_i}\right\} = \frac{3}{2\Theta}\sum_i \tilde{\boldsymbol{\delta}}_i\left\{\sum_j 2(v_j-V_j)\delta_{ij}\right\}\\ &= 2\frac{3}{2\Theta}\sum_i \tilde{\boldsymbol{\delta}}_i(v_i-V_i) = 2\frac{3}{2\Theta}(\boldsymbol{v}-\boldsymbol{V}) = 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(\boldsymbol{v}-\boldsymbol{V})\right\} = 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{u}}.\end{aligned}$$

Hence

$$\boldsymbol{\nabla}_v\,\hat{\Phi}_v(\tilde{u}) = 2\,\hat{\Phi}'_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{u}}. \tag{4.135}$$

Similarly,

$$\boldsymbol{\nabla}_v\,\hat{\Phi}_c(\tilde{u}) = 2\,\hat{\Phi}'_c(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{u}}. \tag{4.136}$$

Next,

$$\boldsymbol{\nabla}_v\,\tilde{u}_i = \boldsymbol{\nabla}_v\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(v_i-V_i)\right\} = \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\sum_j \tilde{\boldsymbol{\delta}}_j\frac{\partial v_i}{\partial v_j} = \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\sum_j \tilde{\boldsymbol{\delta}}_j\delta_{ij}$$

or

$$\boldsymbol{\nabla}_v\,\tilde{u}_i = \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{\delta}}_i, \tag{4.137}$$

$$\begin{aligned}\boldsymbol{\nabla}_v\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\} &= \boldsymbol{\nabla}_v\left\{\frac{1}{2}(\tilde{u}_i\tilde{u}_j+\tilde{u}_j\tilde{u}_i)-\frac{1}{3}\tilde{u}_k\tilde{u}_k\delta_{ij}\right\} = \tilde{u}_i\boldsymbol{\nabla}_v\,\tilde{u}_j+\tilde{u}_j\boldsymbol{\nabla}_v\,\tilde{u}_i-\frac{2}{3}\tilde{u}_k\boldsymbol{\nabla}_v\,\tilde{u}_k\delta_{ij}\\ &= \tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{\delta}}_j+\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{\delta}}_i-\frac{2}{3}\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{\delta}}_k\delta_{ij}\end{aligned}$$

or

$$\boldsymbol{\nabla}_v\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\} = \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(\tilde{u}_i\tilde{\boldsymbol{\delta}}_j+\tilde{u}_j\tilde{\boldsymbol{\delta}}_i-\frac{2}{3}\tilde{u}_k\tilde{\boldsymbol{\delta}}_k\delta_{ij}\right) \tag{4.138}$$

and

$$\begin{aligned}\boldsymbol{\nabla}_v\left\{\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\right\} &= \left(\tilde{u}^2-\frac{5}{2}\right)\boldsymbol{\nabla}_v\,\tilde{u}_i+\tilde{u}_i\boldsymbol{\nabla}_v(\tilde{u}^2)\\ &= \left(\tilde{u}^2-\frac{5}{2}\right)\times\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{\delta}}_i+\tilde{u}_i\times 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{u}}\end{aligned}$$

or

$$\boldsymbol{\nabla}_v\left\{\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\right\} = \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{\boldsymbol{\delta}}_i+2\tilde{u}_i\tilde{\boldsymbol{u}}\right\}. \tag{4.139}$$

Substituting these values in eq. (4.134), we get

$$\begin{aligned}
\boldsymbol{\nabla}_v \varPhi_K &= 2K\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\left[\overline{\tilde{u}_i\tilde{u}_j}\left\{2\hat{\varPhi}'_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{u}}\right\}\right.\\
&\quad\left.+\hat{\varPhi}_v(\tilde{u})\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(\tilde{u}_i\tilde{\boldsymbol{\delta}}_j+\tilde{u}_j\tilde{\boldsymbol{\delta}}_i-\frac{2}{3}\tilde{u}_k\tilde{\boldsymbol{\delta}}_k\delta_{ij}\right)\right\}\right]\\
&\quad+K\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\left[\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\left\{2\hat{\varPhi}'_c(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\boldsymbol{u}}\right\}\right.\\
&\quad\left.+\hat{\varPhi}_c(\tilde{u})\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{\boldsymbol{\delta}}_i+2\tilde{u}_i\tilde{\boldsymbol{u}}\right\}\right\}\right]\\
&= K\frac{2\Theta}{3g}\frac{3}{2\Theta}\frac{\partial V_i}{\partial r_j}\left[4\hat{\varPhi}'_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{\boldsymbol{u}}+2\hat{\varPhi}_v(\tilde{u})\left(\tilde{u}_i\tilde{\boldsymbol{\delta}}_j+\tilde{u}_j\tilde{\boldsymbol{\delta}}_i-\frac{2}{3}\tilde{u}_k\tilde{\boldsymbol{\delta}}_k\delta_{ij}\right)\right]\\
&\quad+K\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial\ln\Theta}{\partial r_i}\left[2\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\hat{\varPhi}'_c(\tilde{u})\tilde{\boldsymbol{u}}+\hat{\varPhi}_c(\tilde{u})\left\{\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{\boldsymbol{\delta}}_i+2\tilde{u}_i\tilde{\boldsymbol{u}}\right\}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\varPhi_K\\
&= K^2\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\left[4\,\hat{\varPhi}'_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\tilde{\boldsymbol{g}}\cdot\tilde{\boldsymbol{u}}+2\hat{\varPhi}_v(\tilde{u})\,\tilde{\boldsymbol{g}}\cdot\left(\tilde{u}_i\tilde{\boldsymbol{\delta}}_j+\tilde{u}_j\tilde{\boldsymbol{\delta}}_i-\frac{2}{3}\tilde{u}_k\tilde{\boldsymbol{\delta}}_k\delta_{ij}\right)\right]\\
&\quad+K^2\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\left[2\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\hat{\varPhi}'_c(\tilde{u})\,\tilde{\boldsymbol{g}}\cdot\tilde{\boldsymbol{u}}+\hat{\varPhi}_c(\tilde{u})\,\tilde{\boldsymbol{g}}\cdot\left\{\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{\boldsymbol{\delta}}_i+2\tilde{u}_i\tilde{\boldsymbol{u}}\right\}\right]\\
&= K^2\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\left[4\hat{\varPhi}'_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\tilde{g}_k\tilde{u}_k+2\hat{\varPhi}_v(\tilde{u})\left(\tilde{u}_i\tilde{g}_j+\tilde{u}_j\tilde{g}_i-\frac{2}{3}\tilde{u}_k\tilde{g}_k\delta_{ij}\right)\right]\\
&\quad+K^2\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\left[2\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\hat{\varPhi}'_c(\tilde{u})\,\tilde{g}_j\tilde{u}_j+\hat{\varPhi}_c(\tilde{u})\left\{\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{g}_i+2\tilde{u}_i\tilde{g}_j\tilde{u}_j\right\}\right]\\
&= K^2\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\left[4\hat{\varPhi}'_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\tilde{g}_k\tilde{u}_k+4\hat{\varPhi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{g}_j}\right]\\
&\quad+K^2\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\left[2\left\{\left(\tilde{u}^2-\frac{5}{2}\right)\hat{\varPhi}'_c(\tilde{u})+\hat{\varPhi}_c(\tilde{u})\right\}\tilde{u}_i\tilde{u}_j\tilde{g}_j+\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{g}_i\right]\\
&= 4K^2\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\hat{\varPhi}'_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{g}_k+4K^2\frac{2\Theta}{3g}\hat{\varPhi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\overline{\tilde{u}_i\tilde{g}_j}\\
&\quad+2K^2\frac{2\Theta}{3g}\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\hat{\varPhi}'_c(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{g}_j+2K^2\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\hat{\varPhi}_c(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{g}_j\\
&\quad+K^2\frac{2\Theta}{3g}\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\tilde{g}_i.
\end{aligned}\tag{4.140}$$

The simplified value of $\tilde{\mathscr{D}}_K \Phi_K + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{\boldsymbol{g}} \cdot \boldsymbol{\nabla}_v \Phi_K$ is given in Appendix E. Substituting the values of the terms in eq. (4.129), from eqs. (4.130)-(4.133) and (E.10), we get

$$
\begin{aligned}
\tilde{\mathscr{L}}(\Phi_{KK}) = {} & 2\left(K\frac{2\Theta}{3g}\right)^2 \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{u}_i \tilde{\mu}_0 \left(\frac{3}{2}\right)^{\frac{1}{2}} \left(\frac{\partial \ln \Theta}{\partial r_j}\frac{\overline{\partial V_i}}{\partial r_j} + 2\frac{\partial}{\partial r_j}\frac{\overline{\partial V_i}}{\partial r_j}\right) \\
& + \left(K\frac{2\Theta}{3g}\right)^2 \left(\tilde{u}^2 - \frac{3}{2}\right)\left(\frac{3}{2}\right)^{\frac{1}{2}} \left(\frac{8}{3}\tilde{\mu}_0 \frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial r_j}\frac{\overline{\partial V_i}}{\partial r_j} + 2\tilde{\kappa}_0 \frac{\partial^2 \ln \Theta}{\partial r_j \partial r_j} + 3\tilde{\kappa}_0 \frac{\partial \ln \Theta}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_j}\right) \\
& + \left(K\frac{2\Theta}{3g}\right)^2 \left[4\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l \frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial r_j}\frac{\partial V_k}{\partial r_l} - \frac{4}{3}\hat{\Phi}_v(\tilde{u})\tilde{u}^2\tilde{u}_i\tilde{u}_j \frac{3}{2\Theta}\frac{\overline{\partial V_i}}{\partial r_j}\frac{\partial V_k}{\partial r_k}\right. \\
& + 2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\overline{\partial V_i}}{\partial r_j}\frac{\partial \ln n}{\partial r_k} + 2\hat{\Phi}_v(\tilde{u})\left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\overline{\partial V_i}}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_k} \\
& - 2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\overline{\partial V_i}}{\partial r_j}\frac{\partial \ln(n\Theta)}{\partial r_k} + 2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i} \\
& - \frac{2}{3}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}^2\tilde{u}_i \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i} + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln n}{\partial r_j} \\
& \left. + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j} - \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_j}\right] \\
& - 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\overline{\partial V_i}}{\partial r_j} \left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln n}{\partial r_k} - \tilde{u}_i\tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k}\right\} \\
& - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right) \left\{\tilde{u}_i\tilde{u}_j \frac{\partial \ln n}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j} - \tilde{u}_i \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i}\right\} \\
& - 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\overline{\partial V_i}}{\partial r_j} \left\{2\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_l} - \tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln(n\Theta)}{\partial r_k}\right\} \\
& - 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \left\{2\tilde{u}_i\tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\overline{\partial V_i}}{\partial r_k}\frac{\partial V_k}{\partial r_j} - \tilde{u}_i \frac{\overline{\partial V_i}}{\partial r_k}\frac{\partial \ln(n\Theta)}{\partial r_k}\right\} \\
& - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u}) \left\{2\tilde{u}_i\tilde{u}_j\tilde{u}_k \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_k} - \tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_j}\right\} \\
& - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right) \left\{2\tilde{u}_i\tilde{u}_j\tilde{u}_k \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_k} - \tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_j}\right\} \\
& - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right) \left\{\tilde{u}_i \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j} - \frac{1}{2}\frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_i}\right\} \\
& - 3\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\overline{\partial V_i}}{\partial r_j} \left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln \Theta}{\partial r_k} - \frac{2}{3}\tilde{u}_i\tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k}\right\} \\
& - 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\overline{\partial V_i}}{\partial r_j} \left\{\tilde{u}^2\tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln \Theta}{\partial r_k} - \frac{2}{3}\tilde{u}^2\tilde{u}_i\tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k}\right\} \\
& - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right) \frac{\partial \ln \Theta}{\partial r_i} \left\{\tilde{u}^2\tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\tilde{u}^2\tilde{u}_i \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right\} \\
& - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u}) \frac{\partial \ln \Theta}{\partial r_i} \left\{\tilde{u}^2\tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\tilde{u}^2\tilde{u}_i \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial \ln\Theta}{\partial r_i}\left\{\tilde{u}_i\tilde{u}_j\frac{\partial \ln\Theta}{\partial r_j}-\frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}\\
&- 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}}-\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}}\right\}\\
&- \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left\{\tilde{u}_i\tilde{u}_j\overline{\frac{\partial \ln(n\Theta)}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_j}}+\tilde{u}_i\tilde{u}_j\overline{\frac{\partial^2 \ln(n\Theta)}{\partial r_i\partial r_j}}\right\}\\
&+ \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{\tilde{u}_i\tilde{u}_j\frac{\partial^2 \ln\Theta}{\partial r_i\partial r_j}-\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_j}-\frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial^2 V_j}{\partial r_i\partial r_j}\right\}\\
&- \frac{1}{2}\tilde{\Omega}(\Phi_K,\Phi_K). \qquad (4.141)
\end{aligned}
$$

4.3.2 Constitutive Relations

Heat Flux

From eq. (2.30), the contribution of Φ_{KK} to the heat flux is

$$
Q_i^{KK} = \frac{n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}^2\tilde{u}_i\,\mathrm{e}^{-\tilde{u}^2}\,\Phi_{KK}.
$$

Using the similar orthogonality relation with respect to Φ_{KK} and following a similar procedure as in getting the expression for heat flux at $O(K\epsilon)$ (cf. eq. (4.24)), we get

$$
Q_i^{KK} = \frac{n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\tilde{\mathscr{L}}(\Phi_{KK}). \qquad (4.142)
$$

Substituting the value of $\tilde{\mathscr{L}}(\Phi_{KK})$ from eq. (4.141) into eq. (4.142), writing $K\frac{2\Theta}{3g}=\ell$, ignoring the terms whose tensorial structure is: $\tilde{u}_i$ or $\tilde{u}_i\tilde{u}_j\tilde{u}_k$ or $\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m$ (because the corresponding integrands are odd functions in components of $\tilde{\boldsymbol{u}}$) and using the orthogonality property of $\hat{\Phi}_c(\tilde{u})$ (which is: $\int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}_i\tilde{u}_j\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)=0$), we get

$$
\begin{aligned}
Q_i^{KK} = {}& \frac{n\ell^2}{2\pi^{3/2}}\frac{2\Theta}{3}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\mathrm{e}^{-\tilde{u}^2}\\
&\times\left[2\hat{\Phi}_v(\tilde{u})\left(\tilde{u}^2-\frac{3}{2}\right)\tilde{u}_j\tilde{u}_k\tilde{u}_l\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \ln\Theta}{\partial r_l}-2\hat{\Phi}_v(\tilde{u})\tilde{u}_j\tilde{u}_k\tilde{u}_l\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \ln(n\Theta)}{\partial r_l}\right.\\
&+2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{\partial V_l}{\partial r_k}\frac{\partial \ln\Theta}{\partial r_j}-\frac{2}{3}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^2\tilde{u}_j\frac{\partial V_k}{\partial r_k}\frac{\partial \ln\Theta}{\partial r_j}\\
&+\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_j\frac{\partial V_k}{\partial r_k}\frac{\partial \ln\Theta}{\partial r_j}+2\hat{\Phi}_v'(\tilde{u})\tilde{u}_j\tilde{u}_k\tilde{u}_l\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \ln(n\Theta)}{\partial r_l}\\
&+2\hat{\Phi}_v(\tilde{u})\tilde{u}_j\overline{\frac{\partial V_j}{\partial r_l}}\frac{\partial \ln(n\Theta)}{\partial r_l}-2\hat{\Phi}_c(\tilde{u})\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{\partial V_j}{\partial r_k}\frac{\partial \ln\Theta}{\partial r_l}\\
&-2\hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{\partial V_j}{\partial r_k}\frac{\partial \ln\Theta}{\partial r_l}-\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_j\frac{\partial V_k}{\partial r_j}\frac{\partial \ln\Theta}{\partial r_k}
\end{aligned}
$$

$$
\begin{aligned}
&-3\hat{\Phi}_v(\tilde{u})\tilde{u}_j\tilde{u}_k\tilde{u}_l\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \ln \Theta}{\partial r_l} - 2\hat{\Phi}_v'(\tilde{u})\tilde{u}^2\tilde{u}_j\tilde{u}_k\tilde{u}_l\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \ln \Theta}{\partial r_l} \\
&+\frac{2}{3}\hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^2\tilde{u}_j\frac{\partial V_k}{\partial r_k}\frac{\partial \ln\Theta}{\partial r_j} + \frac{2}{3}\hat{\Phi}_c(\tilde{u})\tilde{u}^2\tilde{u}_j\frac{\partial V_k}{\partial r_k}\frac{\partial \ln\Theta}{\partial r_j} \\
&+\frac{1}{3}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_j\frac{\partial V_k}{\partial r_k}\frac{\partial \ln\Theta}{\partial r_j} + 2\hat{\Phi}_v(\tilde{u})\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{\partial}{\partial r_l}\overline{\frac{\partial V_j}{\partial r_k}} \\
&-\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_j\frac{\partial V_k}{\partial r_j}\frac{\partial \ln\Theta}{\partial r_k} - \frac{2}{3}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_j\frac{\partial^2 V_k}{\partial r_j\partial r_k}\Bigg] \\
&-\frac{n}{4\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\mathrm{e}^{-\tilde{u}^2}\,\tilde{\Omega}(\Phi_K,\Phi_K)
\end{aligned}
$$

or

$$
\begin{aligned}
Q_i^{KK} = {} & \frac{n\Theta\ell^2}{3\pi^{3/2}}\frac{1}{\Theta}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\hat{\Phi}_c^2(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)^2\tilde{u}_i\tilde{u}_j\,\mathrm{e}^{-\tilde{u}^2} \\
&\times\left[2\frac{\partial V_k}{\partial r_k}\frac{\partial\Theta}{\partial r_j} - \frac{2}{3}\frac{\partial V_k}{\partial r_k}\frac{\partial\Theta}{\partial r_j} - \frac{2}{3}\Theta\frac{\partial^2 V_k}{\partial r_j\partial r_k} - 2\frac{\partial V_k}{\partial r_j}\frac{\partial\Theta}{\partial r_k}\right] \\
&+\frac{n\Theta\ell^2}{3\pi^{3/2}}\frac{1}{n\Theta}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\mathrm{e}^{-\tilde{u}^2}\left[-2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial(n\Theta)}{\partial r_l}\right. \\
&\left.+2\hat{\Phi}_v'(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial(n\Theta)}{\partial r_l} + 2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\frac{\partial V_j}{\partial r_l}}\frac{\partial(n\Theta)}{\partial r_l}\right] \\
&+\frac{2n\Theta\ell^2}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\hat{\Phi}_v(\tilde{u})\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\mathrm{e}^{-\tilde{u}^2}\,\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{\partial}{\partial r_l}\overline{\frac{\partial V_j}{\partial r_k}} \\
&+\frac{n\Theta\ell^2}{3\pi^{3/2}}\frac{1}{\Theta}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial\Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\,\mathrm{e}^{-\tilde{u}^2}\left[2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\right. \\
&\left.-2\hat{\Phi}_c(\tilde{u}) - 2\hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right) + 2\hat{\Phi}_v(\tilde{u})\left(\tilde{u}^2-\frac{3}{2}\right) - 3\hat{\Phi}_v(\tilde{u}) - 2\hat{\Phi}_v'(\tilde{u})\tilde{u}^2\right] \\
&-\frac{n}{4\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\mathrm{e}^{-\tilde{u}^2}\,\tilde{\Omega}(\Phi_K,\Phi_K),
\end{aligned}
$$

where we have used the following simplification:

$$
\begin{aligned}
&\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{\partial V_l}{\partial r_k}\frac{\partial\Theta}{\partial r_j} - \frac{1}{3}\tilde{u}^2\tilde{u}_j\frac{\partial V_k}{\partial r_k}\frac{\partial\Theta}{\partial r_j} = \tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{\partial V_j}{\partial r_k}\frac{\partial\Theta}{\partial r_l} - \frac{1}{3}\tilde{u}^2\tilde{u}_j\frac{\partial V_k}{\partial r_k}\frac{\partial\Theta}{\partial r_j} \\
&= \tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{\partial V_j}{\partial r_k}\frac{\partial\Theta}{\partial r_l} - \frac{1}{3}\tilde{u}^2\tilde{u}_l\frac{\partial V_k}{\partial r_k}\frac{\partial\Theta}{\partial r_l} = \tilde{u}_l\left(\tilde{u}_j\tilde{u}_k\frac{\partial V_j}{\partial r_k} - \frac{1}{3}\delta_{jk}\tilde{u}^2\frac{\partial V_j}{\partial r_k}\right)\frac{\partial\Theta}{\partial r_l} \\
&= \tilde{u}_l\left(\frac{\tilde{u}_j\tilde{u}_k+\tilde{u}_k\tilde{u}_j}{2} - \frac{1}{3}\tilde{u}^2\delta_{jk}\right)\frac{\partial V_j}{\partial r_k}\frac{\partial\Theta}{\partial r_l} = \tilde{u}_l\overline{\tilde{u}_j\tilde{u}_k}\frac{\partial V_j}{\partial r_k}\frac{\partial\Theta}{\partial r_l} = \tilde{u}_j\tilde{u}_k\tilde{u}_l\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial\Theta}{\partial r_l}.
\end{aligned}
$$

The integrals in the expression of Q_i^{KK} can be further simplified by using eqs. (F.9b), (F.16) and (F.17) to obtain

$$
\begin{aligned}
Q_i^{KK} = & \frac{n\ell^2}{3\pi^{3/2}} \frac{4\pi}{3} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, \hat{\Phi}_c^2(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right)^2 \tilde{u}^4\, \mathrm{e}^{-\tilde{u}^2} \\
& \times \left[2 \frac{\partial V_k}{\partial r_k} \frac{\partial \Theta}{\partial r_i} - \left(\frac{2}{3} \frac{\partial V_k}{\partial r_k} \frac{\partial \Theta}{\partial r_i} + \frac{2}{3} \Theta \frac{\partial^2 V_k}{\partial r_i \partial r_k} + 2 \frac{\partial V_k}{\partial r_i} \frac{\partial \Theta}{\partial r_k} \right) \right] \\
& + \frac{\ell^2}{3\pi^{3/2}} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \mathrm{e}^{-\tilde{u}^2} \\
& \times \left[2\{\hat{\Phi}_v'(\tilde{u}) - \hat{\Phi}_v(\tilde{u})\} \tilde{u}^6 \frac{8\pi}{15} \overline{\frac{\partial V_j}{\partial r_i}} \frac{\partial (n\Theta)}{\partial r_j} + 2\hat{\Phi}_v(\tilde{u}) \tilde{u}^4 \frac{4\pi}{3} \overline{\frac{\partial V_j}{\partial r_i}} \frac{\partial (n\Theta)}{\partial r_j} \right] \\
& + \frac{2n\Theta\ell^2}{3\pi^{3/2}} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, \hat{\Phi}_v(\tilde{u}) \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \mathrm{e}^{-\tilde{u}^2}\, \tilde{u}^6 \frac{8\pi}{15} \frac{\partial}{\partial r_j} \overline{\frac{\partial V_j}{\partial r_i}} \\
& + \frac{n\ell^2}{3\pi^{3/2}} \frac{8\pi}{15} \overline{\frac{\partial V_j}{\partial r_i}} \frac{\partial \Theta}{\partial r_j} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}^6\, \mathrm{e}^{-\tilde{u}^2} \left[2\hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \right. \\
& \left. - 2\hat{\Phi}_c(\tilde{u}) - 2\hat{\Phi}_c'(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) + 2\hat{\Phi}_v(\tilde{u}) \left(\tilde{u}^2 - \frac{3}{2}\right) - 3\hat{\Phi}_v(\tilde{u}) - 2\hat{\Phi}_v'(\tilde{u}) \tilde{u}^2 \right] \\
& - \frac{n}{4\pi^{3/2}} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{\Omega}(\Phi_K, \Phi_K) \\
= & \frac{8n\ell^2}{9\pi^{1/2}} \frac{\partial V_j}{\partial r_j} \frac{\partial \Theta}{\partial r_i} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, \hat{\Phi}_c^2(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right)^2 \tilde{u}^4\, \mathrm{e}^{-\tilde{u}^2} \\
& - \frac{4n\ell^2}{9\pi^{1/2}} \left[\frac{2}{3} \frac{\partial}{\partial r_i} \left(\Theta \frac{\partial V_j}{\partial r_j} \right) + 2 \frac{\partial V_j}{\partial r_i} \frac{\partial \Theta}{\partial r_j} \right] \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, \hat{\Phi}_c^2(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right)^2 \tilde{u}^4\, \mathrm{e}^{-\tilde{u}^2} \\
& + \frac{8\ell^2}{45\pi^{1/2}} \overline{\frac{\partial V_j}{\partial r_i}} \frac{\partial (n\Theta)}{\partial r_j} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}^4\, \mathrm{e}^{-\tilde{u}^2} [2\{\hat{\Phi}_v'(\tilde{u}) - \hat{\Phi}_v(\tilde{u})\} \tilde{u}^2 + 5\hat{\Phi}_v(\tilde{u})] \\
& + \frac{16n\Theta\ell^2}{45\pi^{1/2}} \frac{\partial}{\partial r_j} \overline{\frac{\partial V_j}{\partial r_i}} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, \hat{\Phi}_v(\tilde{u}) \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \mathrm{e}^{-\tilde{u}^2}\, \tilde{u}^6 \\
& + \frac{8n\ell^2}{45\pi^{1/2}} \overline{\frac{\partial V_j}{\partial r_i}} \frac{\partial \Theta}{\partial r_j} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}^6\, \mathrm{e}^{-\tilde{u}^2} \left[2\hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \right. \\
& \left. - 2\hat{\Phi}_c(\tilde{u}) - 2\hat{\Phi}_c'(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) + 2\hat{\Phi}_v(\tilde{u}) \left(\tilde{u}^2 - \frac{3}{2}\right) - 3\hat{\Phi}_v(\tilde{u}) - 2\hat{\Phi}_v'(\tilde{u}) \tilde{u}^2 \right] \\
& - \frac{n}{4\pi^{3/2}} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{\Omega}(\Phi_K, \Phi_K). \qquad (4.143)
\end{aligned}
$$

The last term in eq. (4.143) is evaluated separately as following. Substituting the value of $\tilde{\Omega}(\Phi_K, \Phi_K)$ from eq. (2.38), this term is given by

$$
\begin{aligned}
& \frac{n}{4\pi^{3/2}} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{\Omega}(\Phi_K, \Phi_K) \\
& = \frac{n}{2\pi^4} \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)}\, \hat{\Phi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} \\
& \quad \times \{\Phi_K(\tilde{\boldsymbol{u}}_1') \Phi_K(\tilde{\boldsymbol{u}}_2') - \Phi_K(\tilde{\boldsymbol{u}}_1) \Phi_K(\tilde{\boldsymbol{u}}_2)\}.
\end{aligned}
$$

Note that in the above equation, the velocity transformation corresponds to the elastic limit. Following a similar procedure as in the derivation of $Q_{i_3}^{K\epsilon}$, one obtains

$$\frac{n}{4\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\mathrm{e}^{-\tilde{u}^2}\,\tilde{\Omega}(\Phi_K,\Phi_K)=H_1-H_2, \tag{4.144}$$

where

$$H_1=\frac{n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_K(\tilde{\boldsymbol{u}}_2)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta^{(0)}, \tag{4.145}$$

$$H_2=\frac{n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_K(\tilde{\boldsymbol{u}}_2), \tag{4.146}$$

and $I_\delta^{(0)}$ is given in eq. (4.48). We shall simplify H_1 and H_2 as following. Using eqs. (3.18) and (F.3),

$$\begin{aligned}
H_1 &= \frac{n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta^{(0)}\\
&\quad\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_1)\tilde{u}_{1j}\tilde{u}_{1k}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_j}{\partial r_k}}+K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1j}\frac{\partial\ln\Theta}{\partial r_j}\right]\\
&\quad\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2l}\tilde{u}_{2m}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_l}{\partial r_m}}+K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2l}\frac{\partial\ln\Theta}{\partial r_l}\right]\\
&= \frac{n\ell^2}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta^{(0)}\\
&\quad\times\left[4\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1j}\tilde{u}_{1k}\tilde{u}_{2l}\tilde{u}_{2m}\frac{3}{2\Theta}\overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_l}{\partial r_m}}\right.\\
&\quad+2\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1j}\tilde{u}_{1k}\tilde{u}_{2l}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial\ln\Theta}{\partial r_l}\\
&\quad+2\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1j}\tilde{u}_{2l}\tilde{u}_{2m}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_l}{\partial r_m}}\frac{\partial\ln\Theta}{\partial r_j}\\
&\quad\left.+\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1j}\tilde{u}_{2l}\frac{\partial\ln\Theta}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_l}\right].
\end{aligned}$$

We can ignore the first and fourth terms in the square brackets above because ultimately after manipulation corresponding integrands become odd functions in components of $\tilde{\boldsymbol{u}}$ (cf. §4.2). Therefore

$$\begin{aligned}
H_1 &= \frac{n\ell^2}{\pi^4}\frac{2\Theta}{3}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial\ln\Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta^{(0)}\\
&\quad\times\left[\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1j}\tilde{u}_{1k}\tilde{u}_{2l}+\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1l}\tilde{u}_{2j}\tilde{u}_{2k}\right].
\end{aligned}$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}$, we get

$$H_1 = \frac{2n\ell^2}{3\pi^4} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \int \mathrm{d}\tilde{\boldsymbol{s}} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2 - \tilde{u}_2^2}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) I_\delta^{(0)} \times \left[\hat{\Phi}_v(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_i(\tilde{u}_j - \tilde{s}_j)(\tilde{u}_k - \tilde{s}_k)\tilde{u}_{2l} + \hat{\Phi}_c(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2) \left((\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}})^2 - \frac{5}{2}\right) \tilde{u}_i \tilde{u}_{2j} \tilde{u}_{2k}(\tilde{u}_l - \tilde{s}_l)\right].$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}} \cdot \tilde{\boldsymbol{u}}_2 = \tilde{s}\tilde{u}_2 \cos\theta_2'$, i.e.,

$$H_1 = \frac{2n\ell^2}{3\pi^4} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \int \mathrm{d}\tilde{\boldsymbol{s}} \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2' \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2'\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2 - \tilde{u}_2^2}\, \hat{\Phi}_c(\tilde{u}) \times \left(\tilde{u}^2 - \frac{5}{2}\right) I_\delta^{(0)} \left[\hat{\Phi}_v(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_i(\tilde{u}_j - \tilde{s}_j)(\tilde{u}_k - \tilde{s}_k)\tilde{u}_{2l} + \hat{\Phi}_c(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2) \left((\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}})^2 - \frac{5}{2}\right) \tilde{u}_i \tilde{u}_{2j} \tilde{u}_{2k}(\tilde{u}_l - \tilde{s}_l)\right].$$

Note that the components of $\tilde{\boldsymbol{u}}_2$ are the only functions of ϕ_2' (see Appendix H). Therefore the integrations over ϕ_2' give (cf. eqs. (H.6) and (H.18))

$$\begin{aligned} H_1 &= \frac{2n\ell^2}{3\pi^4} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \int \mathrm{d}\tilde{\boldsymbol{s}} \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2'\, \tilde{u}_2^2 \sin\theta_2'\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2 - \tilde{u}_2^2}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) I_\delta^{(0)} \\ &\times \left[\hat{\Phi}_v(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_i(\tilde{u}_j - \tilde{s}_j)(\tilde{u}_k - \tilde{s}_k) \left(2\pi \frac{\tilde{u}_2}{\tilde{s}} \tilde{s}_l \cos\theta_2'\right)\right. \\ &\left. + \hat{\Phi}_c(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2) \left((\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}})^2 - \frac{5}{2}\right) \tilde{u}_i \left\{\pi \frac{\tilde{u}_2^2}{\tilde{s}^2} \tilde{s}_j \tilde{s}_k (3\cos^2\theta_2' - 1)\right\} (\tilde{u}_l - \tilde{s}_l)\right] \\ &= \frac{2n\ell^2}{3\pi^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \int \mathrm{d}\tilde{\boldsymbol{s}} \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2 - \tilde{u}_2^2}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \\ &\times \left[2\hat{\Phi}_v(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \frac{\tilde{u}_2}{\tilde{s}} \left(\int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2'\, \sin\theta_2' \cos\theta_2' I_\delta^{(0)}\right)\right. \\ &\times \tilde{u}_i(\tilde{u}_j - \tilde{s}_j)(\tilde{u}_k - \tilde{s}_k)\tilde{s}_l + \hat{\Phi}_c(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2) \left((\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}})^2 - \frac{5}{2}\right) \frac{\tilde{u}_2^2}{\tilde{s}^2} \\ &\left.\times \left\{\int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2'\, \sin\theta_2' (3\cos^2\theta_2' - 1) I_\delta^{(0)}\right\} \tilde{u}_i \tilde{s}_j \tilde{s}_k (\tilde{u}_l - \tilde{s}_l)\right]. \end{aligned}$$

Using eq. (C.12),

$$\begin{aligned} H_1 &= \frac{2n\ell^2}{3\pi^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \int \mathrm{d}\tilde{\boldsymbol{s}} \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{u}_2=\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2 - \tilde{u}_2^2}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \\ &\times \left[2\,\hat{\Phi}_v(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \frac{1}{\tilde{s}^2} \left(\frac{\tilde{\boldsymbol{s}} \cdot \tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right) \tilde{u}_i(\tilde{u}_j - \tilde{s}_j)(\tilde{u}_k - \tilde{s}_k)\tilde{s}_l\right. \\ &\left. + \hat{\Phi}_c(|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2) \left((\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}})^2 - \frac{5}{2}\right) \frac{\tilde{u}_2}{\tilde{s}^3} \left\{3\left(\frac{\tilde{\boldsymbol{s}} \cdot \tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right)^2 - 1\right\} \tilde{u}_i \tilde{s}_j \tilde{s}_k (\tilde{u}_l - \tilde{s}_l)\right] \end{aligned}$$

or

$$H_1 = \frac{2n\ell^2}{3\pi^3}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)$$
$$\times\left[2\,\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\frac{1}{\tilde{s}^2}\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)\tilde{u}_i(\tilde{u}_j-\tilde{s}_j)(\tilde{u}_k-\tilde{s}_k)\tilde{s}_l\right.$$
$$\left.+\,\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\frac{1}{\tilde{s}^3}\left\{3\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2-\tilde{u}_2^2\right\}\tilde{u}_i\tilde{s}_j\tilde{s}_k(\tilde{u}_l-\tilde{s}_l)\right].$$

Let us replace $\tilde{u}_2^2$ by $\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2$. This shift implies that

$$H_1 = \frac{2n\ell^2}{3\pi^3}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)$$
$$\times\frac{1}{\tilde{s}^2}\left[2\,\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2-\frac{5}{2}\right\}\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)\right.$$
$$\times\,\tilde{u}_i(\tilde{u}_j-\tilde{s}_j)(\tilde{u}_k-\tilde{s}_k)\tilde{s}_l$$
$$+\,\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)$$
$$\left.\times\,\frac{1}{\tilde{s}}\left\{2\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2-\tilde{u}_2^2\right\}\tilde{u}_i\tilde{s}_j\tilde{s}_k(\tilde{u}_l-\tilde{s}_l)\right].$$

The integration over $\tilde{\boldsymbol{s}}$ is performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}=\tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$H_1 = \frac{2n\ell^2}{3\pi^3}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\tilde{u}_2\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}$$
$$\times\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{1}{\tilde{s}^2}\left[2\,\hat{\Phi}_v\left((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\right)\right.$$
$$\times\,\hat{\Phi}_c\left((\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')^{1/2}\right)\left(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta'-\frac{5}{2}\right)\tilde{u}\cos\theta'\tilde{u}_i(\tilde{u}_j-\tilde{s}_j)(\tilde{u}_k-\tilde{s}_k)\tilde{s}_l$$
$$+\,\hat{\Phi}_c\left((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\right)\hat{\Phi}_v\left((\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')^{1/2}\right)\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2-\frac{5}{2}\right)$$
$$\left.\times\,\frac{1}{\tilde{s}}(2\tilde{u}^2\cos^2\theta'-\tilde{u}_2^2)\tilde{u}_i\tilde{s}_j\tilde{s}_k(\tilde{u}_l-\tilde{s}_l)\right].$$

The integrations over ϕ' are given in eqs. (H.25) and (H.26). Hence using these equations,

$$
\begin{aligned}
H_1 = \frac{2n\ell^2}{3\pi^3} & \int \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \int_{\tilde{s}=0}^{\infty} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \mathrm{d}\tilde{s}\, \sin\theta' \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')} \\
& \times \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \left[2\hat{\Phi}_v\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2} \Big) \hat{\Phi}_c\Big((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2} \Big) \right. \\
& \times \left(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta' - \frac{5}{2} \right) \tilde{u}\cos\theta' \Bigg\{ 2\pi \frac{\tilde{s}}{\tilde{u}^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \tilde{u}_i \tilde{u}_j \tilde{u}_k \tilde{u}_l \\
& \times \left(\tilde{u}^2\cos\theta' - 2\tilde{u}\tilde{s}\frac{1}{2}(3\cos^2\theta' - 1) + \tilde{s}^2 \frac{1}{2}(5\cos^3\theta' - 3\cos\theta') \right) \\
& - 2\pi \frac{\tilde{s}^2}{\tilde{u}} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_k} \tilde{u}_i \tilde{u}_j (\tilde{u} - \tilde{s}\cos\theta')(1 - \cos^2\theta') \Bigg\} \\
& + \hat{\Phi}_c\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2} \Big) \hat{\Phi}_v\Big((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2} \Big) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2 - \frac{5}{2} \right) \\
& \times \frac{1}{\tilde{s}} (2\tilde{u}^2\cos^2\theta' - \tilde{u}_2^2) \Bigg\{ 2\pi \frac{\tilde{s}^2}{\tilde{u}^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \tilde{u}_i \tilde{u}_j \tilde{u}_k \tilde{u}_l \left(\tilde{u}\frac{1}{2}(3\cos^2\theta' - 1) - \tilde{s}\frac{1}{2}(5\cos^3\theta' - 3\cos\theta') \right) \\
& \left. - 2\pi \frac{\tilde{s}^3}{\tilde{u}} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_k} \tilde{u}_i \tilde{u}_j \cos\theta' (1 - \cos^2\theta') \Bigg\} \right].
\end{aligned}
$$

Let $\cos\theta' = y$. This implies that

$$
\begin{aligned}
H_1 = \frac{4n\ell^2}{3\pi^2} & \int \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{y=-1}^{1} \mathrm{d}y\, \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2 y^2)}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2} \right) \\
& \times \left[2\hat{\Phi}_v\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2} \Big) \hat{\Phi}_c\Big((\tilde{u}_2^2 + \tilde{u}^2 y^2)^{1/2} \Big) \left(\tilde{u}_2^2 + \tilde{u}^2 y^2 - \frac{5}{2} \right) \tilde{u}y \right. \\
& \times \Bigg\{ \frac{\tilde{s}}{\tilde{u}^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \tilde{u}_i \tilde{u}_j \tilde{u}_k \tilde{u}_l \left(\tilde{u}^2 y - 2\tilde{u}\tilde{s}\frac{1}{2}(3y^2 - 1) + \tilde{s}^2\frac{1}{2}(5y^3 - 3y) \right) \\
& - \frac{\tilde{s}^2}{\tilde{u}} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_k} \tilde{u}_i \tilde{u}_j (\tilde{u} - \tilde{s}y)(1 - y^2) \Bigg\} \\
& + \hat{\Phi}_c\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2} \Big) \hat{\Phi}_v\Big((\tilde{u}_2^2 + \tilde{u}^2 y^2)^{1/2} \Big) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2} \right) \\
& \times \frac{1}{\tilde{s}} (2\tilde{u}^2 y^2 - \tilde{u}_2^2) \Bigg\{ \frac{\tilde{s}^2}{\tilde{u}^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \tilde{u}_i \tilde{u}_j \tilde{u}_k \tilde{u}_l \left(\tilde{u}\frac{1}{2}(3y^2 - 1) - \tilde{s}\frac{1}{2}(5y^3 - 3y) \right) \\
& \left. - \frac{\tilde{s}^3}{\tilde{u}} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_k} \tilde{u}_i \tilde{u}_j y (1 - y^2) \Bigg\} \right].
\end{aligned}
$$

Using eqs. (F.9b) and (F.16), the integrals over $\tilde{\boldsymbol{u}}$ simplify to

$$H_1 = \frac{4n\ell^2}{3\pi^2}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{u}_2\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\,\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)$$
$$\times\Bigg[2\hat{\varPhi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\varPhi}_c\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\tilde{u}y$$
$$\times\Bigg\{\frac{\tilde{s}}{\tilde{u}^3}\frac{8\pi}{15}\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial\Theta}{\partial r_j}\tilde{u}^6\left(\tilde{u}^2y-2\tilde{u}\tilde{s}\frac{1}{2}(3y^2-1)+\tilde{s}^2\frac{1}{2}(5y^3-3y)\right)$$
$$-\frac{\tilde{s}^2}{\tilde{u}}\frac{4\pi}{3}\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial\Theta}{\partial r_j}\tilde{u}^4(\tilde{u}-\tilde{s}y)(1-y^2)\Bigg\}$$
$$+\hat{\varPhi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\varPhi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2-\frac{5}{2}\right)$$
$$\times\frac{1}{\tilde{s}}(2\tilde{u}^2y^2-\tilde{u}_2^2)\Bigg\{\frac{\tilde{s}^2}{\tilde{u}^3}\frac{8\pi}{15}\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial\Theta}{\partial r_j}\tilde{u}^6\left(\tilde{u}\frac{1}{2}(3y^2-1)-\tilde{s}\frac{1}{2}(5y^3-3y)\right)$$
$$-\frac{\tilde{s}^3}{\tilde{u}}\frac{4\pi}{3}\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial\Theta}{\partial r_j}\tilde{u}^4y(1-y^2)\Bigg\}\Bigg]$$

or

$$H_1 = \frac{16n\ell^2}{45\pi}\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial\Theta}{\partial r_j}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\,\hat{\varPhi}_c(\tilde{u})$$
$$\times\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^3\tilde{s}\Bigg[2\hat{\varPhi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\varPhi}_c\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\tilde{u}y$$
$$\times\left\{\big(2\tilde{u}^2y-2\tilde{u}\tilde{s}(3y^2-1)+\tilde{s}^2(5y^3-3y)\big)-5\tilde{s}(\tilde{u}-\tilde{s}y)(1-y^2)\right\}$$
$$+\hat{\varPhi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\varPhi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2-\frac{5}{2}\right)$$
$$\times(2\tilde{u}^2y^2-\tilde{u}_2^2)\left\{\big(\tilde{u}(3y^2-1)-\tilde{s}(5y^3-3y)\big)-5\tilde{s}y(1-y^2)\right\}\Bigg]$$

or

$$H_1 = \frac{32n\ell^2}{45\pi}\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial\Theta}{\partial r_j}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{s}\tilde{u}^3\tilde{u}_2\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\,\hat{\varPhi}_c(\tilde{u})$$
$$\times\left(\tilde{u}^2-\frac{5}{2}\right)\Bigg[\hat{\varPhi}_v\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\varPhi}_c\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2y^2-\frac{5}{2}\right)\tilde{u}y$$
$$\times(2\tilde{u}^2y-\tilde{u}\tilde{s}y^2-3\tilde{u}\tilde{s}+2\tilde{s}^2y)$$
$$+\hat{\varPhi}_c\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\varPhi}_v\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\left(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2-\frac{5}{2}\right)$$
$$\times(2\tilde{u}^2y^2-\tilde{u}_2^2)\left\{\tilde{u}\frac{1}{2}(3y^2-1)-\tilde{s}y\right\}\Bigg]. \tag{4.147}$$

Next, consider eq. (4.146). The integration over $\hat{\boldsymbol{k}}$ is trivial. Hence using eq. (G.1b),

$$H_2 = \frac{n}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\,\varPhi_K(\tilde{\boldsymbol{u}}_1)\varPhi_K(\tilde{\boldsymbol{u}}_2).$$

Substituting the value of $\varPhi_K$ from eq. (3.18) and using eq. (F.3),

$$
\begin{aligned}
H_2 &= \frac{n}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} \\
&\quad \times \left[2K \frac{2\Theta}{3g} \hat{\varPhi}_v(\tilde{u}_1) \tilde{u}_{1j} \tilde{u}_{1k} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_j}{\partial r_k}} + K \frac{2\Theta}{3g} \hat{\varPhi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1j} \frac{\partial \ln \Theta}{\partial r_j}\right] \\
&\quad \times \left[2K \frac{2\Theta}{3g} \hat{\varPhi}_v(\tilde{u}_2) \tilde{u}_{2l} \tilde{u}_{2m} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_l}{\partial r_m}} + K \frac{2\Theta}{3g} \hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_{2l} \frac{\partial \ln \Theta}{\partial r_l}\right] \\
&= \frac{n\ell^2}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} \\
&\quad \times \left[4 \hat{\varPhi}_v(\tilde{u}_1) \hat{\varPhi}_v(\tilde{u}_2) \tilde{u}_{1j} \tilde{u}_{1k} \tilde{u}_{2l} \tilde{u}_{2m} \frac{3}{2\Theta} \overline{\frac{\partial V_j}{\partial r_k}}\, \overline{\frac{\partial V_l}{\partial r_m}} \right. \\
&\quad + 2 \hat{\varPhi}_v(\tilde{u}_1) \hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_{1j} \tilde{u}_{1k} \tilde{u}_{2l} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \ln \Theta}{\partial r_l} \\
&\quad + 2 \hat{\varPhi}_c(\tilde{u}_1) \hat{\varPhi}_v(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1j} \tilde{u}_{2l} \tilde{u}_{2m} \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_l}{\partial r_m}} \frac{\partial \ln \Theta}{\partial r_j} \\
&\quad \left. + \hat{\varPhi}_c(\tilde{u}_1) \hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_{1j} \tilde{u}_{2l} \frac{\partial \ln \Theta}{\partial r_j} \frac{\partial \ln \Theta}{\partial r_l}\right].
\end{aligned}
$$

We can ignore the first and fourth terms in the square brackets above because ultimately after manipulation corresponding integrands become odd functions in components of $\tilde{\boldsymbol{u}}_1$ (cf. §4.2). Therefore

$$
\begin{aligned}
H_2 &= \frac{2n\ell^2}{3\pi^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} \\
&\quad \times \left[\hat{\varPhi}_v(\tilde{u}_1) \hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_{1j} \tilde{u}_{1k} \tilde{u}_{2l} + \hat{\varPhi}_c(\tilde{u}_1) \hat{\varPhi}_v(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1l} \tilde{u}_{2j} \tilde{u}_{2k}\right].
\end{aligned}
$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1 \cdot \tilde{\boldsymbol{u}}_2 = \tilde{u}_1 \tilde{u}_2 \cos\theta_2'$, i.e.,

$$
\begin{aligned}
H_2 &= \frac{2n\ell^2}{3\pi^3} \frac{\partial \Theta}{\partial r_l} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2' \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \\
&\quad \times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \left[\hat{\varPhi}_v(\tilde{u}_1) \hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_{1i} \tilde{u}_{1j} \tilde{u}_{1k} \tilde{u}_{2l} \overline{\frac{\partial V_j}{\partial r_k}} \right. \\
&\quad \left. + \hat{\varPhi}_c(\tilde{u}_1) \hat{\varPhi}_v(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} \left(\tilde{u}_{2j} \tilde{u}_{2k} \overline{\frac{\partial V_j}{\partial r_k}}\right) \tilde{u}_{1l}\right].
\end{aligned}
$$

The integrals over ϕ_2' result into (cf. eqs. (H.18) and (H.6)),

$$
\begin{aligned}
H_2 &= \frac{2n\ell^2}{3\pi^3} \frac{\partial \Theta}{\partial r_l} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\
&\quad \times \hat{\varPhi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \left[\hat{\varPhi}_v(\tilde{u}_1) \hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_{1i} \tilde{u}_{1j} \tilde{u}_{1k} \left(2\pi \frac{\tilde{u}_2}{\tilde{u}_1} \tilde{u}_{1l} \cos\theta_2'\right) \overline{\frac{\partial V_j}{\partial r_k}} \right. \\
&\quad \left. + \hat{\varPhi}_c(\tilde{u}_1) \hat{\varPhi}_v(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} \left\{\pi \frac{\tilde{u}_2^2}{\tilde{u}_1^2} \overline{\frac{\partial V_j}{\partial r_k}} \tilde{u}_{1j} \tilde{u}_{1k} (3\cos^2\theta_2' - 1)\right\} \tilde{u}_{1l}\right]
\end{aligned}
$$

or

$$\begin{aligned}
H_2 &= \frac{4n\ell^2}{3\pi^2}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{1k}\tilde{u}_{1l}\frac{\tilde{u}_2^3}{\tilde{u}_1^2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\\
&\quad\times\left[\tilde{u}_1\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\,\sin\theta_2'\cos\theta_2'(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2)^{1/2}\right)\right.\\
&\quad+\tilde{u}_2\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\\
&\quad\left.\times\left\{\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\,\sin\theta_2'\left(\frac{1}{2}(3\cos^2\theta_2'-1)\right)(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2)^{1/2}\right\}\right]\\
&= \frac{4n\ell^2}{3\pi^2}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{1k}\tilde{u}_{1l}\frac{\tilde{u}_2^3}{\tilde{u}_1^2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\\
&\quad\times\left[\tilde{u}_1\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)R_1(\tilde{u}_1,\tilde{u}_2)+\tilde{u}_2\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)R_2(\tilde{u}_1,\tilde{u}_2)\right],
\end{aligned}$$

where $R_n(\tilde{u}_1,\tilde{u}_2)$ is defined in eq. (4.57) and the values of $R_1(\tilde{u}_1,\tilde{u}_2)$ and $R_2(\tilde{u}_1,\tilde{u}_2)$ are given in eqs. (4.64) and (4.99) respectively. The integration over $\tilde{\boldsymbol{u}}_1$ results into (cf. eq. (F.16)),

$$\begin{aligned}
H_2 &= \frac{4n\ell^2}{3\pi^2}\frac{8\pi}{15}\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial \Theta}{\partial r_j}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_1^6\frac{\tilde{u}_2^3}{\tilde{u}_1^2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\\
&\quad\times\left[\tilde{u}_1\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)R_1(\tilde{u}_1,\tilde{u}_2)+\tilde{u}_2\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)R_2(\tilde{u}_1,\tilde{u}_2)\right]\\
&= \frac{32n\ell^2}{45\pi}\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial \Theta}{\partial r_j}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_1^4\tilde{u}_2^3\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\\
&\quad\times\left[\tilde{u}_1\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)R_1(\tilde{u}_1,\tilde{u}_2)+\tilde{u}_2\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)R_2(\tilde{u}_1,\tilde{u}_2)\right].
\end{aligned} \tag{4.148}$$

Hence from eqs. (4.143), (4.144), (4.147) and (4.148),

$$\boxed{\begin{aligned}
Q_i^{KK} &= \tilde{\theta}_1 n\ell^2\frac{\partial V_j}{\partial r_j}\frac{\partial \Theta}{\partial r_i}+\tilde{\theta}_2 n\ell^2\left\{\frac{2}{3}\frac{\partial}{\partial r_i}\left(\Theta\frac{\partial V_j}{\partial r_j}\right)+2\frac{\partial V_j}{\partial r_i}\frac{\partial \Theta}{\partial r_j}\right\}\\
&\quad+\tilde{\theta}_3\ell^2\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial (n\Theta)}{\partial r_j}+\tilde{\theta}_4 n\ell^2\Theta\frac{\partial}{\partial r_j}\overline{\frac{\partial V_j}{\partial r_i}}+\tilde{\theta}_5 n\ell^2\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial \Theta}{\partial r_j}
\end{aligned}} \tag{4.149}$$

where

$$\tilde{\theta}_1 = \frac{8}{9\pi^{1/2}}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\,\hat{\Phi}_c^2(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)^2\tilde{u}^4\,\mathrm{e}^{-\tilde{u}^2}, \tag{4.150a}$$

$$\tilde{\theta}_2 = -\frac{4}{9\pi^{1/2}}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\,\hat{\Phi}_c^2(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)^2\tilde{u}^4\,\mathrm{e}^{-\tilde{u}^2}, \tag{4.150b}$$

$$\tilde{\theta}_3 = \frac{8}{45\pi^{1/2}}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^4\,\mathrm{e}^{-\tilde{u}^2}\left[2\{\hat{\Phi}_v'(\tilde{u})-\hat{\Phi}_v(\tilde{u})\}\tilde{u}^2+5\hat{\Phi}_v(\tilde{u})\right], \tag{4.150c}$$

$$\tilde{\theta}_4 = \frac{16}{45\pi^{1/2}}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\,\hat{\Phi}_v(\tilde{u})\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^6\,\mathrm{e}^{-\tilde{u}^2}, \tag{4.150d}$$

and

$$\begin{aligned}
\tilde{\theta}_5 = & \frac{8}{45\pi^{1/2}} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}^6\, \mathrm{e}^{-\tilde{u}^2} \left[2\hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) - 2\hat{\Phi}_c(\tilde{u})\right. \\
& \left. -2\hat{\Phi}_c'(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) + 2\hat{\Phi}_v(\tilde{u}) \left(\tilde{u}^2 - \frac{3}{2}\right) - 3\hat{\Phi}_v(\tilde{u}) - 2\hat{\Phi}_v'(\tilde{u})\tilde{u}^2\right] \\
& - \frac{32}{45\pi} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{y=-1}^{1} \mathrm{d}y \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{s}\tilde{u}^3\tilde{u}_2\, \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\, \hat{\Phi}_c(\tilde{u}) \\
& \times \left(\tilde{u}^2 - \frac{5}{2}\right) \left[\hat{\Phi}_v\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\right) \hat{\Phi}_c\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\right) \left(\tilde{u}_2^2 + \tilde{u}^2y^2 - \frac{5}{2}\right) \tilde{u}y\right. \\
& \times (2\tilde{u}^2y - \tilde{u}\tilde{s}y^2 - 3\tilde{u}\tilde{s} + 2\tilde{s}^2y) \\
& + \hat{\Phi}_c\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\right) \hat{\Phi}_v\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\right) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right) \\
& \left. \times (2\tilde{u}^2y^2 - \tilde{u}_2^2) \left\{\tilde{u}\frac{1}{2}(3y^2 - 1) - \tilde{s}y\right\}\right] \\
& + \frac{32}{45\pi} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_1^4\tilde{u}_2^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \\
& \times \left[\tilde{u}_1\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2}\right) R_1(\tilde{u}_1, \tilde{u}_2) + \tilde{u}_2\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right) R_2(\tilde{u}_1, \tilde{u}_2)\right].
\end{aligned} \tag{4.150e}$$

$\tilde{\theta}_i$ are evaluated numerically. Their values are: $\tilde{\theta}_1 \approx 1.2312$, $\tilde{\theta}_2 \approx -0.6156$, $\tilde{\theta}_3 \approx -0.3270$, $\tilde{\theta}_4 \approx 0.2551$ and $\tilde{\theta}_5 \approx 2.5667$.

Pressure Tensor

From eq. (2.28), the contribution of Φ_{KK} to the pressure tensor is given by

$$P_{ij}^{KK} = \frac{2n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i\tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2}\, \Phi_{KK}.$$

Using the similar orthogonality relation with respect to Φ_{KK} and following a similar procedure as in getting the expression for pressure tensor at $O(K\epsilon)$ (cf. eq. (4.67)), we get

$$P_{ij}^{KK} = \frac{2n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\, \mathscr{L}(\Phi_{KK}). \tag{4.151}$$

Substituting the value of $\tilde{\mathscr{L}}(\Phi_{KK})$ from eq. (4.141) into eq. (4.151), writing $K\frac{2\Theta}{3g} = \ell$, and ignoring the terms whose tensorial structure is: $\tilde{u}_i$ or $\tilde{u}_i\tilde{u}_j\tilde{u}_k$ or $\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m$ (because the corresponding integrands are odd functions in components of $\tilde{\boldsymbol{u}}$) or $\overline{\tilde{u}_i\tilde{u}_j}$ (because of symmetry), we get

$$
\begin{aligned}
P_{ij}^{KK} = & \frac{2n\Theta\ell^2}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i\tilde{u}_j} \left[4\left\{\hat{\Phi}_v(\tilde{u}) - \hat{\Phi}_v'(\tilde{u})\right\} \tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n \frac{3}{2\Theta} \overline{\frac{\partial V_k}{\partial r_l}}\, \overline{\frac{\partial V_m}{\partial r_n}} \right. \\
& + 4\hat{\Phi}_v(\tilde{u})\tilde{u}_k\tilde{u}_l \frac{3}{2\Theta} \overline{\frac{\partial V_k}{\partial r_l}} \frac{\partial V_m}{\partial r_m} - 4\hat{\Phi}_v(\tilde{u})\tilde{u}_k\tilde{u}_l \frac{3}{2\Theta} \overline{\frac{\partial V_k}{\partial r_m}} \frac{\partial V_m}{\partial r_l} - 2\hat{\Phi}_v(\tilde{u})\tilde{u}_k\tilde{u}_l \frac{3}{2\Theta} \overline{\frac{\partial V_k}{\partial r_m} \frac{\partial V_m}{\partial r_l}} \\
& + \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \left(\tilde{u}^2 - \frac{3}{2}\right) \tilde{u}_k\tilde{u}_l \frac{1}{\Theta^2} \frac{\partial\Theta}{\partial r_k} \frac{\partial\Theta}{\partial r_l} - \hat{\Phi}_c(\tilde{u}) \tilde{u}^2 \tilde{u}_k\tilde{u}_l \frac{1}{\Theta^2} \frac{\partial\Theta}{\partial r_k} \frac{\partial\Theta}{\partial r_l} \\
& - \frac{1}{2}\hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_k\tilde{u}_l \frac{1}{\Theta^2} \frac{\partial\Theta}{\partial r_k} \frac{\partial\Theta}{\partial r_l} - \hat{\Phi}_c'(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}^2 \tilde{u}_k\tilde{u}_l \frac{1}{\Theta^2} \frac{\partial\Theta}{\partial r_k} \frac{\partial\Theta}{\partial r_l} \\
& - \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_k\tilde{u}_l \frac{1}{n\Theta^2} \frac{\partial(n\Theta)}{\partial r_k} \frac{\partial\Theta}{\partial r_l} + \hat{\Phi}_c(\tilde{u}) \tilde{u}_k\tilde{u}_l \frac{1}{n\Theta^2} \frac{\partial(n\Theta)}{\partial r_k} \frac{\partial\Theta}{\partial r_l} \\
& + \hat{\Phi}_c'(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_k\tilde{u}_l \frac{1}{n\Theta^2} \frac{\partial(n\Theta)}{\partial r_k} \frac{\partial\Theta}{\partial r_l} - \hat{\Phi}_v(\tilde{u}) \tilde{u}_k\tilde{u}_l \frac{1}{\Theta} \overline{\frac{\partial}{\partial r_k} \left(\frac{1}{n} \frac{\partial(n\Theta)}{\partial r_l}\right)} \\
& \left. + \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_k\tilde{u}_l \left(\frac{1}{\Theta} \frac{\partial^2\Theta}{\partial r_k \partial r_l} - \frac{1}{\Theta^2} \frac{\partial\Theta}{\partial r_k} \frac{\partial\Theta}{\partial r_l}\right) \right] \\
& - \frac{n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i\tilde{u}_j} \tilde{\Omega}(\Phi_K, \Phi_K).
\end{aligned} \tag{4.152}
$$

In the above simplification, (along with eqs. (F.3)-(F.2)) the following are used:

$$
\begin{aligned}
& \tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l \overline{\frac{\partial V_i}{\partial r_j}} \frac{\partial V_k}{\partial r_l} - \frac{1}{3}\tilde{u}^2 \tilde{u}_i\tilde{u}_j \overline{\frac{\partial V_i}{\partial r_j}} \frac{\partial V_k}{\partial r_k} = \tilde{u}_i\tilde{u}_j \overline{\frac{\partial V_i}{\partial r_j}} \left[\tilde{u}_k\tilde{u}_l \frac{\partial V_k}{\partial r_l} - \frac{1}{3}\tilde{u}^2 \delta_{kl} \frac{\partial V_k}{\partial r_l}\right] \\
& = \tilde{u}_i\tilde{u}_j \overline{\frac{\partial V_i}{\partial r_j}} \frac{\partial V_k}{\partial r_l} \left[\frac{\tilde{u}_k\tilde{u}_l + \tilde{u}_l\tilde{u}_k}{2} - \frac{1}{3}\delta_{kl}\tilde{u}^2\right] = \tilde{u}_i\tilde{u}_j \overline{\frac{\partial V_i}{\partial r_j}} \frac{\partial V_k}{\partial r_l} \overline{\tilde{u}_k\tilde{u}_l} = \tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l \overline{\frac{\partial V_i}{\partial r_j}}\, \overline{\frac{\partial V_k}{\partial r_l}},
\end{aligned} \tag{4.153a}
$$

$$
\tilde{u}_i\tilde{u}_j \frac{\partial \ln\Theta}{\partial r_i} \frac{\partial \ln n}{\partial r_j} = \tilde{u}_j\tilde{u}_i \frac{\partial \ln\Theta}{\partial r_j} \frac{\partial \ln n}{\partial r_i} = \tilde{u}_i\tilde{u}_j \frac{\partial \ln n}{\partial r_i} \frac{\partial \ln\Theta}{\partial r_j}, \tag{4.153b}
$$

$$
\tilde{u}_k\tilde{u}_l \frac{\partial \ln\Theta}{\partial r_k} \frac{\partial \ln(n\Theta)}{\partial r_l} = \tilde{u}_l\tilde{u}_k \frac{\partial \ln\Theta}{\partial r_l} \frac{\partial \ln(n\Theta)}{\partial r_k} = \tilde{u}_k\tilde{u}_l \frac{1}{n\Theta^2} \frac{\partial(n\Theta)}{\partial r_k} \frac{\partial\Theta}{\partial r_l}, \tag{4.153c}
$$

$$
\begin{aligned}
& \tilde{u}_k\tilde{u}_l \left\{\overline{\frac{\partial \ln(n\Theta)}{\partial r_k} \frac{\partial \ln\Theta}{\partial r_l}} + \overline{\frac{\partial^2 \ln(n\Theta)}{\partial r_k \partial r_l}}\right\} = \overline{\tilde{u}_k\tilde{u}_l} \left\{\frac{\partial \ln(n\Theta)}{\partial r_k} \frac{\partial \ln\Theta}{\partial r_l} + \frac{\partial^2 \ln(n\Theta)}{\partial r_k \partial r_l}\right\} \\
& = \overline{\tilde{u}_k\tilde{u}_l} \left\{\frac{1}{n\Theta^2} \frac{\partial(n\Theta)}{\partial r_k} \frac{\partial\Theta}{\partial r_l} + \frac{\partial}{\partial r_l}\left(\frac{1}{n\Theta} \frac{\partial(n\Theta)}{\partial r_k}\right)\right\} \\
& = \overline{\tilde{u}_k\tilde{u}_l} \left\{\frac{1}{n\Theta^2} \frac{\partial(n\Theta)}{\partial r_k} \frac{\partial\Theta}{\partial r_l} + \frac{1}{\Theta} \frac{\partial}{\partial r_l}\left(\frac{1}{n} \frac{\partial(n\Theta)}{\partial r_k}\right) - \frac{1}{n\Theta^2} \frac{\partial(n\Theta)}{\partial r_k} \frac{\partial\Theta}{\partial r_l}\right\} \\
& = \overline{\tilde{u}_k\tilde{u}_l} \frac{1}{\Theta} \frac{\partial}{\partial r_l}\left(\frac{1}{n} \frac{\partial(n\Theta)}{\partial r_k}\right) = \overline{\tilde{u}_l\tilde{u}_k} \frac{1}{\Theta} \frac{\partial}{\partial r_k}\left(\frac{1}{n} \frac{\partial(n\Theta)}{\partial r_l}\right) = \tilde{u}_k\tilde{u}_l \frac{1}{\Theta} \overline{\frac{\partial}{\partial r_k}\left(\frac{1}{n} \frac{\partial(n\Theta)}{\partial r_l}\right)},
\end{aligned} \tag{4.153d}
$$

$$
\frac{\partial^2 \ln\Theta}{\partial r_k \partial r_l} = \frac{\partial}{\partial r_k}\left(\frac{\partial \ln\Theta}{\partial r_l}\right) = \frac{\partial}{\partial r_k}\left(\frac{1}{\Theta} \frac{\partial\Theta}{\partial r_l}\right) = \frac{1}{\Theta} \frac{\partial^2\Theta}{\partial r_k \partial r_l} - \frac{1}{\Theta^2} \frac{\partial\Theta}{\partial r_k} \frac{\partial\Theta}{\partial r_l}. \tag{4.153e}
$$

Now, eq. (4.152) can be written as

$$
\begin{aligned}
P_{ij}^{KK} &= \frac{4n\ell^2}{\pi^{3/2}}\frac{\partial V_m}{\partial r_m}\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v^2(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l \\
&- \frac{2n\ell^2}{\pi^{3/2}}\left\{\frac{1}{3}\overline{\frac{\partial}{\partial r_k}\left(\frac{1}{n}\frac{\partial(n\Theta)}{\partial r_l}\right)} + \overline{\frac{\partial V_k}{\partial r_m}\frac{\partial V_m}{\partial r_l}} + 2\overline{\frac{\partial V_k}{\partial r_m}}\frac{\partial V_m}{\partial r_l}\right\}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v^2(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l \\
&+ \frac{2n\ell^2}{3\pi^{3/2}}\frac{\partial^2\Theta}{\partial r_k\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u})\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l \\
&+ \frac{2\ell^2}{3\pi^{3/2}\Theta}\frac{\partial(n\Theta)}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u})\left\{\hat{\varPhi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)-\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{7}{2}\right)\right\}\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l \\
&+ \frac{2n\ell^2}{3\pi^{3/2}\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u}) \\
&\times\left\{\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^4-\frac{13}{2}\tilde{u}^2+\frac{15}{2}\right)-\hat{\varPhi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^2\right\}\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l \\
&+ \frac{4n\ell^2}{\pi^{3/2}}\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u})\left\{\hat{\varPhi}_v(\tilde{u})-\hat{\varPhi}_v'(\tilde{u})\right\}\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n \\
&- \frac{n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{\varOmega}(\varPhi_K,\varPhi_K).
\end{aligned}
$$

Using eqs. (F.12) and (F.15), the integrations over $\tilde{\boldsymbol{u}}$ result into

$$
\begin{aligned}
P_{ij}^{KK} &= n\ell^2\frac{\partial V_m}{\partial r_m}\overline{\frac{\partial V_i}{\partial r_j}}\frac{32}{15\sqrt{\pi}}\int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v^2(\tilde{u})\tilde{u}^6 \\
&- n\ell^2\left\{\frac{1}{3}\overline{\frac{\partial}{\partial r_i}\left(\frac{1}{n}\frac{\partial(n\Theta)}{\partial r_j}\right)} + \overline{\frac{\partial V_i}{\partial r_m}\frac{\partial V_m}{\partial r_j}} + 2\overline{\overline{\frac{\partial V_i}{\partial r_m}}\frac{\partial V_m}{\partial r_j}}\right\}\frac{16}{15\sqrt{\pi}}\int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v^2(\tilde{u})\tilde{u}^6 \\
&+ n\ell^2\overline{\frac{\partial^2\Theta}{\partial r_i\partial r_j}}\frac{16}{45\sqrt{\pi}}\int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u})\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^6 \\
&+ \frac{\ell^2}{\Theta}\overline{\frac{\partial(n\Theta)}{\partial r_i}\frac{\partial\Theta}{\partial r_j}}\frac{16}{45\sqrt{\pi}}\int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u})\left\{\hat{\varPhi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)-\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^2-\frac{7}{2}\right)\right\}\tilde{u}^6 \\
&+ \frac{n\ell^2}{\Theta}\overline{\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_j}}\frac{16}{45\sqrt{\pi}}\int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u}) \\
&\times\left\{\hat{\varPhi}_c(\tilde{u})\left(\tilde{u}^4-\frac{13}{2}\tilde{u}^2+\frac{15}{2}\right)-\hat{\varPhi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^2\right\}\tilde{u}^6 \\
&+ n\ell^2\overline{\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}}\frac{128}{105\sqrt{\pi}}\int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u})\left\{\hat{\varPhi}_v(\tilde{u})-\hat{\varPhi}_v'(\tilde{u})\right\}\tilde{u}^8 \\
&- \frac{n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{\varOmega}(\varPhi_K,\varPhi_K). \qquad (4.154)
\end{aligned}
$$

The last term in eq. (4.154) is evaluated separately as following. Substituting the value of $\tilde{\varOmega}(\varPhi_K,\varPhi_K)$ from eq. (2.38), this term is given by

$$
\begin{aligned}
&\frac{n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\varPhi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{\varOmega}(\varPhi_K,\varPhi_K) \\
&= \frac{2n\Theta}{3\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\{\varPhi_K(\tilde{\boldsymbol{u}}_1')\varPhi_K(\tilde{\boldsymbol{u}}_2')-\varPhi_K(\tilde{\boldsymbol{u}}_1)\varPhi_K(\tilde{\boldsymbol{u}}_2)\}.
\end{aligned}
$$

Note that in the above equation, the velocity transformation corresponds to the elastic limit. Following a similar procedure as in the derivation of $Q_{i_3}^{K\epsilon}$, one obtains

$$\frac{n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{\Omega}(\Phi_K,\Phi_K) = M_1 - M_2, \tag{4.155}$$

where

$$M_1 = \frac{2n\Theta}{3\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_K(\tilde{\boldsymbol{u}}_2)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta^{(0)}, \tag{4.156}$$

$$M_2 = \frac{2n\Theta}{3\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\,\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_K(\tilde{\boldsymbol{u}}_2), \tag{4.157}$$

and $I_\delta^{(0)}$ is given in eq. (4.48). We shall simplify M_1 and M_2 as following. Using eqs. (3.18) and (F.3),

$$\begin{aligned}M_1 = {}& \frac{2n\Theta}{3\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)}\\ &\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_1)\tilde{u}_{1k}\tilde{u}_{1l}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1k}\frac{\partial\ln\Theta}{\partial r_k}\right]\\ &\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2m}\tilde{u}_{2n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_m}{\partial r_n}} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2m}\frac{\partial\ln\Theta}{\partial r_m}\right]\end{aligned}$$

or

$$\begin{aligned}M_1 = {}& \frac{2n\Theta\ell^2}{3\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)}\\ &\times\left[4\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_{2m}\tilde{u}_{2n}\frac{3}{2\Theta}\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\right.\\ &+2\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_{2m}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\frac{\partial\ln\Theta}{\partial r_m}\\ &+2\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1k}\tilde{u}_{2m}\tilde{u}_{2n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_m}{\partial r_n}}\frac{\partial\ln\Theta}{\partial r_k}\\ &\left.+\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1k}\tilde{u}_{2m}\frac{\partial\ln\Theta}{\partial r_k}\frac{\partial\ln\Theta}{\partial r_m}\right].\end{aligned}$$

We can ignore the the second and third terms in the square brackets above because ultimately after manipulation corresponding integrands become odd functions in components of $\tilde{\boldsymbol{u}}$ (cf. §4.2). Therefore

$$\begin{aligned}M_1 = {}& \frac{4n\ell^2}{\pi^4}\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)}\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_{2m}\tilde{u}_{2n}\\ &+\frac{2n\Theta\ell^2}{3\pi^4}\frac{\partial\ln\Theta}{\partial r_k}\frac{\partial\ln\Theta}{\partial r_m}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)}\\ &\times\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1k}\tilde{u}_{2m}.\end{aligned}$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}$, we get

$$\begin{aligned}
M_1 = & \frac{4n\ell^2}{\pi^4}\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)} \\
& \times \hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2)(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)\tilde{u}_{2m}\tilde{u}_{2n} \\
& + \frac{2n\ell^2}{3\pi^4\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}I_\delta^{(0)} \\
& \times \hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)(\tilde{u}_k-\tilde{s}_k)\tilde{u}_{2l}.
\end{aligned}$$

The integrations over $\tilde{\boldsymbol{u}}_2$ are performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_2 = \tilde{s}\tilde{u}_2\cos\theta_2'$, i.e.,

$$\begin{aligned}
M_1 = & n\ell^2\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\frac{4}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2} \\
& \times \hat{\Phi}_v(\tilde{u})\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)\tilde{u}_{2m}\tilde{u}_{2n}I_\delta^{(0)} \\
& + \frac{n\ell^2}{\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\frac{2}{3\pi^4}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2} \\
& \times \hat{\Phi}_v(\tilde{u})\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)\tilde{u}_{2l}I_\delta^{(0)}.
\end{aligned}$$

Note that the components of $\tilde{\boldsymbol{u}}_2$ are the only functions of ϕ_2' (see Appendix H). Therefore the integrations over ϕ_2' give (cf. eqs. (H.6) and (H.18))

$$\begin{aligned}
M_1 = & n\ell^2\overline{\frac{\partial V_k}{\partial r_l}}\frac{4}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u})\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|) \\
& \times \hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)\left\{\pi\frac{\tilde{u}_2^2}{\tilde{s}^2}\overline{\frac{\partial V_m}{\partial r_n}}\tilde{s}_m\tilde{s}_n(3\cos^2\theta_2'-1)\right\}I_\delta^{(0)} \\
& + \frac{n\ell^2}{\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\frac{2}{3\pi^4}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u})\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|) \\
& \times \hat{\Phi}_c(\tilde{u}_2)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)\left(2\pi\frac{\tilde{u}_2}{\tilde{s}}\tilde{s}_l\cos\theta_2'\right)I_\delta^{(0)} \\
= & n\ell^2\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\frac{4}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_v(\tilde{u})\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l) \\
& \times \frac{\tilde{s}_m\tilde{s}_n}{\tilde{s}^2}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^4\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{\int_{\theta_2'=0}^{\pi}\sin\theta_2'(3\cos^2\theta_2'-1)I_\delta^{(0)}\,\mathrm{d}\theta_2'\right\}\mathrm{d}\tilde{u}_2 \\
& + \frac{n\ell^2}{\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\frac{4}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_v(\tilde{u})\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k) \\
& \times \frac{\tilde{s}_l}{\tilde{s}}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^3\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left\{\int_{\theta_2'=0}^{\pi}\sin\theta_2'\cos\theta_2' I_\delta^{(0)}\,\mathrm{d}\theta_2'\right\}\mathrm{d}\tilde{u}_2.
\end{aligned}$$

Using eq. (C.12),

$$M_1 = n\ell^2 \overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}} \frac{4}{\pi^3} \int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\, \hat{\Phi}_v(\tilde{u})\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)$$
$$\times \frac{\tilde{s}_m\tilde{s}_n}{\tilde{s}^3}\int_{\tilde{u}_2=\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty} \tilde{u}_2^3\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right)^2-1\right\}\mathrm{d}\tilde{u}_2$$
$$+\frac{n\ell^2}{\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\frac{4}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\, \hat{\Phi}_v(\tilde{u})\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)$$
$$\times \frac{\tilde{s}_l}{\tilde{s}^2}\int_{\tilde{u}_2=\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty} \tilde{u}_2^2\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right)\mathrm{d}\tilde{u}_2.$$

Let us replace $\tilde{u}_2^2$ by $\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2$. This shift implies that

$$M_1 = n\ell^2 \overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}} \frac{4}{\pi^3} \int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\, \hat{\Phi}_v(\tilde{u})\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)$$
$$\times \frac{\tilde{s}_m\tilde{s}_n}{\tilde{s}^3}\int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2\, \mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\, \hat{\Phi}_v\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)\left\{2\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2-\tilde{u}_2^2\right\}\mathrm{d}\tilde{u}_2$$
$$+\frac{n\ell^2}{\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\frac{4}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\, \hat{\Phi}_v(\tilde{u})\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)$$
$$\times \frac{\tilde{s}_l}{\tilde{s}^2}\int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2\, \mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\, \hat{\Phi}_c\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)$$
$$\times\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2-\frac{5}{2}\right\}\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)\mathrm{d}\tilde{u}_2.$$

The integrations over $\tilde{\boldsymbol{s}}$ are performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}=\tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$M_1 = n\ell^2 \overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}} \frac{4}{\pi^3} \int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}$$
$$\times\hat{\Phi}_v(\tilde{u})\hat{\Phi}_v\Big(\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2\right)^{1/2}\Big)\,\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)(\tilde{u}_l-\tilde{s}_l)\tilde{s}_m\tilde{s}_n$$
$$\times\frac{1}{\tilde{s}^3}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')}\,\hat{\Phi}_v\Big(\left(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta'\right)^{1/2}\Big)\left(2\tilde{u}^2\cos^2\theta'-\tilde{u}_2^2\right)\mathrm{d}\tilde{u}_2$$
$$+\frac{n\ell^2}{\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\frac{4}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}$$
$$\times\hat{\Phi}_v(\tilde{u})\hat{\Phi}_c\Big(\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2\right)^{1/2}\Big)\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2-\frac{5}{2}\right)\overline{\tilde{u}_i\tilde{u}_j}(\tilde{u}_k-\tilde{s}_k)\tilde{s}_l$$
$$\times\frac{1}{\tilde{s}^2}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')}\,\hat{\Phi}_c\Big(\left(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta'\right)^{1/2}\Big)\left(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta'-\frac{5}{2}\right)\tilde{u}\cos\theta'\,\mathrm{d}\tilde{u}_2.$$

The values of integrals over ϕ' are given in eqs. (H.17) and (H.23). Hence using these equations,

$$
\begin{aligned}
M_1 = {} & n\ell^2 \frac{4}{\pi^3} \int \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{s}=0}^{\infty} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \mathrm{d}\tilde{s}\, \tilde{s}^2 \sin\theta'\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)}\, \hat{\Phi}_v(\tilde{u}) \hat{\Phi}_v\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2} \Big) \\
& \times \overline{\tilde{u}_i \tilde{u}_j} \Bigg[2\pi \frac{\tilde{s}^2}{\tilde{u}^4} \overline{\frac{\partial V_k}{\partial r_l}}\, \overline{\frac{\partial V_m}{\partial r_n}} \tilde{u}_k \tilde{u}_l \tilde{u}_m \tilde{u}_n \Bigg\{ \tilde{u}^2 \frac{1}{2} (3\cos^2\theta' - 1) - 2\tilde{u}\tilde{s} \frac{1}{2} (5\cos^3\theta' - 3\cos\theta') \\
& + \tilde{s}^2 \frac{1}{8} (35\cos^4\theta' - 30\cos^2\theta' + 3) \Bigg\} \\
& + \pi \frac{\tilde{s}^3}{\tilde{u}^2} \overline{\frac{\partial V_k}{\partial r_m}}\, \overline{\frac{\partial V_m}{\partial r_l}} \tilde{u}_k \tilde{u}_l (1 - \cos^2\theta') \{ \tilde{s}(5\cos^2\theta' - 1) - 4\tilde{u}\cos\theta' \} + \frac{\pi}{2} \tilde{s}^4 (1 - \cos^2\theta')^2 \overline{\frac{\partial V_k}{\partial r_l}}\, \overline{\frac{\partial V_l}{\partial r_k}} \Bigg] \\
& \times \frac{1}{\tilde{s}^3} \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')}\, \hat{\Phi}_v\Big((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2} \Big) (2\tilde{u}^2\cos^2\theta' - \tilde{u}_2^2)\, \mathrm{d}\tilde{u}_2 \\
& + \frac{n\ell^2}{\Theta} \frac{4}{3\pi^3} \int \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{s}=0}^{\infty} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \mathrm{d}\tilde{s}\, \tilde{s}^2 \sin\theta'\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)} \\
& \times \hat{\Phi}_v(\tilde{u}) \hat{\Phi}_c\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2} \Big) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2 - \frac{5}{2} \right) \overline{\tilde{u}_i \tilde{u}_j} \\
& \times \left[-\pi \tilde{s}^2 \frac{\partial \Theta}{\partial r_k} \frac{\partial \Theta}{\partial r_k} (1 - \cos^2\theta') + 2\pi \frac{\tilde{s}}{\tilde{u}^2} \frac{\partial \Theta}{\partial r_k} \frac{\partial \Theta}{\partial r_l} \tilde{u}_k \tilde{u}_l \left\{ \tilde{u}\cos\theta' - \frac{1}{2} \tilde{s} (3\cos^2\theta' - 1) \right\} \right] \\
& \times \frac{1}{\tilde{s}^2} \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')}\, \hat{\Phi}_c\Big((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2} \Big) \left(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta' - \frac{5}{2} \right) \tilde{u}\cos\theta'\, \mathrm{d}\tilde{u}_2 .
\end{aligned}
$$

The values of integrals over $\tilde{\boldsymbol{u}}$ are given in eqs. (F.10), (F.12) and (F.15). Hence using these equations,

$$
\begin{aligned}
M_1 = {} & n\ell^2 \frac{4}{\pi^3} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \mathrm{d}\tilde{s}\, \tilde{s}^2 \sin\theta'\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)} \\
& \times \hat{\Phi}_v(\tilde{u}) \hat{\Phi}_v\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2} \Big) \left[2\pi \frac{\tilde{s}^2}{\tilde{u}^4} \frac{32\pi}{105} \overline{\overline{\frac{\partial V_i}{\partial r_k}}\, \overline{\frac{\partial V_k}{\partial r_j}}} \tilde{u}^8 \left\{ \tilde{u}^2 \frac{1}{2} (3\cos^2\theta' - 1) \right. \right. \\
& \left. - 2\tilde{u}\tilde{s} \frac{1}{2} (5\cos^3\theta' - 3\cos\theta') + \tilde{s}^2 \frac{1}{8} (35\cos^4\theta' - 30\cos^2\theta' + 3) \right\} \\
& \left. + \pi \frac{\tilde{s}^3}{\tilde{u}^2} \frac{8\pi}{15} \overline{\overline{\frac{\partial V_i}{\partial r_m}}\, \overline{\frac{\partial V_m}{\partial r_j}}} \tilde{u}^6 (1 - \cos^2\theta') \{ \tilde{s}(5\cos^2\theta' - 1) - 4\tilde{u}\cos\theta' \} \right] \\
& \times \frac{1}{\tilde{s}^3} \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')}\, \hat{\Phi}_v\Big((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2} \Big) (2\tilde{u}^2\cos^2\theta' - \tilde{u}_2^2)\, \mathrm{d}\tilde{u}_2 \\
& + \frac{n\ell^2}{\Theta} \frac{4}{3\pi^3} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \mathrm{d}\tilde{s}\, \tilde{s}^2 \sin\theta'\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)} \\
& \times \hat{\Phi}_v(\tilde{u}) \hat{\Phi}_c\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2} \Big) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2 - \frac{5}{2} \right) \\
& \times \left[2\pi \frac{\tilde{s}}{\tilde{u}^2} \frac{8\pi}{15} \overline{\frac{\partial \Theta}{\partial r_i} \frac{\partial \Theta}{\partial r_j}} \tilde{u}^6 \left\{ \tilde{u}\cos\theta' - \frac{1}{2} \tilde{s} (3\cos^2\theta' - 1) \right\} \right] \\
& \times \frac{1}{\tilde{s}^2} \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')}\, \hat{\Phi}_c\Big((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2} \Big) \left(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta' - \frac{5}{2} \right) \tilde{u}\cos\theta'\, \mathrm{d}\tilde{u}_2
\end{aligned}
$$

or

$$
\begin{aligned}
M_1 = n\ell^2 \overline{\overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}}} \frac{32}{105\pi} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \mathrm{d}\tilde{s}\, \sin\theta' \tilde{s}\tilde{u}^4\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)} \\
&\times \hat{\Phi}_v(\tilde{u}) \hat{\Phi}_v\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2} \Big) \\
&\times \Big[\big\{ 4\tilde{u}^2 (3\cos^2\theta' - 1) - 8\tilde{u}\tilde{s}(5\cos^3\theta' - 3\cos\theta') + \tilde{s}^2 (35\cos^4\theta' - 30\cos^2\theta' + 3) \big\} \\
&+ 7\tilde{s}(1 - \cos^2\theta') \{ \tilde{s}(5\cos^2\theta' - 1) - 4\tilde{u}\cos\theta' \} \Big] \\
&\times \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')}\, \hat{\Phi}_v\Big((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2} \Big) (2\tilde{u}^2\cos^2\theta' - \tilde{u}_2^2)\, \mathrm{d}\tilde{u}_2 \\
&+ \frac{n\ell^2}{\Theta} \overline{\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_j}} \frac{64}{45\pi} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \mathrm{d}\tilde{s}\, \sin\theta' \tilde{s}\tilde{u}^5\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)}\, \hat{\Phi}_v(\tilde{u}) \\
&\times \hat{\Phi}_c\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2} \Big) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2 - \frac{5}{2} \right) \left\{ \tilde{u}\cos\theta' - \frac{1}{2}\tilde{s}(3\cos^2\theta' - 1) \right\} \\
&\times \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')}\, \hat{\Phi}_c\Big((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2} \Big) \left(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta' - \frac{5}{2} \right) \cos\theta'\, \mathrm{d}\tilde{u}_2.
\end{aligned}
$$

Let $\cos\theta' = y$. This implies that

$$
\begin{aligned}
M_1 = n\ell^2 \overline{\overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}}} \frac{32}{105\pi} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \int_{y=-1}^{1} \mathrm{d}y\, \mathrm{d}\tilde{s}\, \tilde{s}\tilde{u}^4\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \hat{\Phi}_v(\tilde{u}) \\
&\times \hat{\Phi}_v\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2} \Big) \big\{ 4\tilde{u}^2(3y^2 - 1) - 4\tilde{u}\tilde{s}y(3y^2 + 1) + 4\tilde{s}^2(3y^2 - 1) \big\} \\
&\times \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2 y^2)}\, \hat{\Phi}_v\Big((\tilde{u}_2^2 + \tilde{u}^2 y^2)^{1/2} \Big) (2\tilde{u}^2 y^2 - \tilde{u}_2^2)\, \mathrm{d}\tilde{u}_2 \\
&+ \frac{n\ell^2}{\Theta} \overline{\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_j}} \frac{64}{45\pi} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \int_{y=-1}^{1} \mathrm{d}y\, \mathrm{d}\tilde{s}\, \tilde{s}\tilde{u}^5\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \hat{\Phi}_v(\tilde{u}) \\
&\times \hat{\Phi}_c\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2} \Big) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2} \right) \left\{ \tilde{u}y - \frac{1}{2}\tilde{s}(3y^2 - 1) \right\} \\
&\times \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2 y^2)}\, \hat{\Phi}_c\Big((\tilde{u}_2^2 + \tilde{u}^2 y^2)^{1/2} \Big) \left(\tilde{u}_2^2 + \tilde{u}^2 y^2 - \frac{5}{2} \right) y\, \mathrm{d}\tilde{u}_2.
\end{aligned}
$$

or

$$
\begin{aligned}
M_1 = n\ell^2 \overline{\overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}}} \frac{128}{105\pi} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{y=-1}^{1} \mathrm{d}y \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}^4 \tilde{s}\tilde{u}_2 (2\tilde{u}^2 y^2 - \tilde{u}_2^2) \\
&\times \big\{ \tilde{u}^2(3y^2 - 1) - \tilde{u}\tilde{s}y(3y^2 + 1) + \tilde{s}^2(3y^2 - 1) \big\} \hat{\Phi}_v(\tilde{u}) \hat{\Phi}_v\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2} \Big) \\
&\times \hat{\Phi}_v\Big((\tilde{u}_2^2 + \tilde{u}^2 y^2)^{1/2} \Big)\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2 y^2)} \\
&+ \frac{n\ell^2}{\Theta} \overline{\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_j}} \frac{64}{45\pi} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{y=-1}^{1} \mathrm{d}y \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}^5 \tilde{s} y \tilde{u}_2 \\
&\times \hat{\Phi}_v(\tilde{u}) \hat{\Phi}_c\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2} \Big) \hat{\Phi}_c\Big((\tilde{u}_2^2 + \tilde{u}^2 y^2)^{1/2} \Big) \left\{ \tilde{u}y - \frac{1}{2}\tilde{s}(3y^2 - 1) \right\} \\
&\times \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2} \right) \left(\tilde{u}_2^2 + \tilde{u}^2 y^2 - \frac{5}{2} \right) \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2 y^2)}\,. \qquad (4.158)
\end{aligned}
$$

Next, consider eq. (4.157). The integration over $\hat{\boldsymbol{k}}$ is trivial. Hence using eq. (G.1b),

$$M_2 = \frac{2n\Theta}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\,\varPhi_K(\tilde{\boldsymbol{u}}_1)\varPhi_K(\tilde{\boldsymbol{u}}_2)$$

Substituting the value of $\varPhi_K$ from eq. (3.18) and using eq. (F.3),

$$\begin{aligned}
M_2 &= \frac{2n\Theta}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\\
&\quad\times\left[2K\frac{2\Theta}{3g}\hat{\varPhi}_v(\tilde{u}_1)\tilde{u}_{1k}\tilde{u}_{1l}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}+K\frac{2\Theta}{3g}\hat{\varPhi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1k}\frac{\partial\ln\Theta}{\partial r_k}\right]\\
&\quad\times\left[2K\frac{2\Theta}{3g}\hat{\varPhi}_v(\tilde{u}_2)\tilde{u}_{2m}\tilde{u}_{2n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_m}{\partial r_n}}+K\frac{2\Theta}{3g}\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2m}\frac{\partial\ln\Theta}{\partial r_m}\right]\\
&= \frac{2n\Theta\ell^2}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\\
&\quad\times\left[4\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2)\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_{2m}\tilde{u}_{2n}\frac{3}{2\Theta}\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\right.\\
&\quad+2\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_{2m}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\frac{\partial\ln\Theta}{\partial r_m}\\
&\quad+2\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1k}\tilde{u}_{2m}\tilde{u}_{2n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_m}{\partial r_n}}\frac{\partial\ln\Theta}{\partial r_k}\\
&\quad\left.+\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1k}\tilde{u}_{2m}\frac{\partial\ln\Theta}{\partial r_k}\frac{\partial\ln\Theta}{\partial r_m}\right].
\end{aligned}$$

We can ignore the second and third terms in the square brackets above because ultimately after manipulation corresponding integrands become odd functions in components of $\tilde{\boldsymbol{u}}_1$ (cf. §4.2). Therefore

$$\begin{aligned}
M_2 &= \frac{4n\ell^2}{\pi^3}\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_v^2(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_{2m}\tilde{u}_{2n}\\
&\quad+\frac{2n\ell^2}{3\pi^3\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2)\\
&\quad\times\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{2l}.
\end{aligned}$$

The integrations over $\tilde{\boldsymbol{u}}_2$ are performed in a (rotated) spherical coordinate system $(\tilde{u}_2,\theta_2',\phi_2')$ described in part (III) of $Q_{i3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2=\tilde{u}_1\tilde{u}_2\cos\theta_2'$, i.e.,

$$\begin{aligned}
M_2 &= n\ell^2\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\frac{4}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\\
&\quad\times(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2)^{1/2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_v^2(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_{2m}\tilde{u}_{2n}\\
&\quad+\frac{n\ell^2}{\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\frac{2}{3\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\\
&\quad\times(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2)^{1/2}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2)\\
&\quad\times\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{2l}.
\end{aligned}$$

The integrals over ϕ_2' result into (cf. eqs. (H.18) and (H.6)),

$$M_2 = n\ell^2 \overline{\frac{\partial V_k}{\partial r_l}} \frac{4}{\pi^3} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}$$
$$\times\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v^2(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l} \left\{\pi \frac{\tilde{u}_2^2}{\tilde{u}_1^2}\overline{\frac{\partial V_m}{\partial r_n}}\tilde{u}_{1m}\tilde{u}_{1n}(3\cos^2\theta_2' - 1)\right\}$$
$$+ \frac{n\ell^2}{\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\frac{2}{3\pi^3} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}$$
$$\times\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k} \times 2\pi\frac{\tilde{u}_2}{\tilde{u}_1}\tilde{u}_{1l}\cos\theta_2'$$

or

$$M_2 = n\ell^2 \overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}} \frac{8}{\pi^2} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \frac{\tilde{u}_2^4}{\tilde{u}_1^2}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v^2(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_{1m}\tilde{u}_{1n}$$
$$\times \left\{\int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2'\, \sin\theta_2' \frac{1}{2}(3\cos^2\theta_2' - 1)(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}\right\}$$
$$+ \frac{n\ell^2}{\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\frac{4}{3\pi^2} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \frac{\tilde{u}_2^3}{\tilde{u}_1}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right)$$
$$\times\, \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}\left\{\int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2'\, \sin\theta_2' \cos\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}\right\}$$
$$= n\ell^2 \overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}} \frac{8}{\pi^2} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \frac{\tilde{u}_2^4}{\tilde{u}_1^2}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v^2(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}\tilde{u}_{1m}\tilde{u}_{1n}R_2(\tilde{u}_1,\tilde{u}_2)$$
$$+ \frac{n\ell^2}{\Theta}\frac{\partial\Theta}{\partial r_k}\frac{\partial\Theta}{\partial r_l}\frac{4}{3\pi^2} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \frac{\tilde{u}_2^3}{\tilde{u}_1}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right)$$
$$\times\, \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\tilde{u}_{1k}\tilde{u}_{1l}R_1(\tilde{u}_1,\tilde{u}_2),$$

where $R_n(\tilde{u}_1,\tilde{u}_2)$ is defined in eq. (4.57) and the values of $R_1(\tilde{u}_1,\tilde{u}_2)$ and $R_2(\tilde{u}_1,\tilde{u}_2)$ are given in eqs. (4.64) and (4.99) respectively. The integrations over $\tilde{\boldsymbol{u}}_1$ result into (cf. eqs. (F.15) and (F.12)),

$$M_2 = n\ell^2 \overline{\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}} \frac{8}{\pi^2}\frac{32\pi}{105} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \frac{\tilde{u}_2^4}{\tilde{u}_1^2}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v^2(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_1^8 R_2(\tilde{u}_1,\tilde{u}_2)$$
$$+ \frac{n\ell^2}{\Theta}\overline{\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_j}}\frac{4}{3\pi^2}\frac{8\pi}{15} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \frac{\tilde{u}_2^3}{\tilde{u}_1}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2 - \frac{5}{2}\right)$$
$$\times \left(\tilde{u}_2^2 - \frac{5}{2}\right)\tilde{u}_1^6 R_1(\tilde{u}_1,\tilde{u}_2)$$

or

$$M_2 = n\ell^2 \overline{\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}} \frac{256}{105\pi} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_1^6\tilde{u}_2^4 R_2(\tilde{u}_1,\tilde{u}_2)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v^2(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)$$
$$+ \frac{n\ell^2}{\Theta}\overline{\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_j}}\frac{32}{45\pi} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_1^5\tilde{u}_2^3 R_1(\tilde{u}_1,\tilde{u}_2)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)$$
$$\times \left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right). \tag{4.159}$$

Hence, from eqs. (4.154), (4.155), (4.158) and (4.159),

$$\boxed{\begin{aligned}P_{ij}^{KK} &= \tilde{\omega}_1 n\ell^2 \frac{\partial V_k}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}} - \tilde{\omega}_2 n\ell^2 \left\{ \frac{1}{3}\overline{\frac{\partial}{\partial r_i}\left(\frac{1}{n}\frac{\partial (n\Theta)}{\partial r_j}\right)} + \overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}} + 2\overline{\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_j}} \right\} \\ &\quad + \tilde{\omega}_3 n\ell^2 \overline{\frac{\partial^2 \Theta}{\partial r_i \partial r_j}} + \tilde{\omega}_4 \frac{\ell^2}{\Theta}\overline{\frac{\partial (n\Theta)}{\partial r_i}\frac{\partial \Theta}{\partial r_j}} + \tilde{\omega}_5 \frac{n\ell^2}{\Theta}\overline{\frac{\partial \Theta}{\partial r_i}\frac{\partial \Theta}{\partial r_j}} + \tilde{\omega}_6 n\ell^2 \overline{\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}}\end{aligned}} \tag{4.160}$$

where

$$\tilde{\omega}_1 = \frac{32}{15\sqrt{\pi}} \int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v^2(\tilde{u})\tilde{u}^6, \tag{4.161a}$$

$$\tilde{\omega}_2 = \frac{16}{15\sqrt{\pi}} \int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v^2(\tilde{u})\tilde{u}^6, \tag{4.161b}$$

$$\tilde{\omega}_3 = \frac{16}{45\sqrt{\pi}} \int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v(\tilde{u})\hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}^6, \tag{4.161c}$$

$$\tilde{\omega}_4 = \frac{16}{45\sqrt{\pi}} \int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v(\tilde{u}) \left\{ \hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right) - \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{7}{2}\right)\right\}\tilde{u}^6, \tag{4.161d}$$

$$\begin{aligned}\tilde{\omega}_5 &= \frac{16}{45\sqrt{\pi}} \int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v(\tilde{u}) \left\{ \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^4 - \frac{13}{2}\tilde{u}^2 + \frac{15}{2}\right) - \hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}^2\right\}\tilde{u}^6 \\ &\quad - \frac{64}{45\pi}\int_{\tilde{u}=0}^\infty \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^\infty \mathrm{d}\tilde{s} \int_{y=-1}^1 \mathrm{d}y \int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\, \tilde{u}^5 \tilde{s} y \tilde{u}_2\, \hat{\Phi}_v(\tilde{u})\hat{\Phi}_c\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\Big) \\ &\quad \times \hat{\Phi}_c\Big((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\Big)\left\{\tilde{u}y - \frac{1}{2}\tilde{s}(3y^2-1)\right\}\left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right) \\ &\quad \times \left(\tilde{u}_2^2 + \tilde{u}^2y^2 - \frac{5}{2}\right)\mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2y^2)} \\ &\quad + \frac{32}{45\pi}\int_{\tilde{u}_1=0}^\infty \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\, \tilde{u}_1^5 \tilde{u}_2^3 R_1(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2) \\ &\quad \times \left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right),\end{aligned} \tag{4.161e}$$

$$\begin{aligned}\tilde{\omega}_6 &= \frac{128}{105\sqrt{\pi}} \int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v(\tilde{u}) \left\{\hat{\Phi}_v(\tilde{u}) - \hat{\Phi}_v'(\tilde{u})\right\}\tilde{u}^8 \\ &\quad - \frac{128}{105\pi}\int_{\tilde{u}=0}^\infty \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^\infty \mathrm{d}\tilde{s} \int_{y=-1}^1 \mathrm{d}y \int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\, \tilde{u}^4 \tilde{s}\tilde{u}_2(2\tilde{u}^2y^2 - \tilde{u}_2^2) \\ &\quad \times \{\tilde{u}^2(3y^2-1) - \tilde{u}\tilde{s}y(3y^2+1) + \tilde{s}^2(3y^2-1)\}\hat{\Phi}_v(\tilde{u})\hat{\Phi}_v\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\Big) \\ &\quad \times \hat{\Phi}_v\Big((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\Big)\,\mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2y^2)} \\ &\quad + \frac{256}{105\pi}\int_{\tilde{u}_1=0}^\infty \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\, \tilde{u}_1^6 \tilde{u}_2^4 R_2(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v^2(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2).\end{aligned} \tag{4.161f}$$

$\tilde{\omega}_i$ are evaluated numerically. Their values are: $\tilde{\omega}_1 \approx 1.2850$, $\tilde{\omega}_2 \approx 0.6425$, $\tilde{\omega}_3 \approx 0.2551$, $\tilde{\omega}_4 \approx 0.0719$, $\tilde{\omega}_5 \approx 0.0248$ and $\tilde{\omega}_6 \approx 2.3498$.

Collisional Dissipation

Eq. (2.13) implies that the contribution of Φ_{KK} to collisional dissipation Γ is of $O(K^2\epsilon)$. From eq. (2.34), the contribution of Φ_{KK} to collisional dissipation Γ is given by

$$\Gamma_{KK\epsilon} = \frac{\epsilon\Theta}{12\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \int d\tilde{\boldsymbol{u}}_1 d\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, e^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \left\{\Phi_{KK}(\tilde{\boldsymbol{u}}_1) + \Phi_{KK}(\tilde{\boldsymbol{u}}_2) + \Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_K(\tilde{\boldsymbol{u}}_2)\right\}.$$

Again, (similar as above) on interchanging $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$,

$$\int d\tilde{\boldsymbol{u}}_1 d\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_{KK}(\tilde{\boldsymbol{u}}_1) = \int d\tilde{\boldsymbol{u}}_1 d\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_{KK}(\tilde{\boldsymbol{u}}_2).$$

$$\therefore \quad \Gamma_{KK\epsilon} = \frac{\epsilon\Theta}{12\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}(2I_1 + I_2), \tag{4.162}$$

where

$$I_1 = \int d\tilde{\boldsymbol{u}}_1 d\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_{KK}(\tilde{\boldsymbol{u}}_1) \tag{4.163}$$

and

$$I_2 = \int d\tilde{\boldsymbol{u}}_1 d\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, e^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_K(\tilde{\boldsymbol{u}}_2). \tag{4.164}$$

First consider eq. (4.163). Using eq. (3.26), I_1 can be written as

$$I_1 = \int d\tilde{\boldsymbol{u}}_1\, e^{-\tilde{u}_1^2}\, \chi(\tilde{u}_1)\Phi_{KK}(\tilde{\boldsymbol{u}}_1) = \int d\tilde{\boldsymbol{u}}\, e^{-\tilde{u}^2} \chi(\tilde{u})\Phi_{KK}(\tilde{\boldsymbol{u}}). \tag{4.165}$$

Since we do not know the explicit form of Φ_{KK}, we shall use the self adjoint property of $\tilde{\mathscr{L}}$. Using the similar orthogonality relation with respect to Φ_{KK} and following a similar procedure, as below eq. (4.104), we can write

$$I_1 = \int d\tilde{\boldsymbol{u}}\, e^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u})\, \tilde{\mathscr{L}}(\Phi_{KK}). \tag{4.166}$$

Substituting the value of $\tilde{\mathscr{L}}(\Phi_{KK})$ from eq. (4.141) into eq. (4.166), using the orthogonality relations $\left(\int d\tilde{\boldsymbol{u}}\, e^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u}) = \int d\tilde{\boldsymbol{u}}\, \tilde{u}^2\, e^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u}) = 0\right)$, writing $K\frac{2\Theta}{3g} = \ell$ and ignoring the terms whose tensorial structure is: $\tilde{u}_i$ or $\tilde{u}_i\tilde{u}_j\tilde{u}_k$ (because the corresponding integrands are odd functions in components of $\tilde{\boldsymbol{u}}$) or $\overline{\tilde{u}_i\tilde{u}_j}$ (because of symmetry), we get

$$\begin{aligned}
I_1 = \ell^2 \int d\tilde{\boldsymbol{u}}\, e^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u}) \Bigg[& 4\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l \frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_l} - \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\tilde{u}_j \frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_j} \\
& + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{u}_i\tilde{u}_j \frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_j} - 4\hat{\Phi}_v'(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l \frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}} \\
& - 4\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j \frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_j} + \left\{\hat{\Phi}_c(\tilde{u}) + \hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\right\}\tilde{u}_i\tilde{u}_j \frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_j} \\
& + \frac{1}{2}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_i} + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\tilde{u}_j \frac{\partial^2 \ln\Theta}{\partial r_i \partial r_j} \\
& - \left\{\hat{\Phi}_c'(\tilde{u})\tilde{u}^2\left(\tilde{u}^2 - \frac{5}{2}\right) + \hat{\Phi}_c(\tilde{u})\left(\frac{3}{2}\tilde{u}^2 - \frac{5}{4}\right)\right\}\tilde{u}_i\tilde{u}_j \frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_j}\Bigg] \\
& - \frac{1}{2}\int d\tilde{\boldsymbol{u}}\, e^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u})\tilde{\Omega}(\Phi_K, \Phi_K).
\end{aligned} \tag{4.167}$$

In the above simplification, eqs. (F.3)-(F.2), (4.153a) and (4.153b) are used. Now, using eqs. (F.9b), (F.9c), (F.14a) and (F.14b), I_1 can be written as

$$
\begin{aligned}
I_1 = \ell^2 \int_0^\infty & \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\ \bar{\eta}(\tilde{u}) \left[4\hat{\Phi}_v(\tilde{u}) \frac{3}{2\Theta}\frac{8\pi}{15}\tilde{u}^6 \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} - \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{4\pi}{3}\tilde{u}^4 \frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_i} \right.\\
&+\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left(\tilde{u}^2-\frac{3}{2}\right)\frac{4\pi}{3}\tilde{u}^4\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_i} - 4\hat{\Phi}_v'(\tilde{u})\frac{3}{2\Theta}\frac{8\pi}{15}\tilde{u}^6\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}\\
&-4\hat{\Phi}_v(\tilde{u})\frac{3}{2\Theta}\frac{4\pi}{3}\tilde{u}^4\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_i} + \left\{\hat{\Phi}_c(\tilde{u}) + \hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\right\}\frac{4\pi}{3}\tilde{u}^4\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_i}\\
&+\frac{1}{2}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)4\pi\tilde{u}^2\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_i} + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{4\pi}{3}\tilde{u}^4\frac{\partial^2 \ln\Theta}{\partial r_i\partial r_i}\\
&\left.-\left\{\hat{\Phi}_c'(\tilde{u})\tilde{u}^2\left(\tilde{u}^2-\frac{5}{2}\right) + \hat{\Phi}_c(\tilde{u})\left(\frac{3}{2}\tilde{u}^2-\frac{5}{4}\right)\right\}\frac{4\pi}{3}\tilde{u}^4\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_i}\right]\\
&-\frac{1}{2}\int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\ \bar{\eta}(\tilde{u})\tilde{\Omega}(\Phi_K,\Phi_K).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} &= \overline{\frac{\partial V_i}{\partial r_j}}\left[\frac{1}{2}\left(\frac{\partial V_i}{\partial r_j}+\frac{\partial V_j}{\partial r_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial V_k}{\partial r_k}\right] = \frac{1}{2}\left(\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial V_i}{\partial r_j} + \overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_j}{\partial r_i}\right) - \frac{1}{3}\overline{\frac{\partial V_i}{\partial r_i}}\frac{\partial V_k}{\partial r_k}\\
&= \frac{1}{2}\left(\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_j}{\partial r_i} + \overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_j}{\partial r_i}\right) - 0 = \overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_j}{\partial r_i} = \overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_i}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
I_1 &= \ell^2\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}\left[\frac{16\pi}{3}\int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\ \bar{\eta}(\tilde{u})\left\{\frac{2}{5}\{\hat{\Phi}_v(\tilde{u}) - \hat{\Phi}_v'(\tilde{u})\}\tilde{u}^6 - \hat{\Phi}_v(\tilde{u})\tilde{u}^4\right\}\right]\\
&\quad+\ell^2\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_i}\times\frac{4\pi}{3}\int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\ \bar{\eta}(\tilde{u})\tilde{u}^4\left\{\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^4-\frac{11}{2}\tilde{u}^2+5\right) - \hat{\Phi}_c'(\tilde{u})\tilde{u}^2\left(\tilde{u}^2-\frac{5}{2}\right)\right\}\\
&\quad+\ell^2\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_i}\times 2\pi\int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\ \bar{\eta}(\tilde{u})\\
&\quad\times\left[\frac{2}{3}\left\{\hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right) - \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{7}{2}\right)\right\}\tilde{u}^4 + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^2\right]\\
&\quad+\ell^2\frac{\partial^2 \ln\Theta}{\partial r_i\partial r_i}\times\frac{4\pi}{3}\int_0^\infty \mathrm{d}\tilde{u}\ \mathrm{e}^{-\tilde{u}^2}\ \bar{\eta}(\tilde{u})\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^4\\
&\quad-\frac{1}{2}\int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\ \bar{\eta}(\tilde{u})\tilde{\Omega}(\Phi_K,\Phi_K). \qquad (4.168)
\end{aligned}
$$

We shall simplify the last term in eq. (4.168) separately. Substituting the value of $\tilde{\Omega}(\Phi_K,\Phi_K)$ from eq. (2.38),

$$
\begin{aligned}
&\frac{1}{2}\int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\ \bar{\eta}(\tilde{u})\tilde{\Omega}(\Phi_K,\Phi_K)\\
&= \frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\ \bar{\eta}(\tilde{u}_1)\left\{\Phi_K(\tilde{\boldsymbol{u}}_1')\Phi_K(\tilde{\boldsymbol{u}}_2') - \Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_K(\tilde{\boldsymbol{u}}_2)\right\}.
\end{aligned}
$$

Note that in the above equation, the velocity transformation corresponds to the elastic limit. Following a similar procedure as in the derivation of $Q_{i3}^{K\epsilon}$, one obtains

$$\frac{1}{2}\int \mathrm{d}\tilde{\boldsymbol{u}}\ \mathrm{e}^{-\tilde{u}^2}\ \bar{\eta}(\tilde{u})\tilde{\Omega}(\Phi_K,\Phi_K) = J_1 - J_2, \tag{4.169}$$

where

$$J_1 = \frac{1}{\pi^{5/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_K(\tilde{\boldsymbol{u}}_2)\bar{\eta}(\tilde{u})I_\delta^{(0)}, \tag{4.170}$$

$$J_2 = \frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_K(\tilde{\boldsymbol{u}}_2), \tag{4.171}$$

and $I_\delta^{(0)}$ is given in eq. (4.48). We shall simplify J_1 and J_2 as following. Using eqs. (3.18) and (F.3),

$$\begin{aligned}
J_1 &= \frac{1}{\pi^{5/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u})I_\delta^{(0)} \\
&\quad\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_1)\tilde{u}_{1i}\tilde{u}_{1j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\frac{\partial\ln\Theta}{\partial r_i}\right] \\
&\quad\times\left[2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2k}\tilde{u}_{2l}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2k}\frac{\partial\ln\Theta}{\partial r_k}\right] \\
&= \frac{\ell^2}{\pi^{5/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u})I_\delta^{(0)}\left[4\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\tilde{u}_{2l}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\right. \\
&\quad +2\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial\ln\Theta}{\partial r_k} \\
&\quad +2\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{2k}\tilde{u}_{2l}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_k}{\partial r_l}}\frac{\partial\ln\Theta}{\partial r_i} \\
&\quad \left.+\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{2k}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_k}\right].
\end{aligned}$$

We can ignore the second and third terms in the square brackets above because ultimately after manipulation corresponding integrands become odd functions in components of $\tilde{\boldsymbol{u}}$ (cf. §4.2). Therefore

$$\begin{aligned}
J_1 &= \frac{4\ell^2}{\pi^{5/2}}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u})I_\delta^{(0)}\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\tilde{u}_{2l} \\
&\quad+\frac{\ell^2}{\pi^{5/2}}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u})I_\delta^{(0)}\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{1i}\tilde{u}_{2j}.
\end{aligned}$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}$, we get

$$\begin{aligned}
J_1 &= \frac{4\ell^2}{\pi^{5/2}}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\bar{\eta}(\tilde{u})I_\delta^{(0)}\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\tilde{u}_{2k}\tilde{u}_{2l} \\
&\quad+\frac{\ell^2}{\pi^{5/2}}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\bar{\eta}(\tilde{u})I_\delta^{(0)}\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2) \\
&\quad\times\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i)\tilde{u}_{2j}.
\end{aligned}$$

The integrations over $\tilde{\boldsymbol{u}}_2$ are performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_2 = \tilde{s}\tilde{u}_2\cos\theta_2'$, i.e.,

$$\begin{aligned}
J_1 = {}& \frac{4\ell^2}{\pi^{5/2}}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\overline{\eta}(\tilde{u})\\
&\times\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\tilde{u}_{2k}\tilde{u}_{2l}I_\delta^{(0)}\\
&+\frac{\ell^2}{\pi^{5/2}}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\overline{\eta}(\tilde{u})\\
&\times\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i)\tilde{u}_{2j}I_\delta^{(0)}.
\end{aligned}$$

Note that the components of $\tilde{\boldsymbol{u}}_2$ are the only functions of ϕ_2' (see Appendix H). Therefore the integrations over ϕ_2' give (cf. eqs. (H.6) and (H.18))

$$\begin{aligned}
J_1 = {}& \frac{4\ell^2}{\pi^{5/2}}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\overline{\eta}(\tilde{u})\\
&\times\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u}_2)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\left\{\pi\frac{\tilde{u}_2^2}{\tilde{s}^2}\overline{\frac{\partial V_k}{\partial r_l}}\tilde{s}_k\tilde{s}_l(3\cos^2\theta_2'-1)\right\}I_\delta^{(0)}\\
&+\frac{\ell^2}{\pi^{5/2}}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\overline{\eta}(\tilde{u})\\
&\times\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u}_2)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i)\left(2\pi\frac{\tilde{u}_2}{\tilde{s}}\tilde{s}_j\cos\theta_2'\right)I_\delta^{(0)}
\end{aligned}$$

or

$$\begin{aligned}
J_1 = {}& \frac{4\ell^2}{\pi^{3/2}}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\overline{\eta}(\tilde{u})\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\\
&\times\frac{\tilde{s}_k\tilde{s}_l}{\tilde{s}^2}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^4\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{\int_{\theta_2'=0}^{\pi}\sin\theta_2'(3\cos^2\theta_2'-1)I_\delta^{(0)}\,\mathrm{d}\theta_2'\right\}\mathrm{d}\tilde{u}_2\\
&+\frac{2\ell^2}{\pi^{3/2}}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\overline{\eta}(\tilde{u})\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i)\\
&\times\frac{\tilde{s}_j}{\tilde{s}}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^3\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left\{\int_{\theta_2'=0}^{\pi}\sin\theta_2'\cos\theta_2' I_\delta^{(0)}\,\mathrm{d}\theta_2'\right\}\mathrm{d}\tilde{u}_2.
\end{aligned}$$

Using eq. (C.12),

$$\begin{aligned}
J_1 = {}& \frac{4\ell^2}{\pi^{3/2}}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\overline{\eta}(\tilde{u})\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\\
&\times\frac{\tilde{s}_k\tilde{s}_l}{\tilde{s}^3}\int_{\tilde{u}_2=\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\tilde{u}_2^3\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\left\{3\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right)^2-1\right\}\mathrm{d}\tilde{u}_2\\
&+\frac{2\ell^2}{\pi^{3/2}}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\overline{\eta}(\tilde{u})\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i)\\
&\times\frac{\tilde{s}_j}{\tilde{s}^2}\int_{\tilde{u}_2=\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\tilde{u}_2^2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right)\mathrm{d}\tilde{u}_2.
\end{aligned}$$

Let us replace $\tilde{u}_2^2$ by $\tilde{u}_2^2 + \left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2$. This shift implies that

$$
\begin{aligned}
J_1 = {} & \frac{4\ell^2}{\pi^{3/2}}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\bar{\eta}(\tilde{u})\hat{\Phi}_v(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\\
& \times \frac{\tilde{s}_k\tilde{s}_l}{\tilde{s}^3}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\,\hat{\Phi}_v\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)\left\{2\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2-\tilde{u}_2^2\right\}\mathrm{d}\tilde{u}_2\\
& + \frac{2\ell^2}{\pi^{3/2}}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\bar{\eta}(\tilde{u})\hat{\Phi}_c(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\left((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i)\\
& \times \frac{\tilde{s}_j}{\tilde{s}^2}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2\,\mathrm{e}^{-\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}}\,\hat{\Phi}_c\left(\left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2\right\}^{1/2}\right)\\
& \times \left\{\tilde{u}_2^2+\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2-\frac{5}{2}\right\}\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)\mathrm{d}\tilde{u}_2.
\end{aligned}
$$

The integrations over $\tilde{\boldsymbol{s}}$ are performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} = \tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$
\begin{aligned}
J_1 = {} & \frac{4\ell^2}{\pi^{3/2}}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\\
& \times \bar{\eta}(\tilde{u})\hat{\Phi}_v\left(\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2\right)^{1/2}\right)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\tilde{s}_k\tilde{s}_l\frac{1}{\tilde{s}^3}\\
& \times \int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')}\,\hat{\Phi}_v\left(\left(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta'\right)^{1/2}\right)(2\tilde{u}^2\cos^2\theta'-\tilde{u}_2^2)\mathrm{d}\tilde{u}_2\\
& + \frac{2\ell^2}{\pi^{3/2}}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\\
& \times \bar{\eta}(\tilde{u})\hat{\Phi}_c\left(\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2\right)^{1/2}\right)\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2-\frac{5}{2}\right)(\tilde{u}_i-\tilde{s}_i)\tilde{s}_j\frac{1}{\tilde{s}^2}\\
& \times \int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')}\,\hat{\Phi}_c\left(\left(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta'\right)^{1/2}\right)\left(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta'-\frac{5}{2}\right)\tilde{u}\cos\theta'\,\mathrm{d}\tilde{u}_2.
\end{aligned}
$$

Note that the components of $\tilde{\boldsymbol{s}}$ are the only functions of ϕ'. We shall evaluate the parts of J_1 containing velocity gradients and temperature gradients separately.
First consider the part of J_1 containing velocity gradients. Let

$$
\begin{aligned}
J_{1_v} = {} & \frac{4\ell^2}{\pi^{3/2}}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\\
& \times \bar{\eta}(\tilde{u})\hat{\Phi}_v\left(\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2\right)^{1/2}\right)(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\tilde{s}_k\tilde{s}_l\frac{1}{\tilde{s}^3}\\
& \times \int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')}\,\hat{\Phi}_v\left(\left(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta'\right)^{1/2}\right)(2\tilde{u}^2\cos^2\theta'-\tilde{u}_2^2)\,\mathrm{d}\tilde{u}_2\\
= {} & \frac{4\ell^2}{\pi^{3/2}}\frac{3}{2\Theta}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\theta'=0}^{\pi}\mathrm{d}\theta'\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\frac{\tilde{u}_2}{\tilde{s}}\sin\theta'(2\tilde{u}^2\cos^2\theta'-\tilde{u}_2^2)\bar{\eta}(\tilde{u})\\
& \times \hat{\Phi}_v\left(\left(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2\right)^{1/2}\right)\hat{\Phi}_v\left(\left(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta'\right)^{1/2}\right)\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\\
& \times \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')}\,\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\tilde{s}_k\tilde{s}_l.
\end{aligned}
$$

Using eq. (H.23),

$$
\begin{aligned}
J_{1_v} = {} & \frac{4\ell^2}{\pi^{3/2}} \frac{3}{2\Theta} \int \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \frac{\tilde{u}_2}{\tilde{s}} \sin\theta' (2\tilde{u}^2\cos^2\theta' - \tilde{u}_2^2)\bar{\eta}(\tilde{u}) \\
& \times \hat{\Phi}_v\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2}\right) \hat{\Phi}_v\left((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2}\right) \\
& \times \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)} \, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')} \left[2\pi \frac{\tilde{s}^2}{\tilde{u}^4} \overline{\frac{\partial V_i}{\partial r_j}} \, \overline{\frac{\partial V_k}{\partial r_l}} \tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l \right. \\
& \times \left\{ \tilde{u}^2 \frac{1}{2}(3\cos^2\theta' - 1) - 2\tilde{u}\tilde{s}\frac{1}{2}(5\cos^3\theta' - 3\cos\theta') + \tilde{s}^2\frac{1}{8}(35\cos^4\theta' - 30\cos^2\theta' + 3) \right\} \\
& \left. + \pi \frac{\tilde{s}^3}{\tilde{u}^2} \overline{\frac{\partial V_i}{\partial r_k}} \, \overline{\frac{\partial V_k}{\partial r_j}} \tilde{u}_i\tilde{u}_j (1 - \cos^2\theta')\{\tilde{s}(5\cos^2\theta' - 1) - 4\tilde{u}\cos\theta'\} + \frac{\pi}{2}\tilde{s}^4(1 - \cos^2\theta')^2 \overline{\frac{\partial V_i}{\partial r_j}} \, \overline{\frac{\partial V_j}{\partial r_i}} \right].
\end{aligned}
$$

Using eqs. (F.9b) and (F.14b),

$$
\begin{aligned}
J_{1_v} = {} & \frac{4\ell^2}{\pi^{3/2}} \frac{3}{2\Theta} \int_0^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \frac{\tilde{u}_2}{\tilde{s}} \sin\theta' (2\tilde{u}^2\cos^2\theta' - \tilde{u}_2^2)\bar{\eta}(\tilde{u}) \\
& \times \hat{\Phi}_v\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2}\right) \hat{\Phi}_v\left((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2}\right) \\
& \times \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)} \, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')} \left[2\pi \frac{\tilde{s}^2}{\tilde{u}^4} \frac{8\pi}{15} \overline{\frac{\partial V_i}{\partial r_j}} \, \overline{\frac{\partial V_j}{\partial r_i}} \tilde{u}^6 \right. \\
& \times \left\{ \tilde{u}^2 \frac{1}{2}(3\cos^2\theta' - 1) - 2\tilde{u}\tilde{s}\frac{1}{2}(5\cos^3\theta' - 3\cos\theta') + \tilde{s}^2\frac{1}{8}(35\cos^4\theta' - 30\cos^2\theta' + 3) \right\} \\
& + \pi \frac{\tilde{s}^3}{\tilde{u}^2} \overline{\frac{\partial V_i}{\partial r_k}} \, \overline{\frac{\partial V_k}{\partial r_i}} \frac{4\pi}{3} \tilde{u}^4 (1 - \cos^2\theta') \left\{ \tilde{s}(5\cos^2\theta' - 1) - 4\tilde{u}\cos\theta' \right\} \\
& \left. + 4\pi\tilde{u}^2 \frac{\pi}{2}\tilde{s}^4(1 - \cos^2\theta')^2 \overline{\frac{\partial V_i}{\partial r_j}} \, \overline{\frac{\partial V_j}{\partial r_i}} \right] \\
= {} & \frac{4\ell^2}{\pi^{3/2}} \frac{3}{2\Theta} \int_0^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \frac{\tilde{u}_2}{\tilde{s}} \sin\theta' (2\tilde{u}^2\cos^2\theta' - \tilde{u}_2^2)\bar{\eta}(\tilde{u}) \\
& \times \hat{\Phi}_v\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2}\right) \hat{\Phi}_v\left((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2}\right) \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)} \\
& \times \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')} \overline{\frac{\partial V_i}{\partial r_j}} \, \overline{\frac{\partial V_j}{\partial r_i}} \pi^2\tilde{u}^2\tilde{s}^2 \frac{1}{15} \left[16 \left\{ \tilde{u}^2\frac{1}{2}(3\cos^2\theta' - 1) \right. \right. \\
& \left. - 2\tilde{u}\tilde{s}\frac{1}{2}(5\cos^3\theta' - 3\cos\theta') + \tilde{s}^2\frac{1}{8}(35\cos^4\theta' - 30\cos^2\theta' + 3) \right\} \\
& \left. + 20\tilde{s}(1 - \cos^2\theta') \left\{ \tilde{s}(5\cos^2\theta' - 1) - 4\tilde{u}\cos\theta' \right\} + 30\tilde{s}^2(1 - 2\cos^2\theta' + \cos^4\theta') \right] \\
= {} & \frac{4\ell^2\sqrt{\pi}}{15} \frac{3}{2\Theta} \overline{\frac{\partial V_i}{\partial r_j}} \, \overline{\frac{\partial V_j}{\partial r_i}} \int_0^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{\theta'=0}^{\pi} \mathrm{d}\theta' \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \tilde{u}_2\tilde{s}\tilde{u}^2 \sin\theta' (2\tilde{u}^2\cos^2\theta' - \tilde{u}_2^2) \\
& \times \bar{\eta}(\tilde{u})\hat{\Phi}_v\left((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2}\right) \hat{\Phi}_v\left((\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')^{1/2}\right) \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)} \\
& \times \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta')} \left[\tilde{s}^2 \Big\{ 2(35\cos^4\theta' - 30\cos^2\theta' + 3) + 20(1 - \cos^2\theta')(5\cos^2\theta' - 1) \right. \\
& + 30(1 - 2\cos^2\theta' + \cos^4\theta') \Big\} - 16\tilde{u}\tilde{s} \left\{ (5\cos^3\theta' - 3\cos\theta') + 5\cos\theta'(1 - \cos^2\theta') \right\} \\
& \left. + 16 \left\{ \tilde{u}^2\frac{1}{2}(3\cos^2\theta' - 1) \right\} \right]
\end{aligned}
$$

or

$$J_{1_v} = \frac{4\ell^2\sqrt{\pi}}{15}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}}\int_0^\infty \mathrm{d}\tilde{u}\int_{\tilde{s}=0}^\infty \mathrm{d}\tilde{s}\int_{\theta'=0}^\pi \mathrm{d}\theta'\int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\,\tilde{u}_2\tilde{s}\tilde{u}^2\sin\theta'(2\tilde{u}^2\cos^2\theta' - \tilde{u}_2^2)$$
$$\times\,\bar{\eta}(\tilde{u})\hat{\Phi}_v\Big(\left(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2\right)^{1/2}\Big)\,\hat{\Phi}_v\Big(\left(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta'\right)^{1/2}\Big)\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}$$
$$\times\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')}\left[16\tilde{s}^2 - 16\tilde{u}\tilde{s}(2\cos\theta') + 16\left\{\tilde{u}^2\frac{1}{2}(3\cos^2\theta' - 1)\right\}\right].$$

Let $\cos\theta' = y$, hence

$$J_{1_v} = \frac{64\ell^2\sqrt{\pi}}{15}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}}\int_0^\infty \mathrm{d}\tilde{u}\int_{\tilde{s}=0}^\infty \mathrm{d}\tilde{s}\int_{y=-1}^1 \mathrm{d}y\int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\,\tilde{u}_2\tilde{s}\tilde{u}^2(2\tilde{u}^2y^2 - \tilde{u}_2^2)$$
$$\times\,\bar{\eta}(\tilde{u})\hat{\Phi}_v\Big(\left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2\right)^{1/2}\Big)\,\hat{\Phi}_v\Big(\left(\tilde{u}_2^2 + \tilde{u}^2y^2\right)^{1/2}\Big)\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}$$
$$\times\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\left\{\tilde{s}^2 - 2\tilde{u}\tilde{s}y + \frac{1}{2}\tilde{u}^2(3y^2 - 1)\right\}.$$

or

$$J_{1_v} = \frac{64\ell^2\sqrt{\pi}}{15}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}\int_{\tilde{s}=0}^\infty \mathrm{d}\tilde{s}\,\tilde{s}\int_{y=-1}^1 \mathrm{d}y\int_{\tilde{u}=0}^\infty \mathrm{d}\tilde{u}\,\tilde{u}^2\bar{\eta}(\tilde{u})\int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\,\tilde{u}_2(2\tilde{u}^2y^2 - \tilde{u}_2^2)$$
$$\times\left\{\tilde{s}^2 - 2\tilde{u}\tilde{s}y + \frac{1}{2}\tilde{u}^2(3y^2 - 1)\right\}\hat{\Phi}_v\Big(\left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2\right)^{1/2}\Big)\,\hat{\Phi}_v\Big(\left(\tilde{u}_2^2 + \tilde{u}^2y^2\right)^{1/2}\Big)$$
$$\times\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}.$$

Next consider the part of J_1 containing temperature gradient. Let

$$J_{1_t} = \frac{2\ell^2}{\pi^{3/2}}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^\infty\int_{\theta'=0}^\pi\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}$$
$$\times\,\bar{\eta}(\tilde{u})\hat{\Phi}_c\Big(\left(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2\right)^{1/2}\Big)\left(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2 - \frac{5}{2}\right)(\tilde{u}_i - \tilde{s}_i)\tilde{s}_j\frac{1}{\tilde{s}^2}$$
$$\times\int_{\tilde{u}_2=0}^\infty \tilde{u}_2\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')}\,\hat{\Phi}_c\Big(\left(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta'\right)^{1/2}\Big)\left(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta' - \frac{5}{2}\right)\tilde{u}\cos\theta'\,\mathrm{d}\tilde{u}_2.$$

Note that the components of $\tilde{\boldsymbol{s}}$ are the only functions of ϕ'. Therefore the integration over ϕ' is carried out using eq. (H.17). This implies that

$$J_{1_t} = \frac{2\ell^2}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^\infty\int_{\theta'=0}^\pi \mathrm{d}\theta'\mathrm{d}\tilde{s}\,\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\bar{\eta}(\tilde{u})\hat{\Phi}_c\Big(\left(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2\right)^{1/2}\Big)$$
$$\times\left(\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2 - \frac{5}{2}\right)\left[-\pi\tilde{s}^2\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_i}(1 - \cos^2\theta')\right.$$
$$\left.+2\pi\frac{\tilde{s}}{\tilde{u}^2}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\tilde{u}_i\tilde{u}_j\left\{\tilde{u}\cos\theta' - \frac{1}{2}\tilde{s}(3\cos^2\theta' - 1)\right\}\right]$$
$$\times\int_{\tilde{u}_2=0}^\infty \tilde{u}_2\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2\cos^2\theta')}\,\hat{\Phi}_c\Big(\left(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta'\right)^{1/2}\Big)\left(\tilde{u}_2^2 + \tilde{u}^2\cos^2\theta' - \frac{5}{2}\right)\tilde{u}\cos\theta'\,\mathrm{d}\tilde{u}_2.$$

Let $\cos\theta' = y$,

$$J_{1_t} = \frac{2\ell^2}{\pi^{1/2}} \int \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{y=-1}^{1} \mathrm{d}y \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \; \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)} \,\overline{\eta}(\tilde{u})\hat{\Phi}_c\Big(\big(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2\big)^{1/2}\Big)$$
$$\times \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right) \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\, \hat{\Phi}_c\Big(\big(\tilde{u}_2^2 + \tilde{u}^2y^2\big)^{1/2}\Big) \left(\tilde{u}_2^2 + \tilde{u}^2y^2 - \frac{5}{2}\right) \tilde{u}y$$
$$\times \tilde{s} \left[-\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_i}\tilde{s}(1-y^2) + \frac{2}{\tilde{u}^2}\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_j}\tilde{u}_i\tilde{u}_j \left\{\tilde{u}y - \frac{1}{2}\tilde{s}(3y^2-1)\right\}\right].$$

Using eq. (F.9b),

$$J_{1_t} = \frac{2\ell^2}{\pi^{1/2}} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{y=-1}^{1} \mathrm{d}y \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \; \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)} \,\overline{\eta}(\tilde{u})\hat{\Phi}_c\Big(\big(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2\big)^{1/2}\Big)$$
$$\times \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right) \tilde{u}_2\, \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\, \hat{\Phi}_c\Big(\big(\tilde{u}_2^2 + \tilde{u}^2y^2\big)^{1/2}\Big) \left(\tilde{u}_2^2 + \tilde{u}^2y^2 - \frac{5}{2}\right) \tilde{s}\tilde{u}y$$
$$\times \left[-4\pi\tilde{u}^2\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_i}\tilde{s}(1-y^2) + \frac{2}{\tilde{u}^2}\frac{4\pi}{3}\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_i}\tilde{u}^4 \left\{\tilde{u}y - \frac{1}{2}\tilde{s}(3y^2-1)\right\}\right]$$

or

$$J_{1_t} = \frac{2\ell^2}{\pi^{1/2}}\frac{8\pi}{3}\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_i} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\,\tilde{u}^3\overline{\eta}(\tilde{u}) \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s}\,\tilde{s} \int_{y=-1}^{1} \mathrm{d}y\,y \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\,\tilde{u}_2$$
$$\times \hat{\Phi}_c\Big(\big(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2\big)^{1/2}\Big)\, \hat{\Phi}_c\Big(\big(\tilde{u}_2^2 + \tilde{u}^2y^2\big)^{1/2}\Big) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right)$$
$$\times \left(\tilde{u}_2^2 + \tilde{u}^2y^2 - \frac{5}{2}\right) \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)} \left[-\frac{3}{2}\tilde{s}(1-y^2) + \tilde{u}y - \tilde{s}\frac{1}{2}(3y^2-1)\right]$$

or

$$J_{1_t} = \frac{16\ell^2\sqrt{\pi}}{3}\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_i} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s}\,\tilde{s} \int_{y=-1}^{1} \mathrm{d}y\,y \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\,\tilde{u}^3\overline{\eta}(\tilde{u}) \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\,\tilde{u}_2(\tilde{u}y - \tilde{s})$$
$$\times \hat{\Phi}_c\Big(\big(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2\big)^{1/2}\Big)\, \hat{\Phi}_c\Big(\big(\tilde{u}_2^2 + \tilde{u}^2y^2\big)^{1/2}\Big) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right)$$
$$\times \left(\tilde{u}_2^2 + \tilde{u}^2y^2 - \frac{5}{2}\right) \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}.$$

Hence

$$J_1 = \frac{64\ell^2\sqrt{\pi}}{15}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s}\,\tilde{s} \int_{y=-1}^{1} \mathrm{d}y \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\,\tilde{u}^2\overline{\eta}(\tilde{u}) \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\,\tilde{u}_2(2\tilde{u}^2y^2 - \tilde{u}_2^2)$$
$$\times \left\{\tilde{s}^2 - 2\tilde{u}\tilde{s}y + \frac{1}{2}\tilde{u}^2(3y^2-1)\right\} \hat{\Phi}_v\Big(\big(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2\big)^{1/2}\Big)\, \hat{\Phi}_v\Big(\big(\tilde{u}_2^2 + \tilde{u}^2y^2\big)^{1/2}\Big)$$
$$\times \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}$$
$$+ \frac{16\ell^2\sqrt{\pi}}{3}\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln\Theta}{\partial r_i} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s}\,\tilde{s} \int_{y=-1}^{1} \mathrm{d}y\,y \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\,\tilde{u}^3\overline{\eta}(\tilde{u}) \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\,\tilde{u}_2(\tilde{u}y - \tilde{s})$$
$$\times \hat{\Phi}_c\Big(\big(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2\big)^{1/2}\Big)\, \hat{\Phi}_c\Big(\big(\tilde{u}_2^2 + \tilde{u}^2y^2\big)^{1/2}\Big) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2}\right)$$
$$\times \left(\tilde{u}_2^2 + \tilde{u}^2y^2 - \frac{5}{2}\right) \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}. \tag{4.172}$$

Next, consider eq. (4.171). The integration over $\hat{\boldsymbol{k}}$ is trivial. Hence using eq. (G.1b),

$$J_2 = \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \tilde{u}_{12} \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1) \varPhi_K(\tilde{\boldsymbol{u}}_1) \varPhi_K(\tilde{\boldsymbol{u}}_2).$$

Substituting the value of $\varPhi_K$ from eq. (3.18) and using eq. (F.3),

$$\begin{aligned}
J_2 = & \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \tilde{u}_{12} \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1) \\
& \times \left[2K \frac{2\Theta}{3g} \hat{\varPhi}_v(\tilde{u}_1) \tilde{u}_{1i} \tilde{u}_{1j} \left(\frac{3}{2\Theta} \right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} + K \frac{2\Theta}{3g} \hat{\varPhi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2} \right) \tilde{u}_{1i} \frac{\partial \ln \Theta}{\partial r_i} \right] \\
& \times \left[2K \frac{2\Theta}{3g} \hat{\varPhi}_v(\tilde{u}_2) \tilde{u}_{2k} \tilde{u}_{2l} \left(\frac{3}{2\Theta} \right)^{\frac{1}{2}} \overline{\frac{\partial V_k}{\partial r_l}} + K \frac{2\Theta}{3g} \hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2} \right) \tilde{u}_{2k} \frac{\partial \ln \Theta}{\partial r_k} \right] \\
= & \frac{\ell^2}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \tilde{u}_{12} \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1) \\
& \times \left[4\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2) \tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\tilde{u}_{2l} \frac{3}{2\Theta} \overline{\frac{\partial V_i}{\partial r_j}} \, \overline{\frac{\partial V_k}{\partial r_l}} \right. \\
& + 2\hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_2^2 - \frac{5}{2} \right) \tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k} \left(\frac{3}{2\Theta} \right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} \frac{\partial \ln \Theta}{\partial r_k} \\
& + 2\hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2} \right) \tilde{u}_{1i}\tilde{u}_{2k}\tilde{u}_{2l} \left(\frac{3}{2\Theta} \right)^{\frac{1}{2}} \overline{\frac{\partial V_k}{\partial r_l}} \frac{\partial \ln \Theta}{\partial r_i} \\
& \left. + \hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2} \right) \left(\tilde{u}_2^2 - \frac{5}{2} \right) \tilde{u}_{1i}\tilde{u}_{2k} \frac{\partial \ln \Theta}{\partial r_i} \frac{\partial \ln \Theta}{\partial r_k} \right].
\end{aligned}$$

We can ignore the second and third terms in the square brackets above because ultimately after manipulation corresponding integrands become odd functions in components of $\tilde{\boldsymbol{u}}_1$ (cf. §4.2). Therefore

$$\begin{aligned}
J_2 = & \frac{4\ell^2}{\pi^{3/2}} \frac{3}{2\Theta} \overline{\frac{\partial V_i}{\partial r_j}} \, \overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \tilde{u}_{12} \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1) \hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2) \tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\tilde{u}_{2l} \\
& + \frac{\ell^2}{\pi^{3/2}} \frac{\partial \ln \Theta}{\partial r_i} \frac{\partial \ln \Theta}{\partial r_j} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \tilde{u}_{12} \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1) \hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2} \right) \left(\tilde{u}_2^2 - \frac{5}{2} \right) \tilde{u}_{1i}\tilde{u}_{2j}.
\end{aligned}$$

The integrations over $\tilde{\boldsymbol{u}}_2$ are performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1 \cdot \tilde{\boldsymbol{u}}_2 = \tilde{u}_1 \tilde{u}_2 \cos\theta_2'$, i.e.,

$$\begin{aligned}
J_2 = & \frac{4\ell^2}{\pi^{3/2}} \frac{3}{2\Theta} \overline{\frac{\partial V_i}{\partial r_j}} \, \overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2' \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^2 \sin\theta_2' \\
& \times (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1) \hat{\varPhi}_v(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_2) \tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{2k}\tilde{u}_{2l} \\
& + \frac{\ell^2}{\pi^{3/2}} \frac{\partial \ln \Theta}{\partial r_i} \frac{\partial \ln \Theta}{\partial r_j} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2' \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^2 \sin\theta_2' \\
& \times (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1) \hat{\varPhi}_c(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2} \right) \left(\tilde{u}_2^2 - \frac{5}{2} \right) \tilde{u}_{1i}\tilde{u}_{2j}.
\end{aligned}$$

The integrals over ϕ_2' result into (cf. eqs. (H.6) and (H.18)),

$$
\begin{aligned}
J_2 &= \frac{4\ell^2}{\pi^{3/2}} \frac{3}{2\Theta} \overline{\frac{\partial V_i}{\partial r_j}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \\
&\quad \times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \bar{\eta}(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1i}\tilde{u}_{1j} \left\{ \pi \frac{\tilde{u}_2^2}{\tilde{u}_1^2} \overline{\frac{\partial V_k}{\partial r_l}} \tilde{u}_{1k}\tilde{u}_{1l}(3\cos^2\theta_2' - 1) \right\} \\
&\quad + \frac{\ell^2}{\pi^{3/2}} \frac{\partial \ln\Theta}{\partial r_i} \frac{\partial \ln\Theta}{\partial r_j} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \\
&\quad \times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \bar{\eta}(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_{1i} \left(2\pi \frac{\tilde{u}_2}{\tilde{u}_1} \tilde{u}_{1j} \cos\theta_2' \right) \\
&= \frac{8\ell^2}{\pi^{1/2}} \frac{3}{2\Theta} \overline{\frac{\partial V_i}{\partial r_j}}\, \overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \frac{\tilde{u}_2^4}{\tilde{u}_1^2} \bar{\eta}(\tilde{u}_1)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{1k}\tilde{u}_{1l} \\
&\quad \times \left\{ \int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2' \sin\theta_2' \frac{1}{2}(3\cos^2\theta_2' - 1)(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \right\} \\
&\quad + \frac{2\ell^2}{\pi^{1/2}} \frac{\partial \ln\Theta}{\partial r_i} \frac{\partial \ln\Theta}{\partial r_j} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \frac{\tilde{u}_2^3}{\tilde{u}_1} \bar{\eta}(\tilde{u}_1)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right) \\
&\quad \times \tilde{u}_{1i}\tilde{u}_{1j} \left\{ \int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2' \sin\theta_2' \cos\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \right\} \\
&= \frac{8\ell^2}{\pi^{1/2}} \frac{3}{2\Theta} \overline{\frac{\partial V_i}{\partial r_j}}\, \overline{\frac{\partial V_k}{\partial r_l}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \frac{\tilde{u}_2^4}{\tilde{u}_1^2} \bar{\eta}(\tilde{u}_1)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{1i}\tilde{u}_{1j}\tilde{u}_{1k}\tilde{u}_{1l} R_2(\tilde{u}_1,\tilde{u}_2) \\
&\quad + \frac{2\ell^2}{\pi^{1/2}} \frac{\partial \ln\Theta}{\partial r_i} \frac{\partial \ln\Theta}{\partial r_j} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \frac{\tilde{u}_2^3}{\tilde{u}_1} \bar{\eta}(\tilde{u}_1)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right) \\
&\quad \times \tilde{u}_{1i}\tilde{u}_{1j} R_1(\tilde{u}_1,\tilde{u}_2),
\end{aligned}
$$

where $R_n(\tilde{u}_1,\tilde{u}_2)$ is defined in eq. (4.57) and the values of $R_1(\tilde{u}_1,\tilde{u}_2)$ and $R_2(\tilde{u}_1,\tilde{u}_2)$ are given in eqs. (4.64) and (4.99), respectively. The integrations over $\tilde{\boldsymbol{u}}_1$ are carried out by using eqs. (F.14b) and (F.9b), respectively. Hence J_2 changes to

$$
\begin{aligned}
J_2 &= \frac{8\ell^2}{\pi^{1/2}} \frac{3}{2\Theta} \frac{8\pi}{15} \overline{\frac{\partial V_i}{\partial r_j}}\, \overline{\frac{\partial V_i}{\partial r_j}} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \frac{\tilde{u}_2^4}{\tilde{u}_1^2} \bar{\eta}(\tilde{u}_1)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_1^6 R_2(\tilde{u}_1,\tilde{u}_2) \\
&\quad + \frac{2\ell^2}{\pi^{1/2}} \frac{\partial \ln\Theta}{\partial r_i} \frac{\partial \ln\Theta}{\partial r_i} \frac{4\pi}{3} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \frac{\tilde{u}_2^3}{\tilde{u}_1} \bar{\eta}(\tilde{u}_1)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2) \\
&\quad \times \left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right) \tilde{u}_1^4 R_1(\tilde{u}_1,\tilde{u}_2)
\end{aligned}
$$

or

$$
\begin{aligned}
J_2 &= \frac{64\ell^2\sqrt{\pi}}{15} \frac{3}{2\Theta} \overline{\frac{\partial V_i}{\partial r_j}}\, \overline{\frac{\partial V_i}{\partial r_j}} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_1^4\tilde{u}_2^4 R_2(\tilde{u}_1,\tilde{u}_2)\bar{\eta}(\tilde{u}_1)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2) \\
&\quad + \frac{8\ell^2\sqrt{\pi}}{3} \frac{\partial \ln\Theta}{\partial r_i} \frac{\partial \ln\Theta}{\partial r_i} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_1^3\tilde{u}_2^3 R_1(\tilde{u}_1,\tilde{u}_2)\bar{\eta}(\tilde{u}_1)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\
&\quad \times \hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2}\right)\left(\tilde{u}_2^2 - \frac{5}{2}\right).
\end{aligned}
\tag{4.173}
$$

From eqs. (4.168), (4.169), (4.172) and (4.173),

$$I_1 = \ell^2 \left[\tilde{\alpha}_1 \frac{3}{2\Theta} \overline{\frac{\partial V_i}{\partial r_j}}\, \overline{\frac{\partial V_i}{\partial r_j}} + \tilde{\alpha}_2 \frac{\partial \ln \Theta}{\partial r_i} \frac{\partial \ln \Theta}{\partial r_i} + \tilde{\alpha}_3 \frac{\partial \ln \Theta}{\partial r_i} \frac{\partial \ln (n\Theta)}{\partial r_i} + \tilde{\alpha}_4 \frac{\partial^2 \ln \Theta}{\partial r_i \partial r_i} \right], \tag{4.174}$$

where

$$\begin{aligned}
\tilde{\alpha}_1 = {} & \frac{16\pi}{3} \int_0^\infty \mathrm{d}\tilde{u}\; \mathrm{e}^{-\tilde{u}^2}\, \overline{\eta}(\tilde{u}) \left\{ \frac{2}{5} \{ \hat{\Phi}_v(\tilde{u}) - \hat{\Phi}_v'(\tilde{u}) \} \tilde{u}^6 - \hat{\Phi}_v(\tilde{u}) \tilde{u}^4 \right\} \\
& - \frac{64\sqrt{\pi}}{15} \int_{\tilde{s}=0}^\infty \mathrm{d}\tilde{s}\; \tilde{s} \int_{y=-1}^1 \mathrm{d}y \int_{\tilde{u}=0}^\infty \mathrm{d}\tilde{u}\; \tilde{u}^2 \overline{\eta}(\tilde{u}) \int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\; \tilde{u}_2 (2\tilde{u}^2 y^2 - \tilde{u}_2^2) \\
& \times \left\{ \tilde{s}^2 - 2\tilde{u}\tilde{s}y + \frac{1}{2} \tilde{u}^2 (3y^2 - 1) \right\} \hat{\Phi}_v \left((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2} \right) \hat{\Phi}_v \left((\tilde{u}_2^2 + \tilde{u}^2 y^2)^{1/2} \right) \\
& \times \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2 y^2)} \\
& + \frac{64\sqrt{\pi}}{15} \int_{\tilde{u}_1=0}^\infty \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\; \tilde{u}_1^4 \tilde{u}_2^4 R_2(\tilde{u}_1, \tilde{u}_2) \overline{\eta}(\tilde{u}_1)\, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1) \hat{\Phi}_v(\tilde{u}_2),
\end{aligned} \tag{4.175a}$$

$$\begin{aligned}
\tilde{\alpha}_2 = {} & \frac{4\pi}{3} \int_0^\infty \mathrm{d}\tilde{u}\; \mathrm{e}^{-\tilde{u}^2}\, \overline{\eta}(\tilde{u}) \tilde{u}^4 \left\{ \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^4 - \frac{11}{2} \tilde{u}^2 + 5 \right) - \hat{\Phi}_c'(\tilde{u}) \tilde{u}^2 \left(\tilde{u}^2 - \frac{5}{2} \right) \right\} \\
& - \frac{16\sqrt{\pi}}{3} \int_{\tilde{s}=0}^\infty \mathrm{d}\tilde{s}\; \tilde{s} \int_{y=-1}^1 \mathrm{d}y\, y \int_{\tilde{u}=0}^\infty \mathrm{d}\tilde{u}\; \tilde{u}^3 \overline{\eta}(\tilde{u}) \int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\; \tilde{u}_2 (\tilde{u}y - \tilde{s}) \\
& \times \hat{\Phi}_c \left((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2} \right) \hat{\Phi}_c \left((\tilde{u}_2^2 + \tilde{u}^2 y^2)^{1/2} \right) \left(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2 - \frac{5}{2} \right) \\
& \times \left(\tilde{u}_2^2 + \tilde{u}^2 y^2 - \frac{5}{2} \right) \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2 y^2)} \\
& + \frac{8\sqrt{\pi}}{3} \int_{\tilde{u}_1=0}^\infty \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\; \tilde{u}_1^3 \tilde{u}_2^3 R_1(\tilde{u}_1, \tilde{u}_2) \overline{\eta}(\tilde{u}_1)\, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)} \\
& \times \hat{\Phi}_c(\tilde{u}_1) \hat{\Phi}_c(\tilde{u}_2) \left(\tilde{u}_1^2 - \frac{5}{2} \right) \left(\tilde{u}_2^2 - \frac{5}{2} \right),
\end{aligned} \tag{4.175b}$$

$$\tilde{\alpha}_3 = 2\pi \int_0^\infty \mathrm{d}\tilde{u}\; \mathrm{e}^{-\tilde{u}^2}\, \overline{\eta}(\tilde{u}) \left[\frac{2}{3} \left\{ \hat{\Phi}_c'(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2} \right) - \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{7}{2} \right) \right\} \tilde{u}^4 + \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2} \right) \tilde{u}^2 \right], \tag{4.175c}$$

and

$$\tilde{\alpha}_4 = \frac{4\pi}{3} \int_0^\infty \mathrm{d}\tilde{u}\; \mathrm{e}^{-\tilde{u}^2}\, \overline{\eta}(\tilde{u}) \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2} \right) \tilde{u}^4. \tag{4.175d}$$

$\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $\tilde{\alpha}_3$ and $\tilde{\alpha}_4$ are evaluated numerically. Their values are: $\tilde{\alpha}_1 \approx 12.6475$, $\tilde{\alpha}_2 \approx 73.1660$, $\tilde{\alpha}_3 \approx -19.0093$ and $\tilde{\alpha}_4 \approx 15.7585$.

Next, we consider eq. (4.164). The integrations in the right-hand side of eq. (4.164) are carried out by following exactly same process as in the simplification of J_2. The simplified value of I_2 is (cf. eq. (4.173))

$$
\begin{aligned}
I_2 &= \frac{64\pi^2\ell^2}{15}\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_1^4\tilde{u}_2^4\,S_2(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)\\
&+\frac{8\pi^2\ell^2}{3}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_i}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_1^3\tilde{u}_2^3\,S_1(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\\
&\times\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right),
\end{aligned}
\tag{4.176}
$$

where

$$
\begin{aligned}
S_n(\tilde{u}_1,\tilde{u}_2) &\equiv \int_0^{\pi}\mathrm{d}\theta_2'\,\sin\theta_2'\,P_n(\cos\theta_2')\,(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2\cos\theta_2'+\tilde{u}_2^2)^{3/2}\\
&= \int_{-1}^{1}\mathrm{d}y\,P_n(y)\,(\tilde{u}_1^2-2\tilde{u}_1\tilde{u}_2 y+\tilde{u}_2^2)^{3/2}.
\end{aligned}
\tag{4.177}
$$

Eq. (4.176) can be written as

$$
I_2 = \ell^2\left(\tilde{\beta}_1\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}+\tilde{\beta}_2\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_i}\right),
\tag{4.178}
$$

where

$$
\tilde{\beta}_1 = \frac{64\pi^2}{15}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_1^4\tilde{u}_2^4\,S_2(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_2)
\tag{4.179a}
$$

and

$$
\tilde{\beta}_2 = \frac{8\pi^2}{3}\int_{\tilde{u}_1=0}^{\infty}\mathrm{d}\tilde{u}_1\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_1^3\tilde{u}_2^3\,S_1(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_1^2-\frac{5}{2}\right)\left(\tilde{u}_2^2-\frac{5}{2}\right).
\tag{4.179b}
$$

$\tilde{\beta}_1$ and $\tilde{\beta}_2$ are evaluated numerically. Their values are: $\tilde{\beta}_1 \approx 12.9619$ and $\tilde{\beta}_2 \approx 4.8612$.

Substituting the values of I_1 and I_2 from eqs. (4.174) and (4.178) respectively, into eq. (4.162),

$$
\begin{aligned}
\Gamma_{KK\epsilon} &= \frac{\epsilon\ell\Theta}{12\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left[(2\tilde{\alpha}_1+\tilde{\beta}_1)\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}+(2\tilde{\alpha}_2+\tilde{\beta}_2)\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_i}\right.\\
&\left.+2\tilde{\alpha}_3\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln(n\Theta)}{\partial r_i}+2\tilde{\alpha}_4\frac{\partial^2\ln\Theta}{\partial r_i\partial r_i}\right]\\
&= \frac{\epsilon\ell\Theta^{3/2}}{12\pi^3}\left(\frac{2}{3}\right)^{\frac{1}{2}}\left[(2\tilde{\alpha}_1+\tilde{\beta}_1)\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}+(2\tilde{\alpha}_2+\tilde{\beta}_2)\frac{1}{\Theta^2}\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_i}\right.\\
&\left.+2\tilde{\alpha}_3\frac{1}{n\Theta^2}\frac{\partial\Theta}{\partial r_i}\frac{\partial(n\Theta)}{\partial r_i}+2\tilde{\alpha}_4\left(\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial r_i\partial r_i}-\frac{1}{\Theta^2}\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_i}\right)\right]
\end{aligned}
$$

or

$$
\boxed{\Gamma_{KK\epsilon} = \tilde{\rho}_1\epsilon\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}+\tilde{\rho}_2\frac{\epsilon\ell}{\Theta^{1/2}}\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_i}+\tilde{\rho}_3\frac{\epsilon\ell}{n\Theta^{1/2}}\frac{\partial\Theta}{\partial r_i}\frac{\partial(n\Theta)}{\partial r_i}+\tilde{\rho}_4\epsilon\ell\Theta^{1/2}\frac{\partial^2\Theta}{\partial r_i\partial r_i}}
\tag{4.180}
$$

where

$$\tilde{\rho}_1 = \frac{1}{12\pi^3}\left(\frac{3}{2}\right)^{\frac{1}{2}}(2\tilde{\alpha}_1 + \tilde{\beta}_1) \approx 0.1259,$$

$$\tilde{\rho}_2 = \frac{1}{12\pi^3}\left(\frac{2}{3}\right)^{\frac{1}{2}}(2\tilde{\alpha}_2 + \tilde{\beta}_2 - 2\tilde{\alpha}_4) \approx 0.2626,$$

$$\tilde{\rho}_3 = \frac{1}{6\pi^3}\left(\frac{2}{3}\right)^{\frac{1}{2}}\tilde{\alpha}_3 \approx -0.0834,$$

$$\tilde{\rho}_4 = \frac{1}{6\pi^3}\left(\frac{2}{3}\right)^{\frac{1}{2}}\tilde{\alpha}_4 \approx 0.0692.$$

4.4 Constitutive Relations at $O(\epsilon\epsilon)$

In the following, we shall first simplify the expanded Boltzmann equation at this order.

4.4.1 Simplified form of eq. (2.36)

Collecting $O(\epsilon\epsilon)$ terms in eq. (2.36), we have

$$\begin{aligned}
&\tilde{\mathscr{D}}_{\epsilon\epsilon}\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_{\epsilon\epsilon}V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathscr{D}}_{\epsilon\epsilon}\ln\Theta \\
&+ \Phi_\epsilon\left\{\tilde{\mathscr{D}}_\epsilon\ln n + 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{\mathscr{D}}_\epsilon V_i + \left(\tilde{u}^2 - \frac{3}{2}\right)\tilde{\mathscr{D}}_\epsilon\ln\Theta\right\} + \tilde{\mathscr{D}}_\epsilon\Phi_\epsilon \\
&= \epsilon^2\tilde{\mathscr{L}}(\varphi_1^{(2)}) + \epsilon^2\tilde{\Xi}(\varphi_1^{(1)}) + \epsilon^2\tilde{\Lambda}(\varphi_1^{(1)}) + \frac{1}{2}\epsilon^2\tilde{\Omega}(\varphi_1^{(1)},\varphi_1^{(1)}) \\
&\quad + \frac{\epsilon^2}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right).
\end{aligned} \tag{4.181}$$

Clearly (cf. eqs. (2.20)-(2.22)), $\tilde{\mathscr{D}}_{\epsilon\epsilon}\ln n = \tilde{\mathscr{D}}_{\epsilon\epsilon}V_i = 0$ and $\tilde{\mathscr{D}}_{\epsilon\epsilon}\ln\Theta = -\epsilon\tilde{\Gamma}_\epsilon$; but from eq. (2.23),

$$\epsilon\tilde{\Gamma}_\epsilon = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{\Theta}\Gamma_{\epsilon\epsilon}$$

and using eq. (3.39),

$$\epsilon\tilde{\Gamma}_\epsilon \approx \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{\Theta}\left\{-0.0102\,\frac{\epsilon^2}{\ell}\Theta^{3/2}\right\} = -0.0102\left(\frac{3}{2}\right)^{\frac{1}{2}}\epsilon^2\left(K\frac{2\Theta}{3g}\times\frac{1}{\ell}\right) = -a\epsilon^2,$$

where $a = 0.0102\sqrt{(3/2)} \approx 0.0125$. Therefore $\tilde{\mathscr{D}}_{\epsilon\epsilon}\ln\Theta = a\epsilon^2$. The quantity in curly brackets in eq. (4.181), has appeared in the right-hand side of eq. (3.28) and its value is $(\tilde{u}^2-\frac{3}{2})\left\{-\epsilon\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2}\right\}$ (see the equation following eq. (3.28)). From eq. (4.9), the quantity $\tilde{\mathscr{D}}_\epsilon\Phi_\epsilon$ in eq. (4.181) is given by

$$\tilde{\mathscr{D}}_\epsilon\Phi_\epsilon = \epsilon\hat{\Phi}'_e(\tilde{u})\tilde{\mathscr{D}}_\epsilon(\tilde{u}^2),$$

where prime denotes the differentiation with respect to $\tilde{u}^2$. Using eq. (4.14),

$$\tilde{\mathscr{D}}_\epsilon \varPhi_\epsilon = \epsilon\,\hat{\varPhi}_e'(\tilde{u}) \left\{ \epsilon \tilde{u}^2 \frac{2}{3} \left(\frac{2}{\pi}\right)^{1/2} \right\} = \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2} \epsilon^2 \tilde{u}^2 \hat{\varPhi}_e'(\tilde{u}).$$

Substituting these values in eq. (4.181), we get

$$\begin{aligned}
&\left(\tilde{u}^2 - \frac{3}{2}\right) \times a\epsilon^2 + \epsilon\,\hat{\varPhi}_e(\tilde{u}) \times \left(\tilde{u}^2 - \frac{3}{2}\right) \left\{ -\epsilon \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2} \right\} + \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2} \epsilon^2 \tilde{u}^2 \hat{\varPhi}_e'(\tilde{u}) \\
&= \epsilon^2 \tilde{\mathscr{L}}(\varphi_1^{(2)}) + \epsilon^2 \tilde{\varXi}(\varphi_1^{(1)}) + \epsilon^2 \tilde{\varLambda}(\varphi_1^{(1)}) + \frac{1}{2}\epsilon^2 \tilde{\varOmega}(\varphi_1^{(1)}, \varphi_1^{(1)}) \\
&\quad + \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right)
\end{aligned}$$

or

$$\begin{aligned}
\tilde{\mathscr{L}}(\varphi_1^{(2)}) &= a\left(\tilde{u}^2 - \frac{3}{2}\right) + \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2} \left[\tilde{u}^2 \hat{\varPhi}_e'(\tilde{u}) - \left(\tilde{u}^2 - \frac{3}{2}\right) \hat{\varPhi}_e(\tilde{u})\right] \\
&\quad - \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right) \\
&\quad - \tilde{\varXi}(\varphi_1^{(1)}) - \tilde{\varLambda}(\varphi_1^{(1)}) - \frac{1}{2}\tilde{\varOmega}(\varphi_1^{(1)}, \varphi_1^{(1)}).
\end{aligned}$$

The value of integral in the right-hand side of above equation is given in eq. (G.7). Hence using eq. (G.7), above equation can be written as

$$\tilde{\mathscr{L}}(\varphi_1^{(2)}) = \tilde{S}(\tilde{u}) - \tilde{\varXi}(\varphi_1^{(1)}) - \tilde{\varLambda}(\varphi_1^{(1)}) - \frac{1}{2}\tilde{\varOmega}(\varphi_1^{(1)}, \varphi_1^{(1)}), \tag{4.182}$$

where

$$\begin{aligned}
\tilde{S}(\tilde{u}) &= a\left(\tilde{u}^2 - \frac{3}{2}\right) + \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2} \left\{\tilde{u}^2 \hat{\varPhi}_e'(\tilde{u}) - \left(\tilde{u}^2 - \frac{3}{2}\right) \hat{\varPhi}_e(\tilde{u})\right\} \\
&\quad - \left\{ \left(\frac{-15 - 68\tilde{u}^2 + 4\tilde{u}^4}{192\,\pi^{1/2}}\right) \mathrm{e}^{-\tilde{u}^2} + \frac{(63 - 102\tilde{u}^2 - 132\tilde{u}^4 + 8\tilde{u}^6)\,\mathrm{erf}(\tilde{u})}{384\tilde{u}} \right\}
\end{aligned} \tag{4.183}$$

is function of speed $\tilde{u}$ only.

4.4.2 Constitutive Relations

Heat Flux

From eq. (2.30), the contribution of $\varPhi_{\epsilon\epsilon}$ to the heat flux is

$$Q_i^{\epsilon\epsilon} = \frac{n}{2\pi^{3/2}} \left(\frac{2\varTheta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \varPhi_{\epsilon\epsilon}.$$

Using the similar orthogonality relation with respect to $\varPhi_{\epsilon\epsilon}$ and following a similar procedure as in getting the expression for heat flux at $O(K\epsilon)$ (cf. eq. (4.24)), we get

$$Q_i^{\epsilon\epsilon} = \frac{\epsilon^2 n}{2\pi^{3/2}} \left(\frac{2\varTheta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \tilde{\mathscr{L}}(\varphi_1^{(2)}). \tag{4.184}$$

Substituting the value of $\tilde{\mathscr{L}}(\varphi_1^{(2)})$ from eq. (4.182) in the above equation, we get

$$Q_i^{\epsilon\epsilon} = \frac{\epsilon^2 n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2} \left\{\tilde{S}(\tilde{u}) - \tilde{\varXi}(\varphi_1^{(1)}) - \tilde{\varLambda}(\varphi_1^{(1)}) - \frac{1}{2}\tilde{\varOmega}(\varphi_1^{(1)}, \varphi_1^{(1)})\right\}$$
$$\equiv Q_{i_1}^{\epsilon\epsilon} + Q_{i_2}^{\epsilon\epsilon} + Q_{i_3}^{\epsilon\epsilon}, \quad \text{(let)} \tag{4.185}$$

where

$$Q_{i_1}^{\epsilon\epsilon} = \frac{\epsilon^2 n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{S}(\tilde{u}), \tag{4.186}$$

$$Q_{i_2}^{\epsilon\epsilon} = -\frac{\epsilon^2 n}{2\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2} \left\{\tilde{\varXi}(\varphi_1^{(1)}) + \tilde{\varLambda}(\varphi_1^{(1)})\right\}, \tag{4.187}$$

$$Q_{i_3}^{\epsilon\epsilon} = -\frac{\epsilon^2 n}{4\pi^{3/2}}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i\, \mathrm{e}^{-\tilde{u}^2}\, \tilde{\varOmega}(\varphi_1^{(1)}, \varphi_1^{(1)}). \tag{4.188}$$

Now we shall evaluate the quantities $Q_{i_1}^{\epsilon\epsilon}$, $Q_{i_2}^{\epsilon\epsilon}$ and $Q_{i_3}^{\epsilon\epsilon}$ as following.
Clearly, from eqs. (4.183) and (4.186),

$$Q_{i_1}^{\epsilon\epsilon} = 0, \tag{4.189}$$

because the integrand is an odd function in components of $\tilde{\boldsymbol{u}}$. Substituting the values of $\tilde{\varXi}$ and $\tilde{\varLambda}$ from eqs. (2.39) and (2.40) respectively, in eq. (4.187), one obtains

$$\begin{aligned}
Q_{i_2}^{\epsilon\epsilon} = &-\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\
&\times \{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')\}\hat{\varPhi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} \\
&- \frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\
&\times \{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')\}\hat{\varPhi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i}.
\end{aligned}$$

Following a similar procedure as in the derivation of $Q_{i_2}^{K\epsilon}$, we get

$$Q_{i_2}^{\epsilon\epsilon} = -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\}\hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i I_\delta, \tag{4.190}$$

where I_δ is given in eq. (4.34). The integral in eq. (4.190) is then split into two parts. The first part is

$$\begin{aligned}
(I) &= -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i I_\delta \\
&= -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i I_\delta.
\end{aligned} \tag{4.191}$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}$ in the above equation, we get

$$(I) = -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\, \hat{\Phi}_e(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta.$$

Using eq. (4.77),

$$(I) = -\frac{\epsilon^2 n}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\frac{1}{\tilde{s}}\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\, \hat{\Phi}_e(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\, \mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}.$$

The integration over $\tilde{\boldsymbol{s}}$ is performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} = \tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$\begin{aligned}(I) = &-\frac{\epsilon^2 n}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\\ &\times\hat{\Phi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big)\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}.\end{aligned}$$

The integration over ϕ' is just 2π and let $\cos\theta' = y$. Therefore

$$\begin{aligned}(I) = &-\frac{\epsilon^2 n}{\pi^2}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\\ &\times\hat{\Phi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2} = 0\end{aligned}$$

because the integrand above is an odd function in components of $\tilde{\boldsymbol{u}}$. The second part of $Q_{i_2}^{\epsilon\epsilon}$ is

$$\begin{aligned}(II) &= -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta\\ &= -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta.\end{aligned}\tag{4.192}$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}$ in the above equation, we get

$$(II) = -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\, \hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i I_\delta.$$

Using eq. (4.77),

$$\begin{aligned}(II) = &-\frac{\epsilon^2 n}{\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\\ &\times\frac{1}{\tilde{s}}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.\end{aligned}$$

The integration over $\tilde{\boldsymbol{s}}$ is performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} = \tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$(II) = -\frac{\epsilon^2 n}{\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}$$
$$\times\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{1}{\tilde{s}}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.$$

The integration over ϕ' is just 2π and let $\cos\theta' = y$. Therefore

$$(II) = -\frac{2\epsilon^2 n}{\pi^2}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}$$
$$\times\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2. \tag{4.193}$$

This implies that

$$(II) = 0$$

because the integrand above is an odd function in components of $\tilde{\boldsymbol{u}}$. Hence

$$Q_{i_2}^{\epsilon\epsilon} = 0. \tag{4.194}$$

The third contribution to $Q_i^{\epsilon\epsilon}$ is given by eq. (4.188). Substituting the value of $\tilde{\Omega}$ from eq. (2.38) in eq. (4.188), one obtains

$$Q_{i_3}^{\epsilon\epsilon} = -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}$$
$$\times\,\{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')-\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\}. \tag{4.195}$$

Note that eq. (4.195) uses elastic velocity transformation. Following a similar procedure as in the derivation of $Q_{i_3}^{K\epsilon}$, one obtains

$$Q_{i_3}^{\epsilon\epsilon} = -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)I_\delta^{(0)}$$
$$+\,\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}$$
$$\times\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2), \tag{4.196}$$

where $I_\delta^{(0)}$ is given in eq. (4.48). The term $Q_{i_3}^{\epsilon\epsilon}$ is split into two parts. The first part is

$$(I) = -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)I_\delta^{(0)}$$
$$= -\frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\,\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_2)I_\delta^{(0)}. \tag{4.197}$$

Note that except for the extra term $\hat{\Phi}_e(\tilde{u}_1)$ and the definition of $I_\delta^{(0)}$, the integrand in eq. (4.197) is similar to that in eq. (4.192). Hence, by following a similar procedure as below eq. (4.192), we get (cf. eq. (4.193))

$$(I) = -\frac{2\epsilon^2 n}{\pi^2}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{y=-1}^{1} \mathrm{d}y\, \tilde{s}\, \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\, \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \tilde{u}_i \\ \times \hat{\Phi}_e\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\Big) \int_{\tilde{u}_2=|\tilde{u}y|}^{\infty} \tilde{u}_2 \hat{\Phi}_e(\tilde{u}_2)\, \mathrm{e}^{-\tilde{u}_2^2}\, \mathrm{d}\tilde{u}_2 = 0$$

because the integrand above is still an odd function in components of $\tilde{\boldsymbol{u}}$. The second part of eq. (4.196) is

$$(II) = \frac{\epsilon^2 n}{2\pi^4}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2).$$

In the above equation, the integration over $\hat{\boldsymbol{k}}$ is trivial. Hence, using eq. (G.1b),

$$\begin{aligned}(II) &= \frac{\epsilon^2 n}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2) \\ &= \frac{\epsilon^2 n}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} \hat{\Phi}_e(\tilde{u}_1) \hat{\Phi}_e(\tilde{u}_2).\end{aligned}$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1 \cdot \tilde{\boldsymbol{u}}_2 = \tilde{u}_1 \tilde{u}_2 \cos\theta_2'$, i.e.,

$$(II) = \frac{\epsilon^2 n}{2\pi^3}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2' \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \\ \times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_1) \hat{\Phi}_e(\tilde{u}_2) \hat{\Phi}_c(\tilde{u}_1) \left(\tilde{u}_1^2 - \frac{5}{2}\right) \tilde{u}_{1i} = 0$$

because the integrand above is an odd function in components of $\tilde{\boldsymbol{u}}_1$. Hence

$$Q_{i_3}^{\epsilon\epsilon} = 0. \tag{4.198}$$

Therefore, from eqs. (4.185), (4.189), (4.194) and (4.198),

$$\boxed{Q_i^{\epsilon\epsilon} = 0} \tag{4.199}$$

Pressure Tensor

From eq. (2.28), the contribution of $\Phi_{\epsilon\epsilon}$ to the pressure tensor is given by

$$P_{ij}^{\epsilon\epsilon} = \frac{2n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i \tilde{u}_j\, \mathrm{e}^{-\tilde{u}^2}\, \Phi_{\epsilon\epsilon}.$$

Using the similar orthogonality relation with respect to $\Phi_{\epsilon\epsilon}$ and following a similar procedure as in getting the expression for pressure tensor at $O(K\epsilon)$ (cf. eq. (4.67)), we get

$$P_{ij}^{\epsilon\epsilon} = \frac{2\epsilon^2 n\Theta}{3\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \hat{\Phi}_v(\tilde{u}) \overline{\tilde{u}_i \tilde{u}_j}\, \tilde{\mathscr{L}}(\varphi_1^{(2)}). \tag{4.200}$$

Substituting the value of $\tilde{\mathscr{L}}(\varphi_1^{(2)})$ from eq. (4.182) in the above equation, we get

$$\begin{aligned}P_{ij}^{\epsilon\epsilon} &= \frac{2\epsilon^2 n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left\{\tilde{S}(\tilde{u}) - \tilde{\Xi}(\varphi_1^{(1)}) - \tilde{\Lambda}(\varphi_1^{(1)}) - \frac{1}{2}\tilde{\Omega}(\varphi_1^{(1)},\varphi_1^{(1)})\right\}\\ &\equiv P_{ij_1}^{\epsilon\epsilon} + P_{ij_2}^{\epsilon\epsilon} + P_{ij_3}^{\epsilon\epsilon}, \quad \text{(let)}\end{aligned} \tag{4.201}$$

where

$$P_{ij_1}^{\epsilon\epsilon} = \frac{2\epsilon^2 n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\tilde{S}(\tilde{u}), \tag{4.202}$$

$$P_{ij_2}^{\epsilon\epsilon} = -\frac{2\epsilon^2 n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left\{\tilde{\Xi}(\varphi_1^{(1)}) + \tilde{\Lambda}(\varphi_1^{(1)})\right\}, \tag{4.203}$$

$$P_{ij_3}^{\epsilon\epsilon} = -\frac{\epsilon^2 n\Theta}{3\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-\tilde{u}^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\tilde{\Omega}(\varphi_1^{(1)},\varphi_1^{(1)}). \tag{4.204}$$

Now we shall evaluate the quantities $P_{ij_1}^{\epsilon\epsilon}$, $P_{ij_2}^{\epsilon\epsilon}$ and $P_{ij_3}^{\epsilon\epsilon}$ as following.
Clearly, from eqs. (4.202) and (F.10),

$$P_{ij_1}^{\epsilon\epsilon} = 0. \tag{4.205}$$

Substituting the values of $\tilde{\Xi}$ and $\tilde{\Lambda}$ from eqs. (2.39) and (2.40) respectively, in eq. (4.203), one obtains

$$\begin{aligned}P_{ij_2}^{\epsilon\epsilon} = &-\frac{2\epsilon^2 n\Theta}{3\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(1-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right)\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\\ &\times\{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')\}\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\\ &-\frac{2\epsilon^2 n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')\}\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}.\end{aligned}$$

Following a similar procedure as in the simplification of $Q_{i_2}^{K\epsilon}$, we get

$$P_{ij_2}^{\epsilon\epsilon} = -\frac{2\epsilon^2 n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta, \tag{4.206}$$

where I_δ is given in eq. (4.34). The integral in eq. (4.206) is then split into two parts. The first part is

$$\begin{aligned}(I) &= -\frac{2\epsilon^2 n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta\\ &= -\frac{2\epsilon^2 n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta.\end{aligned} \tag{4.207}$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}$ in the above equation, we get

$$(I) = -\frac{2\epsilon^2 n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_e(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta.$$

Using eq. (4.77),

$$(I) = -\frac{2\epsilon^2 n\Theta}{3\pi^3}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_e(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}.$$

The integration over $\tilde{\boldsymbol{s}}$ is performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q^{K\epsilon}_{i2}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}=\tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$(I)=-\frac{2\epsilon^2 n\Theta}{3\pi^3}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\frac{1}{\tilde{s}}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}$$
$$\times\hat{\Phi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)^{1/2}\Big)\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}.$$

The integration over ϕ' is just 2π and let $\cos\theta'=y$. Therefore

$$(I)=-\frac{4\epsilon^2 n\Theta}{3\pi^2}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}$$
$$\times\hat{\Phi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}.$$

The integration over $\tilde{\boldsymbol{u}}$ in the above equation vanishes using eq. (F.10). Hence

$$(I)=0.$$

The second part of $P^{\epsilon\epsilon}_{ij_2}$ is

$$\begin{aligned}(II)&=-\frac{2\epsilon^2 n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta\\&=-\frac{2\epsilon^2 n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta.\end{aligned}\tag{4.208}$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}$ in the above equation, we get

$$(II)=-\frac{2\epsilon^2 n\Theta}{3\pi^4}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int\mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_e(\tilde{u}_2)\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,I_\delta.$$

Using eq. (4.77),

$$(II)=-\frac{4\epsilon^2 n\Theta}{3\pi^3}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int\mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\times\frac{1}{\tilde{s}}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.$$

The integration over $\tilde{\boldsymbol{s}}$ is performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q^{K\epsilon}_{i2}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}=\tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$(II)=-\frac{4\epsilon^2 n\Theta}{3\pi^3}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}$$
$$\times\frac{1}{\tilde{s}}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.$$

The integration over ϕ' is just 2π and let $\cos\theta' = y$. Therefore

$$(II) = -\frac{8\epsilon^2 n\Theta}{3\pi^2}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}$$
$$\times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2. \tag{4.209}$$

The integration over $\tilde{\boldsymbol{u}}$ in the above equation vanishes using eq. (F.10). Hence

$$(II) = 0.$$

Therefore

$$P^{\epsilon\epsilon}_{ij_2} = 0. \tag{4.210}$$

The third contribution to $P^{\epsilon\epsilon}_{ij}$ is given by eq. (4.204). Substituting the value of $\tilde{\Omega}$ from eq. (2.38) in eq. (4.204), one obtains

$$P^{\epsilon\epsilon}_{ij_3} = -\frac{2\epsilon^2 n\Theta}{3\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}$$
$$\times\{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') - \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\}. \tag{4.211}$$

Note that eq. (4.211) uses elastic velocity transformation. Following a similar procedure as in the derivation of $Q^{K\epsilon}_{i_3}$, one obtains

$$P^{\epsilon\epsilon}_{ij_3} = -\frac{2\epsilon^2 n\Theta}{3\pi^4}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)I_\delta^{(0)}$$
$$+\frac{2\epsilon^2 n\Theta}{3\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2), \tag{4.212}$$

where $I_\delta^{(0)}$ is given in eq. (4.48). The term $P^{\epsilon\epsilon}_{ij_3}$ is split into two parts. The first part is

$$(I) = -\frac{2\epsilon^2 n\Theta}{3\pi^4}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)I_\delta^{(0)}$$
$$= -\frac{2\epsilon^2 n\Theta}{3\pi^4}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\,\hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_2)I_\delta^{(0)}. \tag{4.213}$$

Note that except for the extra term $\hat{\Phi}_e(\tilde{u}_1)$ and the definition of $I_\delta^{(0)}$, the integrand in eq. (4.213) is similar to that in eq. (4.208). Hence, by following a similar procedure as below eq. (4.208), we get (cf. eq. (4.209))

$$(I) = -\frac{8\epsilon^2 n\Theta}{3\pi^2}\int\mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\hat{\Phi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)$$
$$\times\int_{\tilde{u}_2=|\tilde{u}y|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.$$

Again, the integration over $\tilde{\boldsymbol{u}}$ in the above equation vanishes using eq. (F.10). Hence

$$(I) = 0.$$

The second part of eq. (4.212) is

$$(II) = \frac{2\epsilon^2 n\Theta}{3\pi^4} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\, \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2).$$

In the above equation, the integration over $\hat{\boldsymbol{k}}$ is trivial. Hence, using eq. (G.1b),

$$\begin{aligned}(II) &= \frac{2\epsilon^2 n\Theta}{3\pi^3} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\, \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\\ &= \frac{2\epsilon^2 n\Theta}{3\pi^3} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\, \hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_2).\end{aligned}$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2 = \tilde{u}_1\tilde{u}_2\cos\theta_2'$, i.e.,

$$\begin{aligned}(II) = \frac{2\epsilon^2 n\Theta}{3\pi^3} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2}\\ \times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\, \hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_2).\end{aligned}$$

Again, the integration over $\tilde{\boldsymbol{u}}_1$ in the above equation vanishes using eq. (F.10). Hence

$$(II) = 0.$$

Hence

$$P_{ij_3}^{\epsilon\epsilon} = 0. \tag{4.214}$$

Therefore, from eqs. (4.201), (4.205), (4.210) and (4.214),

$$\boxed{P_{ij}^{\epsilon\epsilon} = 0} \tag{4.215}$$

Collisional Dissipation

Eq. (2.13) implies that the contribution of $\Phi_{\epsilon\epsilon}$ to collisional dissipation Γ is of $O(\epsilon^3)$. Since the collisional dissipation is of $O(\epsilon^3)$, it is ignored in Sela & Goldhirsch (1998), but here we shall evaluate this also. From eq. (2.34), the contribution of $\Phi_{\epsilon\epsilon}$ to collisional dissipation Γ is given by

$$\Gamma_{\epsilon\epsilon\epsilon} = \frac{\epsilon\Theta}{12\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \left\{\Phi_{\epsilon\epsilon}(\tilde{\boldsymbol{u}}_1) + \Phi_{\epsilon\epsilon}(\tilde{\boldsymbol{u}}_2) + \Phi_\epsilon(\tilde{\boldsymbol{u}}_1)\Phi_\epsilon(\tilde{\boldsymbol{u}}_2)\right\}.$$

Again, (similar as above) on interchanging $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_{\epsilon\epsilon}(\tilde{\boldsymbol{u}}_1) = \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_{\epsilon\epsilon}(\tilde{\boldsymbol{u}}_2).$$

$$\therefore \quad \Gamma_{\epsilon\epsilon\epsilon} = \frac{\epsilon\Theta}{12\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} (2I_1 + I_2), \tag{4.216}$$

where

$$I_1 = \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_{\epsilon\epsilon}(\tilde{\boldsymbol{u}}_1) \tag{4.217}$$

and

$$I_2 = \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \Phi_{\epsilon}(\tilde{\boldsymbol{u}}_1)\Phi_{\epsilon}(\tilde{\boldsymbol{u}}_2). \tag{4.218}$$

First consider eq. (4.217). Using eq. (3.26), I_1 can be written as

$$I_1 = \int \mathrm{d}\tilde{\boldsymbol{u}}_1\, \mathrm{e}^{-\tilde{u}_1^2}\, \chi(\tilde{u}_1)\Phi_{\epsilon\epsilon}(\tilde{\boldsymbol{u}}_1) = \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\chi(\tilde{u})\Phi_{\epsilon\epsilon}(\tilde{\boldsymbol{u}}) = \epsilon^2 \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\chi(\tilde{u})\varphi_1^{(2)}(\tilde{\boldsymbol{u}}). \tag{4.219}$$

Since we do not know the explicit form of $\Phi_{\epsilon\epsilon}$, we shall use the self adjoint property of $\tilde{\mathscr{L}}$. Using the similar orthogonality relation with respect to $\Phi_{\epsilon\epsilon}$ and following a similar procedure, as below eq. (4.104), we can write

$$I_1 = \epsilon^2 \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u})\, \tilde{\mathscr{L}}(\varphi_1^{(2)}).$$

Substituting the value of $\tilde{\mathscr{L}}(\varphi_1^{(2)})$ from eq. (4.182),

$$\begin{aligned} I_1 &= \epsilon^2 \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u}) \left\{ \tilde{S}(\tilde{u}) - \tilde{\Xi}(\varphi_1^{(1)}) - \tilde{\Lambda}(\varphi_1^{(1)}) - \frac{1}{2}\tilde{\Omega}(\varphi_1^{(1)}, \varphi_1^{(1)}) \right\} \\ &= I_{1_a} + I_{1_b} + I_{1_c}, \end{aligned} \tag{4.220}$$

where

$$I_{1_a} = \epsilon^2 \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u})\tilde{S}(\tilde{u}), \tag{4.221}$$

$$I_{1_b} = -\epsilon^2 \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u})\{\tilde{\Xi}(\varphi_1^{(1)}) + \tilde{\Lambda}(\varphi_1^{(1)})\}, \tag{4.222}$$

$$I_{1_c} = -\frac{\epsilon^2}{2} \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u})\, \tilde{\Omega}(\varphi_1^{(1)}, \varphi_1^{(1)}). \tag{4.223}$$

Now we shall evaluate the quantities I_{1_a}, I_{1_b} and I_{1_c} as following. Substituting the value of $\tilde{S}(\tilde{u})$ from eq. (4.183) and using the orthogonality relations: $\int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u}) = \int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2\, \mathrm{e}^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u}) = 0$, eq. (4.221) changes to

$$\begin{aligned} I_{1_a} = \epsilon^2 \int \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u}) \Bigg[\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2} \left\{ \tilde{u}^2 \hat{\Phi}_e'(\tilde{u}) - \left(\tilde{u}^2 - \frac{3}{2}\right)\hat{\Phi}_e(\tilde{u}) \right\} \\ - \left\{ \left(\frac{-15 - 68\tilde{u}^2 + 4\tilde{u}^4}{192\,\pi^{1/2}} \right) \mathrm{e}^{-\tilde{u}^2} + \frac{(63 - 102\tilde{u}^2 - 132\tilde{u}^4 + 8\tilde{u}^6)\,\mathrm{erf}(\tilde{u})}{384\tilde{u}} \right\} \Bigg] = \vartheta_1 \epsilon^2, \end{aligned} \tag{4.224}$$

where (after changing the above integral into spherical polar coordinates)

$$\begin{aligned} \vartheta_1 = 4\pi \int_0^\infty \mathrm{d}\tilde{u}\, \mathrm{e}^{-\tilde{u}^2}\, \bar{\eta}(\tilde{u}) \Bigg[\frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2} \left\{ \tilde{u}^2 \hat{\Phi}_e'(\tilde{u}) - \left(\tilde{u}^2 - \frac{3}{2}\right)\hat{\Phi}_e(\tilde{u}) \right\} \\ - \left\{ \left(\frac{-15 - 68\tilde{u}^2 + 4\tilde{u}^4}{192\,\pi^{1/2}} \right) \mathrm{e}^{-\tilde{u}^2} + \frac{(63 - 102\tilde{u}^2 - 132\tilde{u}^4 + 8\tilde{u}^6)\,\mathrm{erf}(\tilde{u})}{384\tilde{u}} \right\} \Bigg]. \end{aligned} \tag{4.225}$$

ϑ_1 is evaluated numerically. Its value is: $\vartheta_1 \approx 10.7915$. The second contribution to $\Gamma_{\epsilon\epsilon\epsilon}$ is given by eq. (4.222). Substituting the values of $\tilde{\Xi}$ and $\tilde{\Lambda}$ from eqs. (2.39) and (2.40) respectively, in eq. (4.222), one obtains

$$I_{1_b} = -\frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')\}\bar{\eta}(\tilde{u}_1)$$
$$- \frac{\epsilon^2}{\pi^{5/2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')\}\bar{\eta}(\tilde{u}_1).$$

Following a similar procedure as in the derivation of $Q_{i_2}^{K\epsilon}$, we get

$$I_{1_b} = -\frac{\epsilon^2}{\pi^{5/2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\}\bar{\eta}(\tilde{u}) I_\delta, \tag{4.226}$$

where I_δ is given in eq. (4.34). The integral in eq. (4.206) is then split into two parts. The first part is

$$\begin{aligned}(I) &= -\frac{\epsilon^2}{\pi^{5/2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\bar{\eta}(\tilde{u}) I_\delta \\ &= -\frac{\epsilon^2}{\pi^{5/2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\varPhi}_e(\tilde{u}_1)\bar{\eta}(\tilde{u}) I_\delta.\end{aligned} \tag{4.227}$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{s}}$ in the above equation, we get

$$(I) = -\frac{\epsilon^2}{\pi^{5/2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon} \int \mathrm{d}\tilde{\boldsymbol{s}} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\tilde{\boldsymbol{u}}\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\, \hat{\varPhi}_e(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\bar{\eta}(\tilde{u}) I_\delta.$$

Using eq. (4.77),

$$(I) = -\frac{\epsilon^2}{\pi^{3/2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2} \int \mathrm{d}\tilde{\boldsymbol{s}} \mathrm{d}\tilde{\boldsymbol{u}}\, \frac{1}{\tilde{s}}\, \mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\, \hat{\varPhi}_e(|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}|)\bar{\eta}(\tilde{u})\, \mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}.$$

The integration over $\tilde{\boldsymbol{s}}$ is performed in a (rotated) spherical coordinate system $(\tilde{s}, \theta', \phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} = \tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$\begin{aligned}(I) = -\frac{\epsilon^2}{\pi^{3/2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2} \int \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi} \mathrm{d}\phi' \mathrm{d}\theta' \mathrm{d}\tilde{s}\, \tilde{s}^2\sin\theta' \frac{1}{\tilde{s}}\, \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)} \\ \times \hat{\varPhi}_e\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \tilde{s}^2)^{1/2}\Big)\, \bar{\eta}(\tilde{u})\, \mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right)^2}.\end{aligned}$$

The integration over ϕ' is just 2π and let $\cos\theta' = y$. Therefore

$$\begin{aligned}(I) = -\frac{2\epsilon^2}{\pi^{1/2}} \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2} \int \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s} \int_{y=-1}^{1} \mathrm{d}y\, \tilde{s}\, \mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)} \\ \times \hat{\varPhi}_e\Big((\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)^{1/2}\Big)\, \bar{\eta}(\tilde{u})\, \mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}.\end{aligned}$$

Changing into spherical coordinate system, the integration over $\tilde{\boldsymbol{u}}$ results into

$$(I) = -8\pi^{1/2}\epsilon^2 \lim_{\epsilon\to 0} \frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\,\bar{\eta}(\tilde{u}) \times \hat{\Phi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right)^2}.$$

Using eq. (4.37),

$$(I) = -4\pi^{1/2}\epsilon^2\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}(1-\tilde{s}\tilde{u}y)\,\bar{\eta}(\tilde{u}) \times \hat{\Phi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\tilde{u}^2y^2} = \vartheta_2\epsilon^2, \tag{4.228}$$

where

$$\vartheta_2 = -4\pi^{1/2}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}(1-\tilde{s}\tilde{u}y)\,\bar{\eta}(\tilde{u}) \times \hat{\Phi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-\tilde{u}^2y^2}. \tag{4.229}$$

ϑ_2 is evaluated numerically. Its value is: $\vartheta_2 \approx -0.9578$. The second part of I_{1_b} is

$$\begin{aligned}(II) &= -\frac{\epsilon^2}{\pi^{5/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\bar{\eta}(\tilde{u})I_\delta \\ &= -\frac{\epsilon^2}{\pi^{5/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_e(\tilde{u}_2)\bar{\eta}(\tilde{u})I_\delta.\end{aligned} \tag{4.230}$$

Replacing $\tilde{\boldsymbol{u}}_1$ by $\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}$ in the above equation, we get

$$(II) = -\frac{\epsilon^2}{\pi^{5/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2-\tilde{u}_2^2}\,\hat{\Phi}_e(\tilde{u}_2)\bar{\eta}(\tilde{u})I_\delta.$$

Using eq. (4.77),

$$(II) = -\frac{2\epsilon^2}{\pi^{3/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{s}}\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}})^2}\,\bar{\eta}(\tilde{u})\times\frac{1}{\tilde{s}}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.$$

The integration over $\tilde{\boldsymbol{s}}$ is performed in a (rotated) spherical coordinate system $(\tilde{s},\theta',\phi')$ described in part (I) of $Q_{i_2}^{K\epsilon}$ such that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} = \tilde{s}\tilde{u}\cos\theta'$, i.e.,

$$(II) = -\frac{2\epsilon^2}{\pi^{3/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\int_{\theta'=0}^{\pi}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\mathrm{d}\theta'\mathrm{d}\tilde{s}\,\tilde{s}^2\sin\theta'\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}\cos\theta'+\tilde{s}^2)}\,\bar{\eta}(\tilde{u}) \times\frac{1}{\tilde{s}}\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}\cos\theta'\right|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.$$

The integration over ϕ' is just 2π and let $\cos\theta' = y$. Therefore

$$(II) = -\frac{4\epsilon^2}{\pi^{1/2}}\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\frac{1}{q^2}\int \mathrm{d}\tilde{\boldsymbol{u}}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\bar{\eta}(\tilde{u}) \times\int_{\tilde{u}_2=\left|\frac{(1-q)}{q}\tilde{s}+\tilde{u}y\right|}^{\infty}\tilde{u}_2\hat{\Phi}_e(\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2. \tag{4.231}$$

Let us replace $\tilde{u}_2^2$ by $\tilde{u}_2^2 + \left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2$. This shift results into

$$(II) = -\frac{4\epsilon^2}{\pi^{1/2}} \lim_{\epsilon\to 0} \frac{\partial}{\partial\epsilon} \frac{1}{q^2} \int \mathrm{d}\tilde{\boldsymbol{u}} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{y=-1}^{1} \mathrm{d}y\, \tilde{s}\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \bar{\eta}(\tilde{u})$$
$$\times \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_2\, \mathrm{e}^{-\left\{\tilde{u}_2^2 + \left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2\right\}}\, \hat{\Phi}_e\left(\left\{\tilde{u}_2^2 + \left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2\right\}^{1/2}\right).$$

Changing into spherical coordinate system, the integration over $\tilde{\boldsymbol{u}}$ results into

$$(II) = -16\pi^{1/2}\epsilon^2 \lim_{\epsilon\to 0} \frac{\partial}{\partial\epsilon} \frac{1}{q^2} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{y=-1}^{1} \mathrm{d}y\, \tilde{s}\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \bar{\eta}(\tilde{u})$$
$$\times \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_2\, \mathrm{e}^{-\left\{\tilde{u}_2^2 + \left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2\right\}}\, \hat{\Phi}_e\left(\left\{\tilde{u}_2^2 + \left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2\right\}^{1/2}\right). \quad (4.232)$$

The differentiation with respect to ϵ at $\epsilon = 0$ is carried out next. Note that $q \to 1$ as $\epsilon \to 0$. Let

$$\lim_{\epsilon\to 0} \frac{\partial}{\partial\epsilon} \left[\frac{1}{q^2} \hat{\Phi}_e\left(\left\{\tilde{u}_2^2 + \left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2\right\}^{\frac{1}{2}}\right) \mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2}\right] = C$$

or

$$C = \lim_{\epsilon\to 0} \left(-\frac{2}{q^3}\frac{\partial q}{\partial\epsilon}\right) \hat{\Phi}_e\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\right) \mathrm{e}^{-\tilde{u}^2y^2}$$
$$+ \lim_{\epsilon\to 0} \left[\hat{\Phi}_e'\left(\left\{\tilde{u}_2^2 + \left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2\right\}^{1/2}\right) 2\left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)\left(-\frac{\tilde{s}}{q^2}\frac{\partial q}{\partial\epsilon}\right)\right] \mathrm{e}^{-\tilde{u}^2y^2}$$
$$+ \lim_{\epsilon\to 0} \left[-\,\mathrm{e}^{-\left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)^2} 2\left(\frac{(1-q)}{q}\tilde{s} + \tilde{u}y\right)\left(-\frac{\tilde{s}}{q^2}\frac{\partial q}{\partial\epsilon}\right)\right] \hat{\Phi}_e\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\right).$$

Here prime on $\hat{\Phi}_e$ denotes the differentiation with respect to square of its argument. Noting that $\lim_{\epsilon\to 0} \dfrac{\partial q}{\partial\epsilon} = -\dfrac{1}{4}$,

$$C = \frac{1}{2}\hat{\Phi}_e\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\right) \mathrm{e}^{-\tilde{u}^2y^2} + \frac{1}{2}\hat{\Phi}_e'\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\right) \tilde{u}y\tilde{s}\, \mathrm{e}^{-\tilde{u}^2y^2}$$
$$- \frac{1}{2}\tilde{u}y\tilde{s}\, \hat{\Phi}_e\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\right) \mathrm{e}^{-\tilde{u}^2y^2}$$
$$= \frac{1}{2}\mathrm{e}^{-\tilde{u}^2y^2} \left[(1 - \tilde{s}\tilde{u}y)\, \hat{\Phi}_e\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\right) + \tilde{s}\tilde{u}y\, \hat{\Phi}_e'\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{1/2}\right)\right].$$

Hence

$$(II) = -8\pi^{1/2}\epsilon^2 \int_{\tilde{s}=0}^{\infty} \mathrm{d}\tilde{s} \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u} \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \int_{y=-1}^{1} \mathrm{d}y\, \tilde{s}\tilde{u}_2\bar{\eta}(\tilde{u})$$
$$\times \left[(1 - \tilde{s}\tilde{u}y)\, \hat{\Phi}_e\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{\frac{1}{2}}\right) + \tilde{s}\tilde{u}y\, \hat{\Phi}_e'\left((\tilde{u}_2^2 + \tilde{u}^2y^2)^{\frac{1}{2}}\right)\right]$$
$$\times\, \mathrm{e}^{-(\tilde{u}^2 - 2\tilde{u}\tilde{s}y + \tilde{s}^2)}\, \mathrm{e}^{-(\tilde{u}_2^2 + \tilde{u}^2y^2)} = \vartheta_3\epsilon^2, \quad (4.233)$$

where

$$\vartheta_3 = -8\pi^{1/2}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\bar{\eta}(\tilde{u})$$
$$\times\left[(1-\tilde{s}\tilde{u}y)\,\hat{\varPhi}_e\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{\frac{1}{2}}\Big)+\tilde{s}\tilde{u}y\,\hat{\varPhi}_e'\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{\frac{1}{2}}\Big)\right]$$
$$\times\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\,. \tag{4.234}$$

ϑ_3 is evaluated numerically. Its value is: $\vartheta_3 \approx -0.2918$. Therefore

$$I_{1_b} = (\vartheta_2+\vartheta_3)\epsilon^2. \tag{4.235}$$

The third contribution to $\varGamma_{\epsilon\epsilon\epsilon}$ is given by eq. (4.223). Substituting the value of $\tilde{\varOmega}$ from eq. (2.38) in eq. (4.223), one obtains

$$I_{1_c} = -\frac{\epsilon^2}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')-\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\}. \tag{4.236}$$

Note that eq. (4.236) uses elastic velocity transformation. Following a similar procedure as in the derivation of $Q_{i3}^{K\epsilon}$, one obtains

$$I_{1_c} = -\frac{\epsilon^2}{\pi^{5/2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u})\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)I_\delta^{(0)}$$
$$+\frac{\epsilon^2}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2), \tag{4.237}$$

where $I_\delta^{(0)}$ is given in eq. (4.48). The term I_{1_c} is split into two parts. The first part is

$$(I) = -\frac{\epsilon^2}{\pi^{5/2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u})\,\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)I_\delta^{(0)}$$
$$= -\frac{\epsilon^2}{\pi^{5/2}}\int\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\bar{\eta}(\tilde{u})\,\hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_2)I_\delta^{(0)}. \tag{4.238}$$

Note that except for the extra term $\hat{\varPhi}_e(\tilde{u}_1)$ and the definition of $I_\delta^{(0)}$, the integrand in eq. (4.238) is similar to that in eq. (4.230). Hence, by following a similar procedure as below eq. (4.230), we get (cf. eq. (4.232))

$$(I) = -16\pi^{1/2}\epsilon^2\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\bar{\eta}(\tilde{u})\hat{\varPhi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)$$
$$\times\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\,\hat{\varPhi}_e\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big) = \vartheta_4\epsilon^2, \tag{4.239}$$

where

$$\vartheta_4 = -16\pi^{1/2}\int_{\tilde{s}=0}^{\infty}\mathrm{d}\tilde{s}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\int_{y=-1}^{1}\mathrm{d}y\,\tilde{s}\tilde{u}_2\bar{\eta}(\tilde{u})\hat{\varPhi}_e\Big((\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)^{1/2}\Big)$$
$$\times\,\hat{\varPhi}_e\Big((\tilde{u}_2^2+\tilde{u}^2y^2)^{1/2}\Big)\,\mathrm{e}^{-(\tilde{u}^2-2\tilde{u}\tilde{s}y+\tilde{s}^2)}\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}^2y^2)}\,. \tag{4.240}$$

ϑ_4 is evaluated numerically. Its value is: $\vartheta_4 \approx 0.0860$. The second part of eq. (4.237) is

$$(II) = \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} \, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2).$$

In the above equation, the integration over $\hat{\boldsymbol{k}}$ is trivial. Hence, using eq. (G.1b),

$$\begin{aligned}(II) &= \frac{\epsilon^2}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \tilde{u}_{12} \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2) \\ &= \frac{\epsilon^2}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \tilde{u}_{12} \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_2). \end{aligned} \tag{4.241}$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1 \cdot \tilde{\boldsymbol{u}}_2 = \tilde{u}_1\tilde{u}_2 \cos\theta_2'$, i.e.,

$$\begin{aligned}(II) = \frac{\epsilon^2}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2' \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^2 \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \\ \times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_2).\end{aligned}$$

The integration over ϕ_2' is just 2π and let $\cos\theta_2' = y$. Therefore

$$\begin{aligned}(II) &= \frac{2\epsilon^2}{\pi^{1/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^2 \left\{ \int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2' \, \sin\theta_2' (\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)^{1/2} \right\} \\ &\quad \times \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_2) \\ &= \frac{2\epsilon^2}{\pi^{1/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^2 R_0(\tilde{u}_1, \tilde{u}_2) \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_2),\end{aligned}$$

where $R_n(\tilde{u}_1, \tilde{u}_2)$ is defined in eq. (4.57) and the value of $R_0(\tilde{u}_1, \tilde{u}_2)$ is given in eq. (4.61). Changing into spherical polar coordinate system, the integration over $\tilde{\boldsymbol{u}}_1$ results into

$$(II) = 8\pi^{1/2}\epsilon^2 \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^2 R_0(\tilde{u}_1, \tilde{u}_2) \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_2) = \vartheta_5\epsilon^2, \tag{4.242}$$

where

$$\vartheta_5 = 8\pi^{1/2} \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^2 R_0(\tilde{u}_1, \tilde{u}_2) \, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \, \bar{\eta}(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_1)\hat{\varPhi}_e(\tilde{u}_2). \tag{4.243}$$

ϑ_5 is evaluated numerically. Its value is: $\vartheta_5 \approx 0.1984$. Hence

$$I_{1_c} = (\vartheta_4 + \vartheta_5)\epsilon^2. \tag{4.244}$$

Therefore, from eqs. (4.220), (4.224), (4.235) and (4.244), we have

$$I_1 = (\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4 + \vartheta_5)\epsilon^2. \tag{4.245}$$

Next, consider eq. (4.218). Substituting the value of Φ_ϵ, eq. (4.218) changes to

$$I_2 = \epsilon^2 \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_2).$$

The integrals over $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ in the above equation are simplified by following a similar procedure as in the (II) part of I_{1_c}. The simplified value of I_2 is

$$I_2 = \vartheta_6 \epsilon^2, \tag{4.246}$$

where

$$\vartheta_6 = 8\pi^2 \int_{\tilde{u}_1=0}^{\infty} \mathrm{d}\tilde{u}_1 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2\, \tilde{u}_2^2\, S_0(\tilde{u}_1,\tilde{u}_2)\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_1)\hat{\Phi}_e(\tilde{u}_2) \tag{4.247}$$

and $S_n(\tilde{u}_1,\tilde{u}_2)$ is defined in eq. (4.177). ϑ_6 is evaluated numerically. Its value is: $\vartheta_6 \approx 1.2233$. Therefore, from eqs. (4.216), (4.245) and (4.246),

$$\Gamma_{\epsilon\epsilon\epsilon} = \frac{\epsilon\Theta}{12\pi^3\ell}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \{2(\vartheta_1+\vartheta_2+\vartheta_3+\vartheta_4+\vartheta_5)+\vartheta_6\}\epsilon^2 = \tilde{\varsigma}\left(\frac{\epsilon^3}{\ell}\Theta^{3/2}\right), \tag{4.248}$$

where

$$\tilde{\varsigma} = \frac{1}{12\pi^3}\left(\frac{2}{3}\right)^{\frac{1}{2}} \{2(\vartheta_1+\vartheta_2+\vartheta_3+\vartheta_4+\vartheta_5)+\vartheta_6\} \approx 0.0458. \tag{4.249}$$

Eq. (4.248) can be written in another form to get

$$\boxed{\Gamma_{\epsilon\epsilon\epsilon} \approx 0.0458\frac{\epsilon^3}{\ell}\Theta^{3/2} \approx 0.1439\epsilon^3 n d^2 \Theta^{3/2}} \tag{4.250}$$

4.5 Summary

The constitutive relations at $O(K\epsilon)$, $O(K^2)$ and $O(\epsilon^2)$ (second-order in small parameters) have been derived without evaluating the correction terms at these orders. The facts about the constitutive relations obtained are the following.

At $O(K\epsilon)$:

all the analysis is of Navier-Stokes level; expression for heat flux, i.e.,

$$Q_i^{K\epsilon} = -\epsilon\tilde{\kappa}_1 n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i} - \epsilon\tilde{\tau}_1\ell\Theta^{3/2}\frac{\partial n}{\partial r_i}$$

contains a term proportional to gradient of number density and this term is non-Fourier contribution to heat flux; expression for pressure tensor, i.e.,

$$P_{ij}^{K\epsilon} = -2\epsilon\tilde{\mu}_1 n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}$$

is still Newton's law of viscosity; the contribution to dissipation term vanishes, i.e., $\Gamma_{K\epsilon\epsilon} = 0$.

At $O(KK)$:

all the analysis is of Burnett level; expression for heat flux, pressure tensor and collisional dissipation are given by

$$Q_i^{KK} = \tilde{\theta}_1 n\ell^2 \frac{\partial V_j}{\partial r_j}\frac{\partial \Theta}{\partial r_i} + \tilde{\theta}_2 n\ell^2 \left\{ \frac{2}{3}\frac{\partial}{\partial r_i}\left(\Theta \frac{\partial V_j}{\partial r_j}\right) + 2\frac{\partial V_j}{\partial r_i}\frac{\partial \Theta}{\partial r_j} \right\}$$
$$+ \tilde{\theta}_3 \ell^2 \overline{\frac{\partial V_j}{\partial r_i}} \frac{\partial (n\Theta)}{\partial r_j} + \tilde{\theta}_4 n\ell^2 \Theta \frac{\partial}{\partial r_j}\overline{\frac{\partial V_j}{\partial r_i}} + \tilde{\theta}_5 n\ell^2 \overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial \Theta}{\partial r_j},$$

$$P_{ij}^{KK} = \tilde{\omega}_1 n\ell^2 \frac{\partial V_k}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}} - \tilde{\omega}_2 n\ell^2 \left\{ \frac{1}{3}\overline{\frac{\partial}{\partial r_i}\left(\frac{1}{n}\frac{\partial (n\Theta)}{\partial r_j}\right)} + \overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}} + 2\overline{\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_j}} \right\}$$
$$+ \tilde{\omega}_3 n\ell^2 \overline{\frac{\partial^2 \Theta}{\partial r_i \partial r_j}} + \tilde{\omega}_4 \frac{\ell^2}{\Theta}\overline{\frac{\partial (n\Theta)}{\partial r_i}\frac{\partial \Theta}{\partial r_j}} + \tilde{\omega}_5 \frac{n\ell^2}{\Theta}\overline{\frac{\partial \Theta}{\partial r_i}\frac{\partial \Theta}{\partial r_j}} + \tilde{\omega}_6 n\ell^2 \overline{\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}},$$

and

$$\Gamma_{KK\epsilon} = \tilde{\rho}_1 \epsilon \ell \Theta^{1/2} \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \tilde{\rho}_2 \frac{\epsilon \ell}{\Theta^{1/2}}\frac{\partial \Theta}{\partial r_i}\frac{\partial \Theta}{\partial r_i} + \tilde{\rho}_3 \frac{\epsilon \ell}{n\Theta^{1/2}}\frac{\partial \Theta}{\partial r_i}\frac{\partial (n\Theta)}{\partial r_i} + \tilde{\rho}_4 \epsilon \ell \Theta^{1/2}\frac{\partial^2 \Theta}{\partial r_i \partial r_i},$$

respectively. The last three terms in the expression of heat flux obtained in this work are different from that obtained in Kogan (1969) and Sela & Goldhirsch (1998) but match with Chapman & Cowling (1970). These minor corrections might be results of typographical or calculation mistake in Kogan (1969) and Sela & Goldhirsch (1998). These corrections make sense, because otherwise these terms in Kogan (1969) and Sela & Goldhirsch (1998) could be included in the first two terms of expression for heat flux and it could not be required to write them separately.

At $O(\epsilon\epsilon)$:

all the analysis is of Euler level; heat flux and pressure tensor vanish, i.e., $Q_i^{\epsilon\epsilon} = P_{ij}^{\epsilon\epsilon} = 0$; the contribution to dissipation term is of $O(\epsilon^3)$ and given by

$$\Gamma_{\epsilon\epsilon\epsilon} \approx 0.0458 \frac{\epsilon^3}{\ell}\Theta^{3/2} \approx 0.1439 \epsilon^3 n d^2 \Theta^{3/2}.$$

Note that the contribution to dissipation term at this order is ignored in Sela & Goldhirsch (1998).

The numerical values of coefficients obtained in the present work and that in Sela & Goldhirsch (1998) are given in Table 4.1.

Coefficients in:	**Coefficient**	**Sela & Goldhirsch (1998)**	**Present work**
$Q_i^{K\epsilon}$	α_1	−0.3619	−0.3627
	β_1	−0.2110	−0.2110
	α_2	−0.0282	−0.0282
	α_3	0.2849	0.2849
	α_4	−0.0016	−0.0018
	α_5	−0.0016	−0.0018
	α_6	0.0018	0.0025
	α_7	−0.0006	−0.0006
	$\tilde{\kappa}_1$	0.1072	0.1078
	$\tilde{\mu}_1$	0.2110	0.2110
$P_{ij}^{K\epsilon}$	ζ_1	−0.0935	−0.0942
	ζ_2	−0.1349	−0.1349
	ζ_3	0.1094	0.1094
	ζ_4	0.0015	0.0014
	ζ_5	0.0015	0.0014
	ζ_6	0.0010	0.0016
	ζ_7	−0.0003	−0.0004
	$\tilde{\mu}_1$	0.0576	0.0578
Q_i^{KK}	$\tilde{\theta}_1$	1.2291	1.2312
	$\tilde{\theta}_2$	−0.6146	−0.6156
	$\tilde{\theta}_3$	−0.3262	−0.3270
	$\tilde{\theta}_4$	0.2552	0.2551
	$\tilde{\theta}_5$	2.6555	2.5667
P_{ij}^{KK}	$\tilde{\omega}_1$	1.2845	1.2850
	$\tilde{\omega}_2$	0.6422	0.6425
	$\tilde{\omega}_3$	0.2552	0.2551
	$\tilde{\omega}_4$	0.0719	0.0719
	$\tilde{\omega}_5$	0.0231	0.0248
	$\tilde{\omega}_6$	2.3510	2.3498
$\Gamma_{KK\epsilon}$	$\tilde{\alpha}_1$	12.6469	12.6475
	$\tilde{\alpha}_2$	73.1575	73.1660
	$\tilde{\alpha}_3$	−19.0089	−19.0093
	$\tilde{\alpha}_4$	15.7588	15.7585
	$\tilde{\beta}_1$	15.3412	12.9619
	$\tilde{\beta}_2$	−3.4190	4.8612
	$\tilde{\rho}_1$	0.1338	0.1259
	$\tilde{\rho}_2$	0.2444	0.2626
	$\tilde{\rho}_3$	−0.0834	−0.0834
	$\tilde{\rho}_4$	0.0692	0.0692

Table 4.1: Comparision of coefficients

Chapter 5

Conclusion and Future Work

5.1 Conclusion

This thesis is devoted to the study of granular flows belonging to rapid flow regime. The flow considered in this work is restricted to monodisperse collection of smooth inelastic (nearly elastic) hard spheres. The interaction between the particles is assumed to be characterized by instantaneous binary collisions. The work is mostly based on the paper by Sela & Goldhirsch (1998), though a body force term (gravity) is added in the Boltzmann equation. The macroscopic length scale in the problem is defined as a function of thermal velocity and hence it depends on position vector and time (see second paragraph of §2.3), while it has been taken as a constant in Sela & Goldhirsch (1998). Due to this difference, the complicacy in calculations is increased a bit. Although we could also consider the macroscopic length scale as a constant but then to rescale the body force term we could end up with a scale depending upon position vector and time, i.e., the level of complicacy in calculations could remain the same.

To complete the hydrodynamic description of the above mentioned flow, the constitutive relations till second-order (which includes Burnett order description) in small parameters are derived in detail. At $O(K\epsilon)$, the expression for heat flux contains a term proportional to the gradient of number density, which was not present at $O(K)$; therefore this term is a non-Fourier contribution to the heat flux and is identically zero for flows of elastic particles (because it contains ϵ in its coefficient); thus one can say that this term is a consequence of inelasticity. The contribution to collisional dissipation due to $O(\epsilon\epsilon)$ correction to distribution function is also evaluated while it is ignored in Sela & Goldhirsch (1998) because collisional dissipation is of $O(\epsilon^3)$. The constitutive relations obtained in this work match with those obtained in Sela & Goldhirsch (1998) except for few minor corrections in the last three terms of the expression for heat flux at Burnett order. These corrections make sense, because otherwise there was no need to write them separately and they could be included in the first two terms. The expression for heat flux at Burnett order obtained in this work matches with that in Chapman & Cowling (1970). As expected, addition of body force term in the Boltzmann equation does not affect the constitutive relations.

5.1.1 Constitutive Relations

Summarizing the results obtained in previous chapters, the constitutive relations corrected to second order in small parameters are as follows:

Heat Flux

$$
\begin{aligned}
Q_i = & -\tilde{\kappa}\, n\ell\Theta^{1/2}\frac{\partial\Theta}{\partial r_i} - \tilde{\lambda}\,\ell\Theta^{3/2}\frac{\partial n}{\partial r_i} \\
& + \tilde{\theta}_1 n\ell^2\frac{\partial V_j}{\partial r_j}\frac{\partial\Theta}{\partial r_i} + \tilde{\theta}_2 n\ell^2\left\{\frac{2}{3}\frac{\partial}{\partial r_i}\left(\Theta\frac{\partial V_j}{\partial r_j}\right) + 2\frac{\partial V_j}{\partial r_i}\frac{\partial\Theta}{\partial r_j}\right\} \\
& + \tilde{\theta}_3\ell^2\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial(n\Theta)}{\partial r_j} + \tilde{\theta}_4 n\ell^2\Theta\frac{\partial}{\partial r_j}\overline{\frac{\partial V_j}{\partial r_i}} + \tilde{\theta}_5 n\ell^2\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial\Theta}{\partial r_j},
\end{aligned} \tag{5.1}
$$

where $\tilde{\kappa} \approx 0.4100 + 0.1078\epsilon$, $\tilde{\lambda} \approx 0.2110\epsilon$, $\tilde{\theta}_1 \approx 1.2312$, $\tilde{\theta}_2 \approx -0.6156$, $\tilde{\theta}_3 \approx -0.3270$, $\tilde{\theta}_4 \approx 0.2551$ and $\tilde{\theta}_5 \approx 2.5667$.

Pressure Tensor

$$
\begin{aligned}
P_{ij} = & \frac{1}{3}n\Theta\delta_{ij} - 2\tilde{\mu}n\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}} \\
& + \tilde{\omega}_1 n\ell^2\frac{\partial V_k}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}} - \tilde{\omega}_2 n\ell^2\left\{\overline{\frac{1}{3}\frac{\partial}{\partial r_i}\left(\frac{1}{n}\frac{\partial(n\Theta)}{\partial r_j}\right)} + \overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}} + 2\overline{\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_j}}\right\} \\
& + \tilde{\omega}_3 n\ell^2\overline{\frac{\partial^2\Theta}{\partial r_i\partial r_j}} + \tilde{\omega}_4\frac{\ell^2}{\Theta}\overline{\frac{\partial(n\Theta)}{\partial r_i}\frac{\partial\Theta}{\partial r_j}} + \tilde{\omega}_5\frac{n\ell^2}{\Theta}\overline{\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_j}} + \tilde{\omega}_6 n\ell^2\overline{\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}},
\end{aligned} \tag{5.2}
$$

where $\tilde{\mu} = 0.3249 + 0.0578\epsilon$, $\tilde{\omega}_1 \approx 1.2850$, $\tilde{\omega}_2 \approx 0.6425$, $\tilde{\omega}_3 \approx 0.2551$, $\tilde{\omega}_4 \approx 0.0719$, $\tilde{\omega}_5 \approx 0.0248$ and $\tilde{\omega}_6 \approx 2.3498$.

Collisional Dissipation

$$
\Gamma = \frac{\tilde{\delta}}{\ell}\Theta^{3/2} + \tilde{\rho}_1\epsilon\ell\Theta^{1/2}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \tilde{\rho}_2\frac{\epsilon\ell}{\Theta^{1/2}}\frac{\partial\Theta}{\partial r_i}\frac{\partial\Theta}{\partial r_i} + \tilde{\rho}_3\frac{\epsilon\ell}{n\Theta^{1/2}}\frac{\partial\Theta}{\partial r_i}\frac{\partial(n\Theta)}{\partial r_i} + \tilde{\rho}_4\epsilon\ell\Theta^{1/2}\frac{\partial^2\Theta}{\partial r_i\partial r_i}, \tag{5.3}
$$

where $\tilde{\delta} \approx \left(\frac{16}{27\pi}\right)^{1/2}\epsilon - 0.0102\epsilon^2 + 0.0458\epsilon^3$, $\tilde{\rho}_1 \approx 0.1259$, $\tilde{\rho}_2 \approx 0.2626$, $\tilde{\rho}_3 \approx -0.0834$ and $\tilde{\rho}_4 \approx 0.0692$.

5.1.2 Normal Stress Difference in 2-D Granular Poiseuille Flow under Gravity

Consider the 2-D rapid granular (Poiseuille) flow of monodispersed smooth inelastic hard disks enclosed in a slab under the action of gravity. We assume that the granular gas is enclosed between two infinite parallel plates normal to y-axis. A constant external force per unit mass (e.g., gravity) $\boldsymbol{g} = g\widehat{\boldsymbol{x}}$ is acting along a direction $\widehat{\boldsymbol{x}}$ parallel to the plates. The geometry of the problem is sketched in figure 5.1.
We consider 2-D (xy plane), fully developed $\left(\frac{\partial}{\partial x}(\cdot) = 0\right)$ granular poiseuille flow under gravity. Hence the components of mean velocity $\boldsymbol{V}$ are

$$
V_x \equiv V_x(y) = V \text{ (say)} \quad \text{and} \quad V_y = 0.
$$

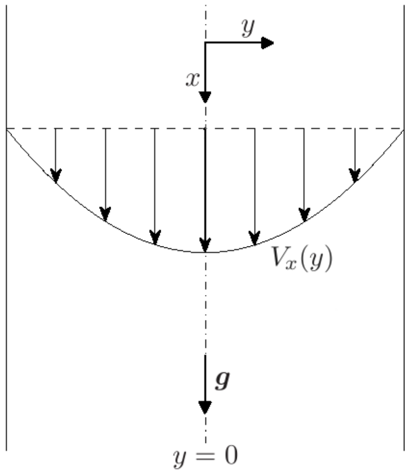


Figure 5.1: Sketch of the planar Poiseuille flow induced by a gravitational force.

Therefore for this problem:

$$\frac{\partial V_i}{\partial r_i} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0 \tag{5.4}$$

and

$$\overline{\frac{\partial V_i}{\partial r_j}} = \frac{1}{2}\left(\frac{\partial V_i}{\partial r_j} + \frac{\partial V_j}{\partial r_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial V_k}{\partial r_k} = \frac{1}{2}\left(\frac{\partial V_i}{\partial r_j} + \frac{\partial V_j}{\partial r_i}\right);$$

hence

$$\overline{\frac{\partial V_x}{\partial x}} = \frac{1}{2}\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_x}{\partial x}\right) = 0 \tag{5.5}$$

and

$$\overline{\frac{\partial V_y}{\partial y}} = \frac{1}{2}\left(\frac{\partial V_y}{\partial y} + \frac{\partial V_y}{\partial y}\right) = 0. \tag{5.6}$$

Normal Stress Difference till Euler-order

To Euler order,

$$P_{ij} = P_{ij}^{0} + P_{ij}^{\epsilon} + P_{ij}^{\epsilon\epsilon}.$$

From eqs. (2.29), (3.37) and (4.215),

$$P_{xx} = \frac{1}{3}n\Theta\delta_{xx} + 0 + 0 = \frac{1}{3}n\Theta,$$
$$P_{yy} = \frac{1}{3}n\Theta\delta_{yy} + 0 + 0 = \frac{1}{3}n\Theta.$$

Hence

$$P_{xx} - P_{yy} = 0.$$

Therefore normal stress difference vanishes, if we consider stress tensor corrected till Euler order.

Normal Stress Difference till Navier-Stokes order

To Navier-Stokes order,

$$P_{ij} = P_{ij}^0 + P_{ij}^K + P_{ij}^\epsilon + P_{ij}^{K\epsilon} + P_{ij}^{\epsilon\epsilon}.$$

From eqs. (2.29), (3.22), (3.37), (4.100) and (4.215),

$$P_{xx} = \frac{1}{3} n\Theta\delta_{xx} - 2\tilde{\mu}_0 n\ell\Theta^{1/2}\overline{\frac{\partial V_x}{\partial x}} + 0 - 2\epsilon\tilde{\mu}_1 n\ell\Theta^{1/2}\overline{\frac{\partial V_x}{\partial x}} + 0.$$

Using eq. (5.5),

$$P_{xx} = \frac{1}{3} n\Theta.$$

Similarly, from eqs. (2.29), (3.22), (3.37), (4.100) and (4.215),

$$P_{yy} = \frac{1}{3} n\Theta\delta_{yy} - 2\tilde{\mu}_0 n\ell\Theta^{1/2}\overline{\frac{\partial V_y}{\partial y}} + 0 - 2\epsilon\tilde{\mu}_1 n\ell\Theta^{1/2}\overline{\frac{\partial V_y}{\partial y}} + 0.$$

Using eq. (5.6),

$$P_{yy} = \frac{1}{3} n\Theta.$$

Hence

$$P_{xx} - P_{yy} = 0.$$

Therefore normal stress difference again vanishes, even if we consider stress tensor corrected till Navier-Stokes order.

Normal Stress Difference till Burnett order

To Burnett order,

$$P_{ij} = P_{ij}^0 + P_{ij}^K + P_{ij}^\epsilon + P_{ij}^{K\epsilon} + P_{ij}^{KK} + P_{ij}^{\epsilon\epsilon}.$$

From eqs. (2.29), (3.22), (3.37), (4.100), (4.160) and (4.215),

$$\begin{aligned} P_{xx} = {} & \frac{1}{3} n\Theta\delta_{xx} - 2\tilde{\mu}_0 n\ell\Theta^{1/2}\overline{\frac{\partial V_x}{\partial x}} + 0 - 2\epsilon\tilde{\mu}_1 n\ell\Theta^{1/2}\overline{\frac{\partial V_x}{\partial x}} \\ & + \tilde{\omega}_1 n\ell^2 \frac{\partial V_k}{\partial r_k}\overline{\frac{\partial V_x}{\partial x}} - \tilde{\omega}_2 n\ell^2 \left\{ \overline{\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{1}{n}\frac{\partial (n\Theta)}{\partial x}\right)} + \overline{\frac{\partial V_x}{\partial r_k}\frac{\partial V_k}{\partial x}} + 2\overline{\overline{\frac{\partial V_x}{\partial r_k}}\frac{\partial V_k}{\partial x}} \right\} \\ & + \tilde{\omega}_3 n\ell^2 \overline{\frac{\partial^2 \Theta}{\partial x \partial x}} + \tilde{\omega}_4 \frac{\ell^2}{\Theta}\overline{\frac{\partial (n\Theta)}{\partial x}\frac{\partial \Theta}{\partial x}} + \tilde{\omega}_5 \frac{n\ell^2}{\Theta}\overline{\frac{\partial \Theta}{\partial x}\frac{\partial \Theta}{\partial x}} + \tilde{\omega}_6 n\ell^2 \overline{\overline{\frac{\partial V_x}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial x}}} + 0. \end{aligned}$$

Using eqs. (5.4) and (5.5), and the fact that $\frac{\partial}{\partial x}(\cdot) = 0$,

$$P_{xx} = \frac{1}{3} n\Theta.$$

Similarly, from eqs. (2.29), (3.22), (3.37), (4.100), (4.160) and (4.215),

$$P_{yy} = \frac{1}{3}n\Theta\delta_{yy} - 2\tilde{\mu}_0 n\ell\Theta^{1/2}\overline{\frac{\partial V_y}{\partial y}} + 0 - 2\epsilon\tilde{\mu}_1 n\ell\Theta^{1/2}\overline{\frac{\partial V_y}{\partial y}}$$
$$+ \tilde{\omega}_1 n\ell^2 \frac{\partial V_k}{\partial r_k}\overline{\frac{\partial V_y}{\partial y}} - \tilde{\omega}_2 n\ell^2 \left\{ \frac{1}{3}\overline{\frac{\partial}{\partial y}\left(\frac{1}{n}\frac{\partial(n\Theta)}{\partial y}\right)} + \overline{\frac{\partial V_y}{\partial r_k}\frac{\partial V_k}{\partial y}} + 2\overline{\overline{\frac{\partial V_y}{\partial r_k}}\frac{\partial V_k}{\partial y}} \right\}$$
$$+ \tilde{\omega}_3 n\ell^2 \overline{\frac{\partial^2\Theta}{\partial y\partial y}} + \tilde{\omega}_4 \frac{\ell^2}{\Theta}\overline{\frac{\partial(n\Theta)}{\partial y}\frac{\partial\Theta}{\partial y}} + \tilde{\omega}_5 \frac{n\ell^2}{\Theta}\overline{\frac{\partial\Theta}{\partial y}\frac{\partial\Theta}{\partial y}} + \tilde{\omega}_6 n\ell^2 \overline{\overline{\frac{\partial V_y}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial y}}} + 0.$$

Using eqs. (5.4) and (5.6), and the fact that $V_y = 0$, Using eqs. (5.6),

$$P_{yy} = \frac{1}{3}n\Theta - \frac{1}{3}\tilde{\omega}_2 n\ell^2 \overline{\frac{\partial}{\partial y}\left(\frac{1}{n}\frac{\partial(n\Theta)}{\partial y}\right)} + \tilde{\omega}_3 n\ell^2 \overline{\frac{\partial^2\Theta}{\partial y\partial y}} + \tilde{\omega}_4 \frac{\ell^2}{\Theta}\overline{\frac{\partial(n\Theta)}{\partial y}\frac{\partial\Theta}{\partial y}} + \tilde{\omega}_5 \frac{n\ell^2}{\Theta}\overline{\frac{\partial\Theta}{\partial y}\frac{\partial\Theta}{\partial y}}.$$

Hence

$$P_{xx} - P_{yy} = \frac{1}{3}\tilde{\omega}_2 n\ell^2 \overline{\frac{\partial}{\partial y}\left(\frac{1}{n}\frac{\partial(n\Theta)}{\partial y}\right)} - \tilde{\omega}_3 n\ell^2 \overline{\frac{\partial^2\Theta}{\partial y\partial y}} - \tilde{\omega}_4 \frac{\ell^2}{\Theta}\overline{\frac{\partial(n\Theta)}{\partial y}\frac{\partial\Theta}{\partial y}} - \tilde{\omega}_5 \frac{n\ell^2}{\Theta}\overline{\frac{\partial\Theta}{\partial y}\frac{\partial\Theta}{\partial y}}.$$

Therefore normal stress difference is non zero at Burnett order, i.e., normal stress difference is Burnett order effect for 2-D fully developed rapid granular (Poiseuille) flow of monodispersed smooth inelastic hard disks under gravity. This result agrees with Tij & Santos (2004).

5.2 Future Work

This work can be applied to obtain the hydrodynamic profiles for the above defined 2-D granular Poiseuille flow problem and to verify the temperature anisotropy, which is captured by kinetic theory but not by Navier-Stokes level hydrodynamic theory (Tij & Santos 2004). The correction terms at second-order in small parameters can be evaluated and thus velocity distribution function correct to Burnett order would be known. Then the symmetry in tails of the velocity distribution function can be checked and compared with the simulation results (e.g., Alam & Chikkadi 2010).

Appendix A

Right-hand side of equation (2.19)

With the help of eq. (2.15), the right-hand side (let us say, it is equal to A) of eq. (2.19) can be written as

$$A = \frac{1}{\tilde{f}_0(\tilde{u}_1)} \mathscr{B}(\tilde{f}, \tilde{f}, e) = \frac{1}{\pi \tilde{f}_0(\tilde{u}_1)} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} \, (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}) \left(\frac{1}{e^2} \tilde{f}(\tilde{\boldsymbol{u}}_1') \tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1) \tilde{f}(\tilde{\boldsymbol{u}}_2) \right).$$

Substituting $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1 + \Phi(\tilde{\boldsymbol{u}}))$, we have

$$\begin{aligned} A = \frac{1}{\pi \tilde{f}_0(\tilde{u}_1)} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} \, (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}) \Big\{ & \frac{1}{e^2} \tilde{f}_0(\tilde{u}_1')(1 + \Phi(\tilde{\boldsymbol{u}}_1')) \tilde{f}_0(\tilde{u}_2')(1 + \Phi(\tilde{\boldsymbol{u}}_2')) \\ & - \tilde{f}_0(\tilde{u}_1)(1 + \Phi(\tilde{\boldsymbol{u}}_1)) \tilde{f}_0(\tilde{u}_2)(1 + \Phi(\tilde{\boldsymbol{u}}_2)) \Big\}. \end{aligned}$$

Using eq. (2.17), above expression changes to

$$\begin{aligned} A = \frac{1}{\pi^{5/2} \, \mathrm{e}^{-\tilde{u}_1^2}} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} \, (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}) \Big\{ & \frac{1}{e^2} \, \mathrm{e}^{-(\tilde{u}_1'^2 + \tilde{u}_2'^2)} (1 + \Phi(\tilde{\boldsymbol{u}}_1'))(1 + \Phi(\tilde{\boldsymbol{u}}_2')) \\ & - \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)} (1 + \Phi(\tilde{\boldsymbol{u}}_1))(1 + \Phi(\tilde{\boldsymbol{u}}_2)) \Big\}. \end{aligned}$$

Using the relation (cf. eq. (2.3)): $\tilde{u}_1'^2 + \tilde{u}_2'^2 = \tilde{u}_1^2 + \tilde{u}_2^2 + \frac{1}{2}(\epsilon + \epsilon^2 + \ldots)(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})^2$,

$$\begin{aligned} A = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} \, (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}) \, \mathrm{e}^{-\tilde{u}_2^2} \Big[& \frac{1}{e^2} \exp \left\{ -\frac{1}{2}(\epsilon + \epsilon^2 + \ldots)(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})^2 \right\} \\ & \times (1 + \Phi(\tilde{\boldsymbol{u}}_1'))(1 + \Phi(\tilde{\boldsymbol{u}}_2')) - (1 + \Phi(\tilde{\boldsymbol{u}}_1))(1 + \Phi(\tilde{\boldsymbol{u}}_2)) \Big]. \end{aligned}$$

Here,

$$\begin{aligned} & \frac{1}{e^2} \exp \left\{ -\frac{1}{2}(\epsilon + \epsilon^2 + \ldots)(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})^2 \right\} \\ & = (1 - \epsilon)^{-1} \exp \left\{ -\frac{1}{2}(\epsilon + \epsilon^2 + \ldots)(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})^2 \right\} \\ & = (1 + \epsilon + \epsilon^2 + \ldots) \left\{ 1 - \frac{1}{2}(\epsilon + \epsilon^2 + \ldots)(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{2!4}(\epsilon + \epsilon^2 + \ldots)^2 (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})^4 - \ldots \right\} \\ & = 1 + \epsilon \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})^2 \right) + \epsilon^2 \left(1 - (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})^4 \right) + \ldots \end{aligned}$$

Therefore

$$
\begin{aligned}
A = &\frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \\
&\times \Bigg[\left\{ 1 + \epsilon \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 \right) + \epsilon^2 \left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4 \right) + \dots \right\} \\
&\times \left\{ 1 + \Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') + \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2') \right\} - \left\{ 1 + \Phi(\tilde{\boldsymbol{u}}_1) + \Phi(\tilde{\boldsymbol{u}}_2) + \Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2) \right\} \Bigg] \\
= &\frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \Bigg[\left\{ 1 + \Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') + \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2') \right\} \\
&+ \epsilon \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 \right) \left\{ 1 + \Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') + \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2') \right\} \\
&+ \epsilon^2 \left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4 \right) \left\{ 1 + \Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') + \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2') \right\} + \dots \\
&- \left\{ 1 + \Phi(\tilde{\boldsymbol{u}}_1) + \Phi(\tilde{\boldsymbol{u}}_2) + \Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2) \right\} \Bigg] \\
= &\frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \\
&\times \left\{ \Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') + \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2') - \Phi(\tilde{\boldsymbol{u}}_1) - \Phi(\tilde{\boldsymbol{u}}_2) - \Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2) \right\} \\
&+ \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 \right) \\
&+ \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 \right) \left\{ \Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') + \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2') \right\} \\
&+ \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4 \right) \\
&+ \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4 \right) \\
&\times \left\{ \Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') + \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2') \right\} + \text{h.o.t.}
\end{aligned}
$$

Substituting expansion for Φ given in eq. (2.27),

$$
\begin{aligned}
A = &\frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \\
&\times \Bigg[\left\{ \Phi_K(\tilde{\boldsymbol{u}}_1') + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \Phi_{KK}(\tilde{\boldsymbol{u}}_1') + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_1') + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_1') + \dots \right\} \\
&+ \left\{ \Phi_K(\tilde{\boldsymbol{u}}_2') + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') + \Phi_{KK}(\tilde{\boldsymbol{u}}_2') + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_2') + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_2') + \dots \right\} \\
&+ \left\{ \Phi_K(\tilde{\boldsymbol{u}}_1') + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \Phi_{KK}(\tilde{\boldsymbol{u}}_1') + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_1') + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_1') + \dots \right\} \\
&\times \left\{ \Phi_K(\tilde{\boldsymbol{u}}_2') + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') + \Phi_{KK}(\tilde{\boldsymbol{u}}_2') + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_2') + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_2') + \dots \right\} \\
&- \left\{ \Phi_K(\tilde{\boldsymbol{u}}_1) + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) + \Phi_{KK}(\tilde{\boldsymbol{u}}_1) + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_1) + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_1) + \dots \right\} \\
&- \left\{ \Phi_K(\tilde{\boldsymbol{u}}_2) + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2) + \Phi_{KK}(\tilde{\boldsymbol{u}}_2) + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_2) + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_2) + \dots \right\} \\
&- \left\{ \Phi_K(\tilde{\boldsymbol{u}}_1) + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) + \Phi_{KK}(\tilde{\boldsymbol{u}}_1) + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_1) + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_1) + \dots \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\{ \varPhi_K(\tilde{\boldsymbol{u}}_2) + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2) + \varPhi_{KK}(\tilde{\boldsymbol{u}}_2) + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_2) + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_2) + \dots \right\} \Bigg] \\
&+ \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \\
&+ \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \\
&\times \Bigg[\left\{ \varPhi_K(\tilde{\boldsymbol{u}}_1') + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varPhi_{KK}(\tilde{\boldsymbol{u}}_1') + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_1') + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_1') + \dots \right\} \\
&+ \left\{ \varPhi_K(\tilde{\boldsymbol{u}}_2') + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') + \varPhi_{KK}(\tilde{\boldsymbol{u}}_2') + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_2') + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_2') + \dots \right\} \\
&+ \left\{ \varPhi_K(\tilde{\boldsymbol{u}}_1') + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varPhi_{KK}(\tilde{\boldsymbol{u}}_1') + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_1') + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_1') + \dots \right\} \\
&\times \left\{ \varPhi_K(\tilde{\boldsymbol{u}}_2') + \epsilon\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') + \varPhi_{KK}(\tilde{\boldsymbol{u}}_2') + \epsilon\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_2') + \epsilon^2\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_2') + \dots \right\} \Bigg] \\
&+ \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right) + \text{h.o.t.}
\end{aligned}
$$

or

$$
\begin{aligned}
A = &\frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left\{ \varPhi_K(\tilde{\boldsymbol{u}}_1') + \varPhi_K(\tilde{\boldsymbol{u}}_2') - \varPhi_K(\tilde{\boldsymbol{u}}_1) - \varPhi_K(\tilde{\boldsymbol{u}}_2) \right\} \\
&+ \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left\{ \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') - \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) - \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2) \right\} \\
&+ \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left\{ \varphi_K^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_K^{(1)}(\tilde{\boldsymbol{u}}_2') - \varphi_K^{(1)}(\tilde{\boldsymbol{u}}_1) - \varphi_K^{(1)}(\tilde{\boldsymbol{u}}_2) \right\} \\
&+ \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left\{ \varPhi_{KK}(\tilde{\boldsymbol{u}}_1') + \varPhi_{KK}(\tilde{\boldsymbol{u}}_2') - \varPhi_{KK}(\tilde{\boldsymbol{u}}_1) - \varPhi_{KK}(\tilde{\boldsymbol{u}}_2) \right\} \\
&+ \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left\{ \varphi_1^{(2)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(2)}(\tilde{\boldsymbol{u}}_2') - \varphi_1^{(2)}(\tilde{\boldsymbol{u}}_1) - \varphi_1^{(2)}(\tilde{\boldsymbol{u}}_2) \right\} \\
&+ \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left\{ \varPhi_K(\tilde{\boldsymbol{u}}_1')\varPhi_K(\tilde{\boldsymbol{u}}_2') - \varPhi_K(\tilde{\boldsymbol{u}}_1)\varPhi_K(\tilde{\boldsymbol{u}}_2) \right\} \\
&+ \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \\
&\times \left\{ \varPhi_K(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') + \varPhi_K(\tilde{\boldsymbol{u}}_2')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') - \varPhi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2) - \varPhi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) \right\} \\
&+ \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left\{ \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') - \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2) \right\} \\
&+ \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \\
&+ \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \left\{ \varPhi_K(\tilde{\boldsymbol{u}}_1') + \varPhi_K(\tilde{\boldsymbol{u}}_2') \right\} \\
&+ \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \left\{ \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') \right\} \\
&+ \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right) + \text{h.o.t.}
\end{aligned}
$$

Since $\tilde{\boldsymbol{u}}_1'$, $\tilde{\boldsymbol{u}}_2'$ are functions of $\tilde{\boldsymbol{u}}_1$, $\tilde{\boldsymbol{u}}_2$ and also of ϵ. That means pre-collisional velocities are implicit functions of ϵ and therefore, for further simplification one can apply Taylor series expansion around $\epsilon = 0$ $\left(\text{i.e, } f(\epsilon) = f(0) + \epsilon f'(0) + \frac{1}{2!}\epsilon^2 f''(0) + \dots\right)$. Thus we get

$$
\begin{aligned}
A = \frac{1}{\pi^{5/2}} & \left[\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2') - \Phi_K(\tilde{\boldsymbol{u}}_1) - \Phi_K(\tilde{\boldsymbol{u}}_2)\right\}\right. \\
& \left. +\epsilon \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2')\right\}\right] \\
& + \frac{\epsilon}{\pi^{5/2}}\left[\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') - \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1) - \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\right\}\right. \\
& \left. +\epsilon \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')\right\}\right] \\
& + \frac{\epsilon}{\pi^{5/2}}\left[\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\varphi_K^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_K^{(1)}(\tilde{\boldsymbol{u}}_2') - \varphi_K^{(1)}(\tilde{\boldsymbol{u}}_1) - \varphi_K^{(1)}(\tilde{\boldsymbol{u}}_2)\right\}\right] \\
& + \frac{1}{\pi^{5/2}}\left[\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\Phi_{KK}(\tilde{\boldsymbol{u}}_1') + \Phi_{KK}(\tilde{\boldsymbol{u}}_2') - \Phi_{KK}(\tilde{\boldsymbol{u}}_1) - \Phi_{KK}(\tilde{\boldsymbol{u}}_2)\right\}\right] \\
& + \frac{\epsilon^2}{\pi^{5/2}}\left[\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\varphi_1^{(2)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(2)}(\tilde{\boldsymbol{u}}_2') - \varphi_1^{(2)}(\tilde{\boldsymbol{u}}_1) - \varphi_1^{(2)}(\tilde{\boldsymbol{u}}_2)\right\}\right] \\
& + \frac{1}{\pi^{5/2}}\left[\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\Phi_K(\tilde{\boldsymbol{u}}_1')\Phi_K(\tilde{\boldsymbol{u}}_2') - \Phi_K(\tilde{\boldsymbol{u}}_1)\Phi_K(\tilde{\boldsymbol{u}}_2)\right\}\right] \\
& + \frac{\epsilon}{\pi^{5/2}}\left[\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\Phi_K(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') + \Phi_K(\tilde{\boldsymbol{u}}_2')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1')\right.\right. \\
& \left.\left. - \Phi_K(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2) - \Phi_K(\tilde{\boldsymbol{u}}_2)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\right\}\right] \\
& + \frac{\epsilon^2}{\pi^{5/2}}\left[\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1')\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2') - \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1)\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2)\right\}\right] \\
& + \frac{\epsilon}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \\
& + \frac{\epsilon}{\pi^{5/2}}\left[\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right)\left\{\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2')\right\}\right] \\
& + \frac{\epsilon^2}{\pi^{5/2}}\left[\lim_{\epsilon\to 0}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right)\left\{\varphi_1^{(1)}(\tilde{\boldsymbol{u}}_1') + \varphi_1^{(1)}(\tilde{\boldsymbol{u}}_2')\right\}\right] \\
& + \frac{\epsilon^2}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right) + \text{h.o.t.}
\end{aligned}
$$

Let us define the following operators:

$$
\mathscr{L}(\Phi) \equiv \frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2') - \Phi(\tilde{\boldsymbol{u}}_1) - \Phi(\tilde{\boldsymbol{u}}_2)\right\},
$$

$$
\begin{aligned}
\tilde{\Omega}(\Phi,\psi) \equiv \frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} & \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left\{\Phi(\tilde{\boldsymbol{u}}_1')\psi(\tilde{\boldsymbol{u}}_2') + \Phi(\tilde{\boldsymbol{u}}_2')\psi(\tilde{\boldsymbol{u}}_1')\right. \\
& \left. - \Phi(\tilde{\boldsymbol{u}}_1)\psi(\tilde{\boldsymbol{u}}_2) - \Phi(\tilde{\boldsymbol{u}}_2)\psi(\tilde{\boldsymbol{u}}_1)\right\},
\end{aligned}
$$

$$\tilde{\Xi}(\Phi) \equiv \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \mathrm{e}^{-\tilde{u}_2^2} \left\{\Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2')\right\},$$

and

$$\tilde{\Lambda}(\Phi) = \frac{1}{\pi^{5/2}} \lim_{\epsilon\to 0} \frac{\partial}{\partial\epsilon} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left\{\Phi(\tilde{\boldsymbol{u}}_1') + \Phi(\tilde{\boldsymbol{u}}_2')\right\},$$

where the operators $\tilde{\mathscr{L}}$, $\tilde{\Omega}$ and $\tilde{\Xi}$ are defined for elastically colliding particles*. The operator $\tilde{\mathscr{L}}$ is the (standard) rescaled linearized Boltzmann operator for elastically colliding particles. Hence A can be written as

$$\begin{aligned} A = {} & \tilde{\mathscr{L}}(\Phi_K) + \epsilon\tilde{\Lambda}(\Phi_K) + \epsilon\,\tilde{\mathscr{L}}(\varphi_1^{(1)}) + \epsilon^2\tilde{\Lambda}(\varphi_1^{(1)}) + \epsilon\,\tilde{\mathscr{L}}(\varphi_K^{(1)}) + \tilde{\mathscr{L}}(\Phi_{KK}) + \epsilon^2\tilde{\mathscr{L}}(\varphi_1^{(2)}) \\ & + \frac{1}{2}\tilde{\Omega}(\Phi_K,\Phi_K) + \epsilon\tilde{\Omega}(\Phi_K,\varphi_1^{(1)}) + \frac{1}{2}\epsilon^2\tilde{\Omega}(\varphi_1^{(1)},\varphi_1^{(1)}) \\ & + \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) + \epsilon\tilde{\Xi}(\Phi_K) + \epsilon^2\tilde{\Xi}(\varphi_1^{(1)}) \\ & + \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right) + \text{h.o.t.} \end{aligned}$$

or

$$\begin{aligned} A = {} & \tilde{\mathscr{L}}(\Phi_K) + \epsilon\,\tilde{\mathscr{L}}(\varphi_1^{(1)}) + \frac{\epsilon}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - \frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) \\ & + \epsilon\,\tilde{\mathscr{L}}(\varphi_K^{(1)}) + \epsilon\tilde{\Xi}(\Phi_K) + \epsilon\tilde{\Lambda}(\Phi_K) + \epsilon\tilde{\Omega}(\Phi_K,\varphi_1^{(1)}) + \tilde{\mathscr{L}}(\Phi_{KK}) + \frac{1}{2}\tilde{\Omega}(\Phi_K,\Phi_K) \\ & + \epsilon^2\tilde{\mathscr{L}}(\varphi_1^{(2)}) + \epsilon^2\tilde{\Xi}(\varphi_1^{(1)}) + \epsilon^2\tilde{\Lambda}(\varphi_1^{(1)}) + \frac{1}{2}\epsilon^2\tilde{\Omega}(\varphi_1^{(1)},\varphi_1^{(1)}) \\ & + \frac{\epsilon^2}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \left(1 - (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2 + \frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right) + \text{h.o.t.} \end{aligned} \tag{A.1}$$

*These are equivalent representations of the terms appearing in the expression of A, for example, limit $\epsilon \to 0$ in

$$\frac{1}{\pi^{5/2}} \lim_{\epsilon\to 0} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2} \{\Phi_K(\tilde{\boldsymbol{u}}_1') + \Phi_K(\tilde{\boldsymbol{u}}_2') - \Phi_K(\tilde{\boldsymbol{u}}_1) - \Phi_K(\tilde{\boldsymbol{u}}_2)\}$$

takes care of the inelastic collisons and this expression is equal to $\tilde{\mathscr{L}}(\Phi_K)$.

Appendix B

Proof of Solvability

In this appendix it is shown that the generalized Chapman-Enskog expansion results in soluble equations to all orders in the expansion.
From eq. (2.15),

$$\tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f}) \equiv \tilde{\mathscr{B}}(\tilde{f},\tilde{f},e=1) = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right).$$

Now, let us simplify $\tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f})$ by substituting $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1+\Phi(\tilde{\boldsymbol{u}}))$.

$$\begin{aligned}\tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f}) = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} & \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\Big[\tilde{f}_0(\tilde{u}_1')(1+\Phi(\tilde{\boldsymbol{u}}_1'))\tilde{f}_0(\tilde{u}_2')(1+\Phi(\tilde{\boldsymbol{u}}_2')) \\ & - \tilde{f}_0(\tilde{u}_1)(1+\Phi(\tilde{\boldsymbol{u}}_1))\tilde{f}_0(\tilde{u}_2)(1+\Phi(\tilde{\boldsymbol{u}}_2))\Big],\end{aligned}$$

but for elastic case $\tilde{f}_0(\tilde{u}_1')\tilde{f}_0(\tilde{u}_2') = \tilde{f}_0(\tilde{u}_1)\tilde{f}_0(\tilde{u}_2)$, hence

$$\begin{aligned}\tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f}) = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} & \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\tilde{f}_0(\tilde{u}_1)\tilde{f}_0(\tilde{u}_2)\Big[\{1+\Phi(\tilde{\boldsymbol{u}}_1')+\Phi(\tilde{\boldsymbol{u}}_2')+\Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2')\} \\ & - \{1+\Phi(\tilde{\boldsymbol{u}}_1)+\Phi(\tilde{\boldsymbol{u}}_2)+\Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2)\}\Big].\end{aligned}$$

Taking $\tilde{f}_0(\tilde{u}_1)$ out of the integral and using eq. (2.17) to replace $\tilde{f}_0(\tilde{u}_2)$, we get

$$\begin{aligned}\tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f}) &= \frac{\tilde{f}_0(\tilde{u}_1)}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\{\Phi(\tilde{\boldsymbol{u}}_1')+\Phi(\tilde{\boldsymbol{u}}_2')-\Phi(\tilde{\boldsymbol{u}}_1)-\Phi(\tilde{\boldsymbol{u}}_2)\} \\ &\quad + \frac{\tilde{f}_0(\tilde{u}_1)}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\{\Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2')-\Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2)\} \\ &= \tilde{f}_0(\tilde{u}_1)\,\mathscr{L}(\Phi) + \frac{1}{2}\tilde{f}_0(\tilde{u}_1)\tilde{\Omega}(\Phi,\Phi),\end{aligned}$$

where the definition of $\tilde{\Omega}$ is given in eq. (2.38). Hence

$$\tilde{f}_0(\tilde{u}_1)\,\mathscr{L}(\Phi) = \tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f}) - \frac{1}{2}\tilde{f}_0(\tilde{u}_1)\tilde{\Omega}(\Phi,\Phi).$$

With the help of eq. (2.14) above equation can be written as

$$\begin{aligned}\tilde{f}_0(\tilde{u}_1)\,\mathscr{L}(\Phi) = \tilde{\mathscr{D}}\tilde{f} + \tilde{f}\tilde{\mathscr{D}}\left(\ln n - \frac{3}{2}\ln\Theta\right) - \frac{1}{2}\tilde{f}_0(\tilde{u}_1)\tilde{\Omega}(\Phi,\Phi) + \tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f}) \\ - \tilde{\mathscr{B}}(\tilde{f},\tilde{f},e) + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\tilde{f}. \qquad \text{(B.1)}\end{aligned}$$

Eq. (B.1) is soluble only if right-hand side of it is orthogonal to all the summational invariants (1, $\tilde{\boldsymbol{u}}$ and $\tilde{u}^2$) of $\tilde{\mathscr{L}}$. The integral over $\tilde{\boldsymbol{u}}$ of the first term on the RHS of (B.1) times any of the summational invariants can be carried out as follows:
Consider the integration involving the summational invariant, 1.

$$\begin{aligned}
\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{\mathscr{D}}\tilde{f} &= \int \left\{ \left(\frac{3}{2\Theta}\right)^{\frac{3}{2}} \mathrm{d}\boldsymbol{v} \right\} \left\{ \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right) \right\} \left\{ \frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} f \right\} \\
&= \frac{K}{g}\frac{3}{2\Theta} \int \mathrm{d}\boldsymbol{v} \left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial r_i}\right) \left\{ \frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} f \right\} \\
&= \frac{K}{g}\frac{3}{2\Theta} \left[\frac{\partial}{\partial t}\left\{ \frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int f\, \mathrm{d}\boldsymbol{v} \right\} + \frac{\partial}{\partial r_i}\left\{ \frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \int f v_i\, \mathrm{d}\boldsymbol{v} \right\} \right] \\
&= \frac{K}{g}\frac{3}{2\Theta} \left[\frac{\partial}{\partial t}\left\{ \frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} n \right\} + \frac{\partial}{\partial r_i}\left\{ \frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} n V_i \right\} \right] \\
&= \frac{K}{g}\frac{3}{2\Theta} \left[\frac{\partial}{\partial t}\left\{ \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \right\} + \frac{\partial}{\partial r_i}\left\{ \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} V_i \right\} \right] \\
&= \frac{K}{g}\frac{3}{2\Theta} \left[\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \frac{\partial\Theta}{\partial t} + V_i \left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \frac{\partial\Theta}{\partial r_i} + \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \frac{\partial V_i}{\partial r_i} \right] \\
&= \frac{K}{g}\frac{3}{2\Theta} \left[\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \left(\frac{\partial}{\partial t} + V_i \frac{\partial}{\partial r_i}\right)\Theta + \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} \frac{\partial V_i}{\partial r_i} \right] \\
&= \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \left[\frac{\mathrm{D}\Theta}{\mathrm{D}t} + \frac{2\Theta}{3}\frac{\partial V_i}{\partial r_i} \right] \\
&= \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \left[-\Gamma - \frac{2}{n}\frac{\partial V_i}{\partial r_j} P_{ij} - \frac{2}{n}\frac{\partial Q_i}{\partial r_i} + \frac{2\Theta}{3}\frac{\partial V_i}{\partial r_i} \right]. \quad \text{(using eq. (2.10))}
\end{aligned}$$

But from eq. (2.23), $\frac{3}{2}\epsilon\tilde{\Gamma} = \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\Gamma$. Hence,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{\mathscr{D}}\tilde{f} = -\frac{3}{2}\epsilon\tilde{\Gamma} - \frac{2}{n}\frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \left[\frac{\partial V_i}{\partial r_j}\left(P_{ij} - \frac{1}{3}n\Theta\delta_{ij}\right) + \frac{\partial Q_i}{\partial r_i} \right]$$

or

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{\mathscr{D}}\tilde{f} = -\frac{2}{n}\frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \left(\frac{\partial V_i}{\partial r_j} P'_{ij} + \frac{\partial Q_i}{\partial r_i} \right) - \frac{3}{2}\epsilon\tilde{\Gamma} \tag{B.2}$$

where $P'_{ij} = P_{ij} - \frac{1}{3}n\Theta\delta_{ij}$ is the deviatoric stress tensor. Similarly, the integration involving the summational invariant, $\tilde{\boldsymbol{u}}$ is

$$\begin{aligned}
\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i \tilde{\mathscr{D}}\tilde{f} &= \int \left\{ \left(\frac{3}{2\Theta}\right)^{\frac{3}{2}} \mathrm{d}\boldsymbol{v} \right\} \left\{ \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} (v_i - V_i) \right\} \left\{ \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right) \right\} \tilde{f} \\
&= \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}} \int \mathrm{d}\boldsymbol{v}\, (v_i - V_i) \left(\frac{\partial}{\partial t} + v_j \frac{\partial}{\partial r_j}\right) \left\{ \frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} f \right\}
\end{aligned}$$

or

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}_i\tilde{\mathscr{D}}\tilde{f} = \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\left[\frac{\partial}{\partial t}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int f v_i\,\mathrm{d}\boldsymbol{v}\right\} - V_i\frac{\partial}{\partial t}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int f\,\mathrm{d}\boldsymbol{v}\right\}\right.$$
$$\left. + \frac{\partial}{\partial r_j}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int f v_i v_j\,\mathrm{d}\boldsymbol{v}\right\} - V_i\frac{\partial}{\partial r_j}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\int f v_j\,\mathrm{d}\boldsymbol{v}\right\}\right].$$

Let us evaluate the integral involved in the third term on the RHS of above expression separately,

$$\because \quad \int f u_i\,\mathrm{d}\boldsymbol{v} = \int f(v_i - V_i)\,\mathrm{d}\boldsymbol{v} = \int f v_i\,\mathrm{d}\boldsymbol{v} - V_i\int f\,\mathrm{d}\boldsymbol{v} = n\,V_i - V_i\,n = 0$$
$$\therefore \quad \int f v_i v_j\,\mathrm{d}\boldsymbol{v} = \int f(u_i + V_i)(u_j + V_j)\,\mathrm{d}\boldsymbol{v}$$
$$= \int f u_i u_j\,\mathrm{d}\boldsymbol{v} + V_i\int f u_j\,\mathrm{d}\boldsymbol{v} + V_j\int f u_i\,\mathrm{d}\boldsymbol{v} + V_iV_j\int f\,\mathrm{d}\boldsymbol{v} = P_{ij} + 0 + 0 + nV_iV_j$$
$$\Rightarrow \quad \int f v_i v_j\,\mathrm{d}\boldsymbol{v} = P_{ij} + nV_iV_j.$$

Hence,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}_i\tilde{\mathscr{D}}\tilde{f} = \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\left[\frac{\partial}{\partial t}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}nV_i\right\} - V_i\frac{\partial}{\partial t}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}n\right\}\right.$$
$$\left. + \frac{\partial}{\partial r_j}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}(P_{ij} + nV_iV_j)\right\} - V_i\frac{\partial}{\partial r_j}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}nV_j\right\}\right]$$
$$= \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\left[\frac{\partial}{\partial t}\left\{\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}V_i\right\} - V_i\frac{\partial}{\partial t}\left\{\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\right\} + \frac{\partial}{\partial r_j}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}P_{ij}\right\}\right.$$
$$\left. + \frac{\partial}{\partial r_j}\left\{\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}V_iV_j\right\} - V_i\frac{\partial}{\partial r_j}\left\{\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}V_j\right\}\right]$$
$$= \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\left[\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_i}{\partial t} + P_{ij}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\left(-\frac{1}{n^2}\right)\frac{\partial n}{\partial r_j} + P_{ij}\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial \Theta}{\partial r_j}\right.$$
$$\left. + \frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial P_{ij}}{\partial r_j} + \left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}V_j\frac{\partial V_i}{\partial r_j}\right]$$
$$= \frac{K}{g}\left[\left(\frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\right)V_i - P_{ij}\frac{1}{n^2}\frac{\partial n}{\partial r_j} + P_{ij}\frac{3}{2n\Theta}\frac{\partial \Theta}{\partial r_j} + \frac{1}{n}\frac{\partial P_{ij}}{\partial r_j}\right]$$
$$= \frac{1}{n}\frac{K}{g}\left[n\frac{\mathrm{D}V_i}{\mathrm{D}t} - P_{ij}\frac{1}{n}\frac{\partial n}{\partial r_j} + P_{ij}\frac{3}{2\Theta}\frac{\partial \Theta}{\partial r_j} + \frac{\partial P_{ij}}{\partial r_j}\right]$$
$$= \frac{1}{n}\frac{K}{g}\left[\left(n\,g_i - \frac{\partial P_{ij}}{\partial r_j}\right) - P_{ij}\frac{\partial \ln n}{\partial r_j} + P_{ij}\frac{3}{2}\frac{\partial \ln \Theta}{\partial r_j} + \frac{\partial P_{ij}}{\partial r_j}\right] \quad \text{(Using eq. (2.9))}$$
$$= K\tilde{g}_i + \frac{1}{n}\frac{K}{g}\left[-P_{ij}\frac{\partial \ln n}{\partial r_j} + P_{ij}\frac{3}{2}\frac{\partial \ln \Theta}{\partial r_j}\right]$$

or

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}_i\tilde{\mathscr{D}}\tilde{f} = K\tilde{g}_i + \frac{1}{n}\frac{K}{g}P_{ij}\frac{\partial}{\partial r_j}\left(\frac{3}{2}\ln\Theta - \ln n\right) \tag{B.3}$$

and the integration involving the summational invariant, $\tilde{u}^2$ is

$$\begin{aligned}
\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{\mathscr{D}} \tilde{f} &= \int \left\{ \left(\frac{3}{2\Theta} \right)^{\frac{3}{2}} \mathrm{d}\boldsymbol{v} \right\} \left\{ \frac{3}{2\Theta} (\boldsymbol{v} - \boldsymbol{V})^2 \right\} \left\{ \frac{K}{g} \left(\frac{2\Theta}{3} \right)^{\frac{1}{2}} \left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \right) \right\} \tilde{f} \\
&= \frac{K}{g} \left(\frac{3}{2\Theta} \right)^2 \int \mathrm{d}\boldsymbol{v}\, (v^2 + V^2 - 2v_j V_j) \left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial r_i} \right) \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} f \right\} \\
&= \frac{K}{g} \left(\frac{3}{2\Theta} \right)^2 \left[\frac{\partial}{\partial t} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} \int f v^2 \,\mathrm{d}\boldsymbol{v} \right\} + \frac{\partial}{\partial r_i} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} \int f v^2 v_i \,\mathrm{d}\boldsymbol{v} \right\} \right. \\
&\quad + V^2 \frac{\partial}{\partial t} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} \int f \,\mathrm{d}\boldsymbol{v} \right\} + V^2 \frac{\partial}{\partial r_i} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} \int f v_i \,\mathrm{d}\boldsymbol{v} \right\} \\
&\quad \left. - 2V_j \frac{\partial}{\partial t} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} \int f v_j \,\mathrm{d}\boldsymbol{v} \right\} - 2V_j \frac{\partial}{\partial r_i} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} \int f v_i v_j \,\mathrm{d}\boldsymbol{v} \right\} \right]
\end{aligned}$$

Let us evaluate the following integrals involved in the above equation separately,

$$\begin{aligned}
&\int f v^2 \,\mathrm{d}\boldsymbol{v} = \int f (\boldsymbol{u} + \boldsymbol{V})^2 \,\mathrm{d}\boldsymbol{v} = \int f (u^2 + V^2 + 2u_j V_j) \,\mathrm{d}\boldsymbol{v} \\
&= \int f u^2 \,\mathrm{d}\boldsymbol{v} + V^2 \int f \,\mathrm{d}\boldsymbol{v} + 2V_j \int f u_j \,\mathrm{d}\boldsymbol{v} = n\Theta + V^2 \times n + 2V_j \times 0 = n(\Theta + V^2),
\end{aligned}$$

and

$$\begin{aligned}
&\int f v^2 v_i \,\mathrm{d}\boldsymbol{v} = \int f (\boldsymbol{u} + \boldsymbol{V})^2 (u_i + V_i) \,\mathrm{d}\boldsymbol{v} = \int f (u^2 + V^2 + 2u_j V_j)(u_i + V_i) \,\mathrm{d}\boldsymbol{v} \\
&= \int f u^2 u_i \,\mathrm{d}\boldsymbol{v} + V^2 \int f u_i \,\mathrm{d}\boldsymbol{v} + 2V_j \int f u_i u_j \,\mathrm{d}\boldsymbol{v} + V_i \int f u^2 \,\mathrm{d}\boldsymbol{v} + V^2 V_i \int f \,\mathrm{d}\boldsymbol{v} + 2V_i V_j \int f u_j \,\mathrm{d}\boldsymbol{v} \\
&= 2Q_i + 0 + 2V_j P_{ij} + V_i \times n\Theta + V^2 V_i \times n + 0 = 2(Q_i + V_j P_{ij}) + nV_i(\Theta + V^2).
\end{aligned}$$

Hence

$$\begin{aligned}
&\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{\mathscr{D}} \tilde{f} \\
&= \frac{K}{g} \left(\frac{3}{2\Theta} \right)^2 \left[\frac{\partial}{\partial t} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} n(\Theta + V^2) \right\} + \frac{\partial}{\partial r_i} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} \left(2(Q_i + V_j P_{ij}) + nV_i(\Theta + V^2) \right) \right\} \right. \\
&\quad + V^2 \frac{\partial}{\partial t} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} n \right\} + V^2 \frac{\partial}{\partial r_i} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} nV_i \right\} - 2V_j \frac{\partial}{\partial t} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} nV_j \right\} \\
&\quad \left. - 2V_j \frac{\partial}{\partial r_i} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} (P_{ij} + nV_i V_j) \right\} \right] \\
&= \frac{K}{g} \left(\frac{3}{2\Theta} \right)^2 \left[\frac{\partial}{\partial t} \left\{ \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} (\Theta + V^2) \right\} + 2 \frac{\partial}{\partial r_i} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} (Q_i + V_j P_{ij}) \right\} \right. \\
&\quad + \frac{\partial}{\partial r_i} \left\{ \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} V_i (\Theta + V^2) \right\} + V^2 \frac{\partial}{\partial t} \left\{ \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} \right\} + V^2 \frac{\partial}{\partial r_i} \left\{ \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} V_i \right\} \\
&\quad \left. - 2V_j \frac{\partial}{\partial t} \left\{ \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} V_j \right\} - 2V_j \frac{\partial}{\partial r_i} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} P_{ij} \right\} - 2V_j \frac{\partial}{\partial r_i} \left\{ \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} V_i V_j \right\} \right]
\end{aligned}$$

or

$$
\begin{aligned}
\int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}^2\tilde{\mathscr{D}}\tilde{f} = {} & \frac{K}{g}\left(\frac{3}{2\Theta}\right)^2\Bigg[(\Theta+V^2)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial t}+\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial}{\partial t}(\Theta+V^2) \\
& +\frac{2}{n}(Q_i+V_jP_{ij})\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial r_i}+2\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial}{\partial r_i}\left\{\frac{1}{n}(Q_i+V_jP_{ij})\right\} \\
& +V_i(\Theta+V^2)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial r_i}+\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial}{\partial r_i}\left\{V_i(\Theta+V^2)\right\}+V^2\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial t} \\
& +V^2V_i\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial r_i}+V^2\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_i}{\partial r_i}-2V_j\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_j}{\partial t}-2V^2\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial t} \\
& -\frac{2}{n}V_jP_{ij}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial r_i}-2V_j\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial}{\partial r_i}\left\{\frac{1}{n}P_{ij}\right\}-2V^2V_i\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial r_i} \\
& -2V^2\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_i}{\partial r_i}-2V_iV_j\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_j}{\partial r_i}\Bigg] \\
= {} & \frac{K}{g}\left(\frac{3}{2\Theta}\right)^2\Bigg[(\Theta+V^2)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+V_i\frac{\partial}{\partial r_i}\right)\Theta+\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\Theta}{\partial t} \\
& +\underbrace{2V_j\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_j}{\partial t}}+\frac{2}{n}Q_i\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial r_i}-\frac{2}{n^2}Q_i\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial n}{\partial r_i}+\frac{2}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i} \\
& +\underbrace{2V_j\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial}{\partial r_i}\left\{\frac{1}{n}P_{ij}\right\}}+\frac{2}{n}P_{ij}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_j}{\partial r_i}+\Theta\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_i}{\partial r_i}+\underbrace{V^2\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_i}{\partial r_i}} \\
& +V_i\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial\Theta}{\partial r_i}+\underbrace{2V_iV_j\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_j}{\partial r_i}}+V^2\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial t}+V^2V_i\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial r_i} \\
& +\underbrace{V^2\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_i}{\partial r_i}}-\underbrace{2V_j\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_j}{\partial t}}-2V^2\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial t}-\underbrace{2V_j\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial}{\partial r_i}\left\{\frac{1}{n}P_{ij}\right\}} \\
& -2V^2V_i\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial r_i}-\underbrace{2V^2\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_i}{\partial r_i}}-\underbrace{2V_iV_j\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_j}{\partial r_i}}\Bigg]
\end{aligned}
$$

where the colored underbraces show the corresponding terms cancelling with each other. Hence

$$
\begin{aligned}
\int \mathrm{d}\tilde{\boldsymbol{u}}\,\tilde{u}^2\tilde{\mathscr{D}}\tilde{f} = {} & \frac{K}{g}\left(\frac{3}{2\Theta}\right)^2\Bigg[(\Theta+V^2)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\mathrm{D}\Theta}{\mathrm{D}t}+\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\left(\frac{\partial}{\partial t}+V_i\frac{\partial}{\partial r_i}\right)\Theta+\frac{2}{n}Q_i\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\Theta}{\partial r_i} \\
& -\frac{2}{n^2}Q_i\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial n}{\partial r_i}+\frac{2}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}+\frac{2}{n}P_{ij}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_j}{\partial r_i}+\Theta\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}\frac{\partial V_i}{\partial r_i} \\
& -V^2\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+V_i\frac{\partial}{\partial r_i}\right)\Theta\Bigg] \\
= {} & \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\Bigg[(\Theta+V^2)\frac{3}{2\Theta}\frac{\mathrm{D}\Theta}{\mathrm{D}t}+\frac{\mathrm{D}\Theta}{\mathrm{D}t}+\frac{3}{n\Theta}Q_i\frac{\partial\Theta}{\partial r_i}-\frac{2}{n^2}Q_i\frac{\partial n}{\partial r_i}+\frac{2}{n}\frac{\partial Q_i}{\partial r_i}+\frac{2}{n}P_{ij}\frac{\partial V_i}{\partial r_j} \\
& +\Theta\frac{\partial V_i}{\partial r_i}-V^2\frac{3}{2\Theta}\frac{\mathrm{D}\Theta}{\mathrm{D}t}\Bigg].
\end{aligned}
$$

Using eq. (2.10),

$$\begin{aligned}
\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{\mathscr{D}} \tilde{f} &= \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \left[\Theta \times \frac{3}{2\Theta}\left(-\Gamma - \frac{2}{n}\frac{\partial V_i}{\partial r_j}P_{ij} - \frac{2}{n}\frac{\partial Q_i}{\partial r_i}\right) + \left(-\Gamma - \frac{2}{n}\frac{\partial V_i}{\partial r_j}P_{ij} - \frac{2}{n}\frac{\partial Q_i}{\partial r_i}\right)\right. \\
&\quad \left. + \frac{3}{n\Theta}Q_i\frac{\partial \Theta}{\partial r_i} - \frac{2}{n^2}Q_i\frac{\partial n}{\partial r_i} + \frac{2}{n}\frac{\partial Q_i}{\partial r_i} + \frac{2}{n}P_{ij}\frac{\partial V_i}{\partial r_j} + \Theta\frac{\partial V_i}{\partial r_i}\right] \\
&= \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left[-\frac{5}{2}\Gamma - \frac{3}{n}P_{ij}\frac{\partial V_i}{\partial r_j} - \frac{3}{n}\frac{\partial Q_i}{\partial r_i} + \frac{3}{n\Theta}Q_i\frac{\partial \Theta}{\partial r_i} - \frac{2}{n^2}Q_i\frac{\partial n}{\partial r_i} + \Theta\frac{\partial V_i}{\partial r_i}\right] \\
&= \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left[-\frac{5}{2}\Gamma - \frac{3}{n}\left(P_{ij} - \frac{1}{3}n\Theta\delta_{ij}\right)\frac{\partial V_i}{\partial r_j} - \frac{3}{n}\frac{\partial Q_i}{\partial r_i} + \frac{3}{n}Q_i\frac{\partial \ln\Theta}{\partial r_i} - \frac{2}{n}Q_i\frac{\partial \ln n}{\partial r_i}\right] \\
&= \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{2}{n}\left[-\frac{3}{2}P'_{ij}\frac{\partial V_i}{\partial r_j} - \frac{3}{2}\frac{\partial Q_i}{\partial r_i} + \frac{3}{2}Q_i\frac{\partial \ln\Theta}{\partial r_i} - Q_i\frac{\partial \ln n}{\partial r_i}\right] - \frac{5}{2}\frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\Gamma,
\end{aligned}$$

where $P'_{ij} = P_{ij} - \frac{1}{3}n\Theta\delta_{ij}$ is the deviatoric stress tensor. From eq. (2.23), $\frac{3}{2}\epsilon\tilde{\Gamma} = \frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\Gamma$, therefore

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}^2 \tilde{\mathscr{D}} \tilde{f} = -\frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{2}{n}\left[Q_i\frac{\partial \ln n}{\partial r_i} - \frac{3}{2}Q_i\frac{\partial \ln\Theta}{\partial r_i} + \frac{3}{2}P'_{ij}\frac{\partial V_i}{\partial r_j} + \frac{3}{2}\frac{\partial Q_i}{\partial r_i}\right] - \frac{15}{4}\epsilon\tilde{\Gamma}. \qquad \text{(B.4)}$$

The integral over $\tilde{\boldsymbol{u}}$ of the second term on the RHS of eq. (B.1) times any of the summational invariants can be carried out as follows:
Let us first simplify the term $\tilde{\mathscr{D}}\left(\ln n - \frac{3}{2}\ln\Theta\right)$ using eqs. (2.20) and (2.22).

$$\begin{aligned}
&\tilde{\mathscr{D}}\left(\ln n - \frac{3}{2}\ln\Theta\right) \\
&= K\frac{2\Theta}{3g}\left(\tilde{u}_i\frac{\partial \ln n}{\partial r_i} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right) \\
&\quad - \frac{3}{2}\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial \ln\Theta}{\partial r_j} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}P_{ij} - \frac{2}{n\Theta}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial Q_i}{\partial r_i}\right\} - \epsilon\tilde{\Gamma}\right] \\
&= K\frac{2\Theta}{3g}\left[\tilde{u}_i\frac{\partial \ln n}{\partial r_i} - \frac{2\Theta}{3}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial V_i}{\partial r_j}\delta_{ij} - \frac{3}{2}\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} + \frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}P_{ij}\frac{\partial V_i}{\partial r_j} + \frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}\right] \\
&\quad + \frac{3}{2}\epsilon\tilde{\Gamma} \\
&= K\frac{2\Theta}{3g}\left[\tilde{u}_i\frac{\partial \ln n}{\partial r_i} - \frac{3}{2}\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} + \frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\left(P_{ij} - \frac{1}{3}n\Theta\delta_{ij}\right)\frac{\partial V_i}{\partial r_j} + \frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}\right] + \frac{3}{2}\epsilon\tilde{\Gamma} \\
&= K\frac{2\Theta}{3g}\left[\tilde{u}_i\frac{\partial \ln n}{\partial r_i} - \frac{3}{2}\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} + \frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}P'_{ij}\frac{\partial V_i}{\partial r_j} + \frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}\right] + \frac{3}{2}\epsilon\tilde{\Gamma}.
\end{aligned}$$

Now, the integration (of 2^{nd} term of eq. (B.1)) involving the summational invariant, 1 is:

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}\tilde{\mathscr{D}}\left(\ln n - \frac{3}{2}\ln\Theta\right) = \int\left\{\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\mathrm{d}\boldsymbol{v}\right\}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}f\right\}\tilde{\mathscr{D}}\left(\ln n - \frac{3}{2}\ln\Theta\right)$$

or

$$
\begin{aligned}
&\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}\tilde{\mathscr{D}}\left(\ln n-\frac{3}{2}\ln\Theta\right)\\
&=\frac{1}{n}\int \mathrm{d}\boldsymbol{v}\, f\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial\ln n}{\partial r_i}-\frac{3}{2}\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}+\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}P'_{ij}\frac{\partial V_i}{\partial r_j}+\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}\right\}+\frac{3}{2}\epsilon\tilde{\Gamma}\right]\\
&=K\frac{2\Theta}{3g}\left(\frac{\partial\ln n}{\partial r_i}-\frac{3}{2}\frac{\partial\ln\Theta}{\partial r_i}\right)\frac{1}{n}\int \mathrm{d}\boldsymbol{v}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(v_i-V_i)\right\}f\\
&\quad+\left[K\frac{2\Theta}{3g}\left\{\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}P'_{ij}\frac{\partial V_i}{\partial r_j}+\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}\right\}+\frac{3}{2}\epsilon\tilde{\Gamma}\right]\frac{1}{n}\int \mathrm{d}\boldsymbol{v}\, f\qquad\left(\because\tilde{\Gamma}=\tilde{\Gamma}(\boldsymbol{r},t)\right)\\
&=K\frac{2\Theta}{3g}\left(\frac{\partial\ln n}{\partial r_i}-\frac{3}{2}\frac{\partial\ln\Theta}{\partial r_i}\right)\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\frac{1}{n}\int \mathrm{d}\boldsymbol{v}\, v_i f-\frac{1}{n}V_i\int \mathrm{d}\boldsymbol{v}\, f\right\}\\
&\quad+\left[K\frac{2\Theta}{3g}\left\{\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}P'_{ij}\frac{\partial V_i}{\partial r_j}+\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}\right\}+\frac{3}{2}\epsilon\tilde{\Gamma}\right]\frac{1}{n}\times n\\
&=K\frac{2\Theta}{3g}\left(\frac{\partial\ln n}{\partial r_i}-\frac{3}{2}\frac{\partial\ln\Theta}{\partial r_i}\right)\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\frac{1}{n}\times nV_i-\frac{1}{n}V_i\times n\right\}\\
&\quad+\left[K\frac{2\Theta}{3g}\left\{\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}P'_{ij}\frac{\partial V_i}{\partial r_j}+\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}\right\}+\frac{3}{2}\epsilon\tilde{\Gamma}\right]
\end{aligned}
$$

or

$$
\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{f}\tilde{\mathscr{D}}\left(\ln n-\frac{3}{2}\ln\Theta\right)=\frac{2}{n}\frac{K}{g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(P'_{ij}\frac{\partial V_i}{\partial r_j}+\frac{\partial Q_i}{\partial r_i}\right)+\frac{3}{2}\epsilon\tilde{\Gamma}. \tag{B.5}
$$

Similarly, the integration (of 2nd term of eq. (B.1)) involving the summational invariant, $\tilde{\boldsymbol{u}}$ is:

$$
\begin{aligned}
&\int \mathrm{d}\tilde{\boldsymbol{u}}\ \tilde{u}_k\tilde{f}\tilde{\mathscr{D}}\left(\ln n-\frac{3}{2}\ln\Theta\right)\\
&=\int\left\{\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\mathrm{d}\boldsymbol{v}\right\}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}u_k\right\}\left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}}f\right\}\tilde{\mathscr{D}}\left(\ln n-\frac{3}{2}\ln\Theta\right)\\
&=\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{1}{n}\int \mathrm{d}\boldsymbol{v}\, u_k f\\
&\quad\times\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial\ln n}{\partial r_i}-\frac{3}{2}\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}+\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}P'_{ij}\frac{\partial V_i}{\partial r_j}+\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}\right\}+\frac{3}{2}\epsilon\tilde{\Gamma}\right]\\
&=K\frac{2\Theta}{3g}\left(\frac{\partial\ln n}{\partial r_i}-\frac{3}{2}\frac{\partial\ln\Theta}{\partial r_i}\right)\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{1}{n}\int \mathrm{d}\boldsymbol{v}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}u_i\right\}u_k f\\
&\quad+\left[K\frac{2\Theta}{3g}\left\{\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}P'_{ij}\frac{\partial V_i}{\partial r_j}+\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}\right\}+\frac{3}{2}\epsilon\tilde{\Gamma}\right]\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{1}{n}\int \mathrm{d}\boldsymbol{v}\, u_k f\\
&=K\frac{2\Theta}{3g}\left(\frac{\partial\ln n}{\partial r_i}-\frac{3}{2}\frac{\partial\ln\Theta}{\partial r_i}\right)\frac{3}{2n\Theta}\int \mathrm{d}\boldsymbol{v}\, u_k u_i f\\
&\quad+\left[K\frac{2\Theta}{3g}\left\{\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}P'_{ij}\frac{\partial V_i}{\partial r_j}+\frac{2}{n}\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\frac{\partial Q_i}{\partial r_i}\right\}+\frac{3}{2}\epsilon\tilde{\Gamma}\right]\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{1}{n}\int \mathrm{d}\boldsymbol{v}\, u_k f.
\end{aligned}
$$

Here, $\int \mathrm{d}\boldsymbol{v}\, u_k u_i f = P_{ki}$ and $\int \mathrm{d}\boldsymbol{v}\, u_k f = 0$ as above. Hence,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_k \tilde{f} \tilde{\mathscr{D}} \left(\ln n - \frac{3}{2} \ln \Theta \right) = K \frac{2\Theta}{3g} \left(\frac{\partial \ln n}{\partial r_i} - \frac{3}{2} \frac{\partial \ln \Theta}{\partial r_i} \right) \frac{3}{2n\Theta} P_{ki}$$

or

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, \tilde{u}_i \tilde{f} \tilde{\mathscr{D}} \left(\ln n - \frac{3}{2} \ln \Theta \right) = -\frac{K}{g} \frac{1}{n} P_{ij} \frac{\partial}{\partial r_j} \left(\frac{3}{2} \ln \Theta - \ln n \right) \tag{B.6}$$

and the integration (of 2nd term of eq. (B.1)) involving the summational invariant, $\tilde{u}^2$ is:

$$\begin{aligned}
&\int \mathrm{d}\tilde{\boldsymbol{u}}\ \tilde{u}^2 \tilde{f} \tilde{\mathscr{D}} \left(\ln n - \frac{3}{2} \ln \Theta \right) \\
&= \int \left\{ \left(\frac{3}{2\Theta} \right)^{\frac{3}{2}} \mathrm{d}\boldsymbol{v} \right\} \left\{ \left(\frac{3}{2\Theta} \right) u^2 \right\} \left\{ \frac{1}{n} \left(\frac{2\Theta}{3} \right)^{\frac{3}{2}} f \right\} \tilde{\mathscr{D}} \left(\ln n - \frac{3}{2} \ln \Theta \right) \\
&= \frac{3}{2n\Theta} \int \mathrm{d}\boldsymbol{v}\, u^2 f \\
&\quad \times \left[K \frac{2\Theta}{3g} \left\{ \tilde{u}_i \frac{\partial \ln n}{\partial r_i} - \frac{3}{2} \tilde{u}_i \frac{\partial \ln \Theta}{\partial r_i} + \frac{2}{n} \left(\frac{3}{2\Theta} \right)^{\frac{3}{2}} P'_{ij} \frac{\partial V_i}{\partial r_j} + \frac{2}{n} \left(\frac{3}{2\Theta} \right)^{\frac{3}{2}} \frac{\partial Q_i}{\partial r_i} \right\} + \frac{3}{2} \epsilon \tilde{\Gamma} \right] \\
&= K \frac{2\Theta}{3g} \left(\frac{\partial \ln n}{\partial r_i} - \frac{3}{2} \frac{\partial \ln \Theta}{\partial r_i} \right) \frac{3}{2n\Theta} \int \mathrm{d}\boldsymbol{v} \left\{ \left(\frac{3}{2\Theta} \right)^{\frac{1}{2}} u_i \right\} u^2 f \\
&\quad + \left[K \frac{2\Theta}{3g} \left\{ \frac{2}{n} \left(\frac{3}{2\Theta} \right)^{\frac{3}{2}} P'_{ij} \frac{\partial V_i}{\partial r_j} + \frac{2}{n} \left(\frac{3}{2\Theta} \right)^{\frac{3}{2}} \frac{\partial Q_i}{\partial r_i} \right\} + \frac{3}{2} \epsilon \tilde{\Gamma} \right] \frac{3}{2n\Theta} \int \mathrm{d}\boldsymbol{v}\, u^2 f.
\end{aligned}$$

Using the definitions of granular temperature (eq. (2.7)) and heat flux (eq. (2.12)),

$$\begin{aligned}
&\int \mathrm{d}\tilde{\boldsymbol{u}}\ \tilde{u}^2 \tilde{f} \tilde{\mathscr{D}} \left(\ln n - \frac{3}{2} \ln \Theta \right) \\
&= K \frac{2\Theta}{3g} \left(\frac{\partial \ln n}{\partial r_i} - \frac{3}{2} \frac{\partial \ln \Theta}{\partial r_i} \right) \frac{3}{2n\Theta} \left(\frac{3}{2\Theta} \right)^{\frac{1}{2}} \times 2Q_i \\
&\quad + \left[K \frac{2\Theta}{3g} \left\{ \frac{2}{n} \left(\frac{3}{2\Theta} \right)^{\frac{3}{2}} P'_{ij} \frac{\partial V_i}{\partial r_j} + \frac{2}{n} \left(\frac{3}{2\Theta} \right)^{\frac{3}{2}} \frac{\partial Q_i}{\partial r_i} \right\} + \frac{3}{2} \epsilon \tilde{\Gamma} \right] \frac{3}{2n\Theta} \times n\Theta \\
&= \frac{K}{g} \left(\frac{3}{2\Theta} \right)^{\frac{1}{2}} \frac{2}{n} \left[Q_i \frac{\partial \ln n}{\partial r_i} - \frac{3}{2} Q_i \frac{\partial \ln \Theta}{\partial r_i} + \frac{3}{2} P'_{ij} \frac{\partial V_i}{\partial r_j} + \frac{3}{2} \frac{\partial Q_i}{\partial r_i} \right] + \frac{9}{4} \epsilon \tilde{\Gamma}.
\end{aligned} \tag{B.7}$$

The integral over $\tilde{\boldsymbol{u}}_1$ of the third term on the RHS of eq. (B.1) times any of the summational invariants vanishes as following. Let $\psi(\tilde{\boldsymbol{u}}_1)$ be any of the summational invariants: 1, $\tilde{\boldsymbol{u}}_1$ and $\tilde{u}_1^2$, and $I = \int \psi(\tilde{\boldsymbol{u}}_1) \frac{1}{2} \tilde{f}_0(\tilde{u}_1) \tilde{\Omega}(\Phi, \Phi)\, \mathrm{d}\tilde{\boldsymbol{u}}_1$. Substituting the values of $\tilde{f}_0$ and $\tilde{\Omega}$ from eqs. (2.17) and (2.38) respectively, I changes to

$$I = \frac{1}{\pi^4} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\, (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}) \psi(\tilde{\boldsymbol{u}}_1)\, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)} \left\{ \Phi(\tilde{\boldsymbol{u}}_1') \Phi(\tilde{\boldsymbol{u}}_2') - \Phi(\tilde{\boldsymbol{u}}_1) \Phi(\tilde{\boldsymbol{u}}_2) \right\}. \tag{*1}$$

Further let us interchange the dummy variables $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ in the above equation. Note that, (even for the inelastic case) $\tilde{\boldsymbol{u}}_1 \longleftrightarrow \tilde{\boldsymbol{u}}_2 \Rightarrow \tilde{\boldsymbol{u}}_1' \longleftrightarrow \tilde{\boldsymbol{u}}_2'$, which can be seen as follows. By the definitions of $\tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2'$ (cf. eqs. (2.1) and (2.2)),

$$\tilde{\boldsymbol{u}}_1' = \tilde{\boldsymbol{u}}_1 - \frac{(1+e)}{2e}\left(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}\right)\hat{\boldsymbol{k}}$$
$$\tilde{\boldsymbol{u}}_2' = \tilde{\boldsymbol{u}}_2 + \frac{(1+e)}{2e}\left(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}\right)\hat{\boldsymbol{k}}.$$

Now if we interchange $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$, $\tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2'$ change as follows

$$\tilde{\boldsymbol{u}}_1' \rightarrow \tilde{\boldsymbol{u}}_2 - \frac{(1+e)}{2e}\left(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{21}\right)\hat{\boldsymbol{k}} = \tilde{\boldsymbol{u}}_2 + \frac{(1+e)}{2e}\left(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}\right)\hat{\boldsymbol{k}} = \tilde{\boldsymbol{u}}_2'$$
$$\tilde{\boldsymbol{u}}_2' \rightarrow \tilde{\boldsymbol{u}}_1 + \frac{(1+e)}{2e}\left(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{21}\right)\hat{\boldsymbol{k}} = \tilde{\boldsymbol{u}}_1 - \frac{(1+e)}{2e}\left(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}\right)\hat{\boldsymbol{k}} = \tilde{\boldsymbol{u}}_1'.$$

That means, on interchanging $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$, $\tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2'$ interchange by themselves and vice-versa. Therefore, on interchanging the dummy variables $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ in eq. (*1), we get

$$I = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{21}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{21})\psi(\tilde{\boldsymbol{u}}_2)\,\mathrm{e}^{-(\tilde{u}_2^2+\tilde{u}_1^2)}\left\{\Phi(\tilde{\boldsymbol{u}}_2')\Phi(\tilde{\boldsymbol{u}}_1') - \Phi(\tilde{\boldsymbol{u}}_2)\Phi(\tilde{\boldsymbol{u}}_1)\right\}$$
$$= \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}<0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\psi(\tilde{\boldsymbol{u}}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\left\{\Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2') - \Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2)\right\}.$$

Using eq. (F.8),

$$I = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\psi(\tilde{\boldsymbol{u}}_2)\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\left\{\Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2') - \Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2)\right\}. \qquad (*2)$$

Adding eqs. (*1) and (*2),

$$2I = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\{\psi(\tilde{\boldsymbol{u}}_1)+\psi(\tilde{\boldsymbol{u}}_2)\}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\left\{\Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2') - \Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2)\right\} \quad (*3)$$

and now interchanging the dummy variables $\tilde{\boldsymbol{u}}_1 \longleftrightarrow \tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2 \longleftrightarrow \tilde{\boldsymbol{u}}_2'$,

$$2I = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0} \mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\{\psi(\tilde{\boldsymbol{u}}_1')+\psi(\tilde{\boldsymbol{u}}_2')\}\,\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\left\{\Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2) - \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2')\right\}$$

but since the above integral uses the elastic velocity transformation, we can again change the integration variables using the relations, $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}' = -\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}$, $\tilde{\boldsymbol{u}}_1'+\tilde{\boldsymbol{u}}_2' = \tilde{\boldsymbol{u}}_1+\tilde{\boldsymbol{u}}_2$, $\tilde{u}_1'^2+\tilde{u}_2'^2 = \tilde{u}_1^2+\tilde{u}_2^2$ and $\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2' = \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2$. Thus

$$2I = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}<0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\{\psi(\tilde{\boldsymbol{u}}_1')+\psi(\tilde{\boldsymbol{u}}_2')\}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\left\{\Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2) - \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2')\right\}.$$

Using eq. (F.8),

$$2I = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\{\psi(\tilde{\boldsymbol{u}}_1')+\psi(\tilde{\boldsymbol{u}}_2')\}\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\left\{\Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2) - \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2')\right\} \qquad (*4)$$

Adding eqs. (*3) and (*4), we get

$$4I = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\{\psi(\tilde{\boldsymbol{u}}_1)+\psi(\tilde{\boldsymbol{u}}_2)-\psi(\tilde{\boldsymbol{u}}_1')-\psi(\tilde{\boldsymbol{u}}_2')\}$$
$$\times\left\{\Phi(\tilde{\boldsymbol{u}}_1)\Phi(\tilde{\boldsymbol{u}}_2) - \Phi(\tilde{\boldsymbol{u}}_1')\Phi(\tilde{\boldsymbol{u}}_2')\right\}.$$

Note that the above integral uses elastic velocity transformation and therefore for any of the summational invariants: 1, $\tilde{\boldsymbol{u}}_1$ and $\tilde{u}_1^2$,

$$\psi(\tilde{\boldsymbol{u}}_1) + \psi(\tilde{\boldsymbol{u}}_2) - \psi(\tilde{\boldsymbol{u}}_1') - \psi(\tilde{\boldsymbol{u}}_2') = 0$$

due to conservation of mass, momentum and energy, respectively. Hence

$$4I = 0 \quad \text{or} \quad I = 0$$

or

$$\int \psi(\tilde{\boldsymbol{u}}_1)\frac{1}{2}\tilde{f}_0(\tilde{u}_1)\tilde{\Omega}(\Phi,\Phi)\,\mathrm{d}\tilde{\boldsymbol{u}}_1 = 0 \quad \text{for} \quad \psi(\tilde{\boldsymbol{u}}_1) = 1,\ \tilde{\boldsymbol{u}}_1 \text{ and } \tilde{u}_1^2. \tag{B.8}$$

The integral over $\tilde{\boldsymbol{u}}_1$ of the fourth term on the RHS of eq. (B.1) times any of the summational invariants also vanishes due to the conservation of mass, momentum and energy for the elastic particles. To verify this one can simply replace π^4 and Φ by π and $\tilde{f}$ respectively, and following a similar procedure as above. Thus

$$\int \psi(\tilde{\boldsymbol{u}}_1)\tilde{\mathscr{B}}_{el}(\tilde{f},\tilde{f})\,\mathrm{d}\tilde{\boldsymbol{u}}_1 = 0 \quad \text{for} \quad \psi(\tilde{\boldsymbol{u}}_1) = 1,\ \tilde{\boldsymbol{u}}_1 \text{ and } \tilde{u}_1^2. \tag{B.9}$$

The integral over $\tilde{\boldsymbol{u}}_1$ of the fifth term on the right-hand side of (B.1) times any of the summational invariants can be carried out as follows:
Consider the integration involving the summational invariant, 1.

$$\begin{aligned}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e) &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right)\\ &= I_1 - I_2, \quad \text{(let)}\end{aligned}$$

where

$$\begin{aligned}I_1 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2'),\\ I_2 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2).\end{aligned}$$

Note that the integrals I_1 and I_2 use inelastic velocity transformation. In I_1, one can change the integration variables using the relations, $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12} = -e(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')$ (cf. eq. (2.2)) and $\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2 = e\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'$ (cf. eq. (J.2)). Thus one obtains

$$\begin{aligned}I_1 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'<0} e\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,\left\{-e(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\right\}\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2')\\ &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'<0} \mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2').\end{aligned}$$

Changing the dummy variables $\tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2'$ to $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ respectively,

$$I_1 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}<0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2).$$

Using eq. (F.8),

$$I_1 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2) = I_2.$$

Hence

$$\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e) = I_1 - I_2 = 0. \tag{B.10}$$

Next, consider the integration involving the summational invariant, $\tilde{\boldsymbol{u}}_1$.
Let $\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\tilde{\boldsymbol{u}}_1\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e) = I_3$. Hence

$$I_3 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\tilde{\boldsymbol{u}}_1\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right). \tag{*5}$$

Interchanging the variables $\tilde{\boldsymbol{u}}_1 \longleftrightarrow \tilde{\boldsymbol{u}}_2$, consequently $\tilde{\boldsymbol{u}}_1 \longleftrightarrow \tilde{\boldsymbol{u}}_2$, and hence

$$I_3 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{21}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{21})\tilde{\boldsymbol{u}}_2\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_2')\tilde{f}(\tilde{\boldsymbol{u}}_1') - \tilde{f}(\tilde{\boldsymbol{u}}_2)\tilde{f}(\tilde{\boldsymbol{u}}_1)\right).$$

Using eq. (F.8),

$$I_3 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\tilde{\boldsymbol{u}}_2\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right). \tag{*6}$$

Adding eqs. (*5) and (*6), we get

$$\begin{aligned}2I_3 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})(\tilde{\boldsymbol{u}}_1+\tilde{\boldsymbol{u}}_2)\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right)\\ &= I_4 - I_5, \quad \text{(let)}\end{aligned}$$

where

$$\begin{aligned}I_4 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})(\tilde{\boldsymbol{u}}_1+\tilde{\boldsymbol{u}}_2)\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2'),\\ I_5 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})(\tilde{\boldsymbol{u}}_1+\tilde{\boldsymbol{u}}_2)\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2).\end{aligned}$$

Note that the integrals I_4 and I_5 use inelastic velocity transformation. In I_4, one can change the integration variables using the relations, $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12} = -e(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')$ (cf. eq. (2.2)), $\tilde{\boldsymbol{u}}_1+\tilde{\boldsymbol{u}}_2 = \tilde{\boldsymbol{u}}_1'+\tilde{\boldsymbol{u}}_2'$ (momentum conservation) and $\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2 = e\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'$ (cf. eq. (J.2)). Thus one obtains

$$\begin{aligned}I_4 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'<0} e\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,\left\{-e(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\right\}(\tilde{\boldsymbol{u}}_1'+\tilde{\boldsymbol{u}}_2')\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2')\\ &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'<0} \mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')(\tilde{\boldsymbol{u}}_1'+\tilde{\boldsymbol{u}}_2')\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2').\end{aligned}$$

Changing the dummy variables $\tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2'$ to $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ respectively,

$$I_4 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}<0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})(\tilde{\boldsymbol{u}}_1+\tilde{\boldsymbol{u}}_2)\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2).$$

Using eq. (F.8),

$$I_4 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})(\tilde{\boldsymbol{u}}_1+\tilde{\boldsymbol{u}}_2)\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2) = I_5.$$

Hence

$$2I_3 = I_4 - I_5 = 0$$

or

$$\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\tilde{\boldsymbol{u}}_1\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e) = 0. \tag{B.11}$$

Next, consider the integration involving the summational invariant, $\tilde{u}_1^2$.
Let $\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\tilde{u}_1^2\tilde{\mathscr{B}}(\tilde{f},\tilde{f},e) = I_6$. Hence

$$I_6 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\tilde{u}_1^2\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right). \tag{*7}$$

Interchanging the variables $\tilde{\boldsymbol{u}}_1 \longleftrightarrow \tilde{\boldsymbol{u}}_2$, consequently $\tilde{\boldsymbol{u}}_1 \longleftrightarrow \tilde{\boldsymbol{u}}_2$, and hence

$$I_6 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{21}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{21})\tilde{u}_2^2\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_2')\tilde{f}(\tilde{\boldsymbol{u}}_1') - \tilde{f}(\tilde{\boldsymbol{u}}_2)\tilde{f}(\tilde{\boldsymbol{u}}_1)\right).$$

Using eq. (F.8),

$$I_6 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\tilde{u}_2^2\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right). \tag{*8}$$

Adding eqs. (*7) and (*8), we get

$$\begin{aligned} 2I_6 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})(\tilde{u}_1^2+\tilde{u}_2^2)\left(\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') - \tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2)\right) \\ &= I_7 - I_8, \quad \text{(let)} \end{aligned}$$

where

$$\begin{aligned} I_7 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})(\tilde{u}_1^2+\tilde{u}_2^2)\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2'), \\ I_8 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})(\tilde{u}_1^2+\tilde{u}_2^2)\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2). \end{aligned}$$

Note that the integrals I_7 and I_8 use inelastic velocity transformation. In I_7, one can change the integration variables using the relations, $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12} = -e(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')$ (cf. eq. (2.2)), $\tilde{u}_1^2+\tilde{u}_2^2 = \tilde{u}_1'^2+\tilde{u}_2'^2-\frac{\epsilon}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2$ and $\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2 = e\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'$ (cf. eq. (J.2)). Thus one obtains

$$\begin{aligned} I_7 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'<0} e\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,\left\{-e(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\right\}\left\{\tilde{u}_1'^2+\tilde{u}_2'^2-\frac{\epsilon}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2\right\}\frac{1}{e^2}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2') \\ &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'<0} \mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\left\{\tilde{u}_1'^2+\tilde{u}_2'^2-\frac{\epsilon}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')^2\right\}\tilde{f}(\tilde{\boldsymbol{u}}_1')\tilde{f}(\tilde{\boldsymbol{u}}_2'). \end{aligned}$$

Changing the dummy variables $\tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2'$ to $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ respectively,

$$I_7 = \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}<0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left\{\tilde{u}_1^2+\tilde{u}_2^2-\frac{\epsilon}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right\}\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2).$$

Using eq. (F.8),

$$\begin{aligned} I_7 &= \frac{1}{\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\left\{\tilde{u}_1^2+\tilde{u}_2^2-\frac{\epsilon}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right\}\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2) \\ &= I_8 - \frac{\epsilon}{2\pi}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^3\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2). \end{aligned}$$

Using eq. (G.1c),

$$I_7 = I_8 - \frac{\epsilon}{2\pi}\times\frac{\pi}{2}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2).$$

Using eq. (2.25),

$$I_7 = I_8 - 3\epsilon\times\frac{1}{12}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\tilde{f}(\tilde{\boldsymbol{u}}_1)\tilde{f}(\tilde{\boldsymbol{u}}_2) = I_8 - 3\epsilon\tilde{\Gamma}.$$

Hence

$$2I_6 = I_7 - I_8 = -3\epsilon\tilde{\Gamma}$$

or

$$\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\tilde{u}_1^2\mathscr{B}(\tilde{f},\tilde{f},e) = -\frac{3}{2}\epsilon\tilde{\Gamma}. \tag{B.12}$$

Finally, the integral over $\tilde{\boldsymbol{u}}$ of the sixth term on the RHS of (B.1) times any of the summational invariants can be carried out as following. Let $\psi(\tilde{\boldsymbol{u}})$ be any of the summational invariant: 1, $\tilde{\boldsymbol{u}}$ and $\tilde{u}^2$, and $I_9 = \int\psi(\tilde{\boldsymbol{u}})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\tilde{f}\,\mathrm{d}\tilde{\boldsymbol{u}}$. Hence

$$\begin{aligned} I_9 &= \int\psi(\tilde{\boldsymbol{u}})K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\frac{\partial\tilde{f}}{\partial v_i}\,\mathrm{d}\tilde{\boldsymbol{u}} = K\tilde{g}_i\int\psi(\tilde{\boldsymbol{u}})\frac{\partial\tilde{f}}{\partial\tilde{u}_i}\,\mathrm{d}\tilde{\boldsymbol{u}} = K\tilde{g}_i\int\left\{\frac{\partial}{\partial\tilde{u}_i}\Big(\tilde{f}\,\psi(\tilde{\boldsymbol{u}})\Big)-\tilde{f}\,\frac{\partial\psi(\tilde{\boldsymbol{u}})}{\partial\tilde{u}_i}\right\}\mathrm{d}\tilde{\boldsymbol{u}} \\ &= K\tilde{g}_i\int\frac{\partial}{\partial\tilde{u}_i}\Big(\tilde{f}\,\psi(\tilde{\boldsymbol{u}})\Big)\,\mathrm{d}\tilde{u}_i\mathrm{d}\tilde{u}_j\mathrm{d}\tilde{u}_k - K\tilde{g}_i\int\tilde{f}\,\frac{\partial\psi(\tilde{\boldsymbol{u}})}{\partial\tilde{u}_i}\,\mathrm{d}\tilde{\boldsymbol{u}} \quad (i,j\text{ and }k\text{ are different.}) \\ &= K\tilde{g}_i\int\Big(\tilde{f}\,\psi(\tilde{\boldsymbol{u}})\Big)_{\tilde{u}_i=-\infty}^{\tilde{u}_i=\infty}\mathrm{d}\tilde{u}_j\mathrm{d}\tilde{u}_k - K\tilde{g}_i\int\tilde{f}\,\frac{\partial\psi(\tilde{\boldsymbol{u}})}{\partial\tilde{u}_i}\,\mathrm{d}\tilde{\boldsymbol{u}} = -K\tilde{g}_i\int\tilde{f}\,\frac{\partial\psi(\tilde{\boldsymbol{u}})}{\partial\tilde{u}_i}\,\mathrm{d}\tilde{\boldsymbol{u}} \end{aligned}$$

because $\tilde{f}(\tilde{\boldsymbol{u}}) = \tilde{f}_0(\tilde{u})(1+\Phi)$ and $\tilde{f}_0(\tilde{u}) = \pi^{-3/2}\,\mathrm{e}^{-\tilde{u}^2}$. Now substituting the values of $\psi(\tilde{\boldsymbol{u}})$ as 1, $\tilde{\boldsymbol{u}}$ and $\tilde{u}^2$ separately, one obtains

$$\int K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\,\tilde{f}\,\mathrm{d}\tilde{\boldsymbol{u}} = -K\tilde{g}_i\int\tilde{f}\,\frac{\partial(1)}{\partial\tilde{u}_i}\,\mathrm{d}\tilde{\boldsymbol{u}} = 0, \tag{B.13}$$

$$\begin{aligned}\int \tilde{u}_j K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v \tilde{f}\,\mathrm{d}\tilde{\boldsymbol{u}} &= -K\tilde{g}_i \int \tilde{f}\,\frac{\partial \tilde{u}_j}{\partial \tilde{u}_i}\,\mathrm{d}\tilde{\boldsymbol{u}} = -K\tilde{g}_i \int \tilde{f}\delta_{ij}\mathrm{d}\tilde{\boldsymbol{u}} = -K\tilde{g}_j \int \tilde{f}\mathrm{d}\tilde{\boldsymbol{u}}\\ &= -K\tilde{g}_j \int \left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} f\right\}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\mathrm{d}\boldsymbol{v}\right\}\\ &= -K\tilde{g}_j \frac{1}{n}\int f\,\mathrm{d}\boldsymbol{v} = -K\tilde{g}_j \frac{1}{n}\times n = -K\tilde{g}_j\end{aligned}$$

or

$$\int \tilde{u}_i K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v \tilde{f}\,\mathrm{d}\tilde{\boldsymbol{u}} = -K\tilde{g}_i \tag{B.14}$$

and

$$\begin{aligned}&\int \tilde{u}^2 K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v \tilde{f}\,\mathrm{d}\tilde{\boldsymbol{u}}\\ &= -K\tilde{g}_i \int \tilde{f}\,\frac{\partial \tilde{u}^2}{\partial \tilde{u}_i}\,\mathrm{d}\tilde{\boldsymbol{u}} = -K\tilde{g}_i \int \tilde{f}\,2\tilde{u}_j\frac{\partial \tilde{u}_j}{\partial \tilde{u}_i}\,\mathrm{d}\tilde{\boldsymbol{u}} = -K\tilde{g}_i \int \tilde{f}\,2\tilde{u}_j\delta_{ij}\mathrm{d}\tilde{\boldsymbol{u}} = -2K\tilde{g}_i \int \tilde{f}\tilde{u}_i\mathrm{d}\tilde{\boldsymbol{u}}\\ &= -K\tilde{g}_i \int \left\{\frac{1}{n}\left(\frac{2\Theta}{3}\right)^{\frac{3}{2}} f\right\}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(v_i - V_i)\right\}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{3}{2}}\mathrm{d}\boldsymbol{v}\right\}\\ &= -K\tilde{g}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{1}{n}\int f(v_i - V_i)\,\mathrm{d}\boldsymbol{v} = -K\tilde{g}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(\frac{1}{n}\int f v_i\,\mathrm{d}\boldsymbol{v} - V_i\frac{1}{n}\int f\,\mathrm{d}\boldsymbol{v}\right)\\ &= -K\tilde{g}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(\frac{1}{n}\times nV_i - V_i\frac{1}{n}\times n\right)\end{aligned}$$

or

$$\int \tilde{u}^2 K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v \tilde{f}\,\mathrm{d}\tilde{\boldsymbol{u}} = 0. \tag{B.15}$$

- From eqs. (B.2), (B.5), (B.8), (B.9), (B.10) and (B.13), we see that that the RHS of eq. (B.1) times the summational invariant, 1 is equal to zero.
- From eqs. (B.3), (B.6), (B.8), (B.9), (B.11) and (B.14), we see that that the RHS of eq. (B.1) times the summational invariant, $\tilde{\boldsymbol{u}}$ is equal to zero.
- From eqs. (B.4), (B.7), (B.8), (B.9), (B.12) and (B.15), we see that that the RHS of eq. (B.1) times the summational invariant, $\tilde{u}^2$ is equal to zero.

That means the right-hand side of eq. (B.1) is orthogonal to all the summational invariants of $\tilde{\mathscr{L}}$.

Appendix C

The Integral I_δ

In this appendix we shall evaluate the integral I_δ defined as

$$I_\delta = \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\delta\Big(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1 + q(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big)\,, \tag{C.1}$$

where $q = \frac{1+e}{2}$. Upon expressing the delta function as $\delta(\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{w}\,\mathrm{e}^{i\boldsymbol{w}\cdot\boldsymbol{x}}$ and defining $\tilde{\boldsymbol{s}} \equiv \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1$, we get

$$\begin{aligned} I_\delta &= \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \times \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{w}\,\mathrm{e}^{i\boldsymbol{w}\cdot\left(\tilde{\boldsymbol{s}}+q(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\right)} \\ &= \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{w}\,\mathrm{e}^{i\boldsymbol{w}\cdot\tilde{\boldsymbol{s}}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{iq(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})(\hat{\boldsymbol{k}}\cdot\boldsymbol{w})}\,. \end{aligned} \tag{C.2}$$

The integration over $\hat{\boldsymbol{k}}$ is performed in a spherical polar coordinate system $(\hat{k},\theta,\phi)$, whose z-axis coincides with $\tilde{\boldsymbol{u}}_{12}$ (see fig. C.1). Let w_z and $w_\perp$ be the components of $\boldsymbol{w}$ parallel and perpendicular to $\tilde{\boldsymbol{u}}_{12}$ respectively and χ be the azimuthal angle of the projection of $\boldsymbol{w}$ on the plane normal to $\tilde{\boldsymbol{u}}_{12}$. Hence

$$\begin{gathered} \hat{\boldsymbol{k}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \\ \tilde{\boldsymbol{u}}_{12} = \tilde{u}_{12}\,\hat{\boldsymbol{z}} \\ \boldsymbol{w} = w_\perp\cos\chi\,\hat{\boldsymbol{x}} + w_\perp\sin\chi\,\hat{\boldsymbol{y}} + w_z\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12} = \tilde{u}_{12}\cos\theta \\ \hat{\boldsymbol{k}}\cdot\boldsymbol{w} = w_\perp\sin\theta\,(\cos\phi\cos\chi + \sin\phi\sin\chi) + w_z\cos\theta = w_\perp\sin\theta\cos(\phi-\chi) + w_z\cos\theta. \end{gathered}$$

Substituting these values in (C.2), we get

$$I_\delta = \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{w}\,\mathrm{e}^{i\boldsymbol{w}\cdot\tilde{\boldsymbol{s}}}\int_{\hat{k}=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=\chi}^{2\pi+\chi}\tilde{u}_{12}\cos\theta\,\mathrm{e}^{iq\tilde{u}_{12}\cos\theta\,(w_\perp\sin\theta\cos(\phi-\chi)+w_z\cos\theta)}\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}\hat{k}.$$

Let $\phi - \chi = \omega$, hence

$$\begin{aligned} I_\delta &= \frac{\tilde{u}_{12}}{(2\pi)^3}\int \mathrm{d}\boldsymbol{w}\,\mathrm{e}^{i\boldsymbol{w}\cdot\tilde{\boldsymbol{s}}}\int_{\hat{k}=0}^{1}\mathrm{d}\hat{k}\int_{\theta=0}^{\pi/2}\int_{\omega=0}^{2\pi}\sin\theta\cos\theta\,\mathrm{e}^{iq\tilde{u}_{12}\cos\theta\,(w_\perp\sin\theta\cos\omega+w_z\cos\theta)}\,\mathrm{d}\omega\,\mathrm{d}\theta \\ &= \frac{\tilde{u}_{12}}{(2\pi)^3}\int \mathrm{d}\boldsymbol{w}\,\mathrm{e}^{i\boldsymbol{w}\cdot\tilde{\boldsymbol{s}}}\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\,\mathrm{e}^{iqw_z\tilde{u}_{12}\cos^2\theta}\left\{\int_{\omega=0}^{2\pi}\mathrm{e}^{i(qw_\perp\tilde{u}_{12}\sin\theta\cos\theta)\cos\omega}\,\mathrm{d}\omega\right\}\mathrm{d}\theta. \end{aligned}$$

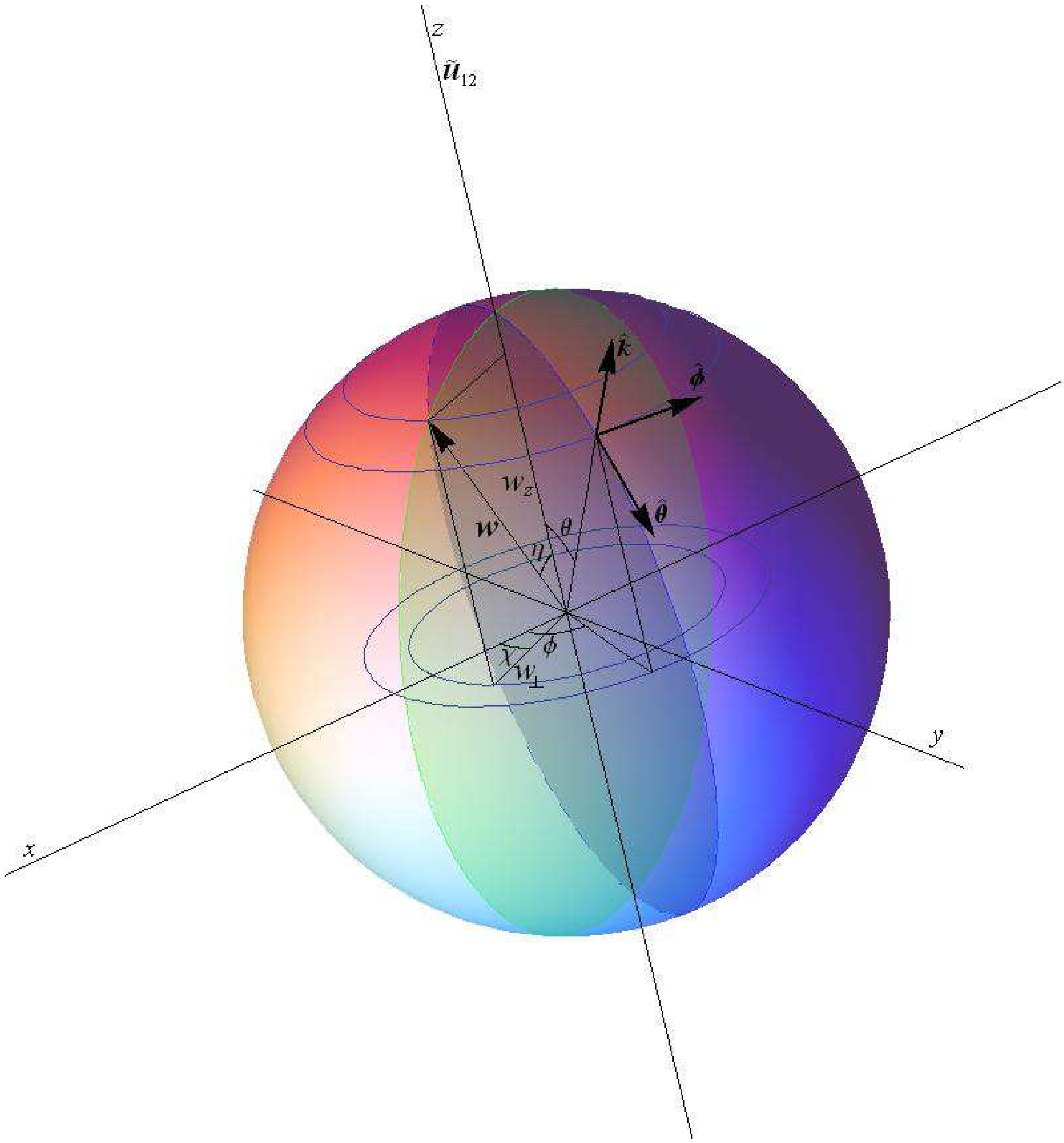


Figure C.1: Spherical Coordinate System $(\hat{k}, \theta, \phi)$

Since the Bessel function of first kind is defined as

$$J_n(x) = \frac{1}{2\pi i^n} \int_0^{2\pi} \mathrm{e}^{ix\cos\phi}\, \mathrm{e}^{in\phi}\, \mathrm{d}\phi \qquad \Rightarrow \qquad J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{ix\cos\phi}\, \mathrm{d}\phi,$$

therefore,

$$I_\delta = \frac{\tilde{u}_{12}}{(2\pi)^2} \int \mathrm{d}\boldsymbol{w}\, \mathrm{e}^{i\boldsymbol{w}\cdot\tilde{\boldsymbol{s}}} \int_{\theta=0}^{\pi/2} \sin\theta\cos\theta\, \mathrm{e}^{iqw_z\tilde{u}_{12}\cos^2\theta}\, J_0(qw_\perp\tilde{u}_{12}\sin\theta\cos\theta)\, \mathrm{d}\theta. \qquad \text{(C.3)}$$

The integration over $\boldsymbol{w}$ is performed in a cylindrical coordinate system $(w_\perp, \chi, w_z)$, whose z-axis coincides with $\tilde{\boldsymbol{u}}_{12}$ (see fig. C.2).

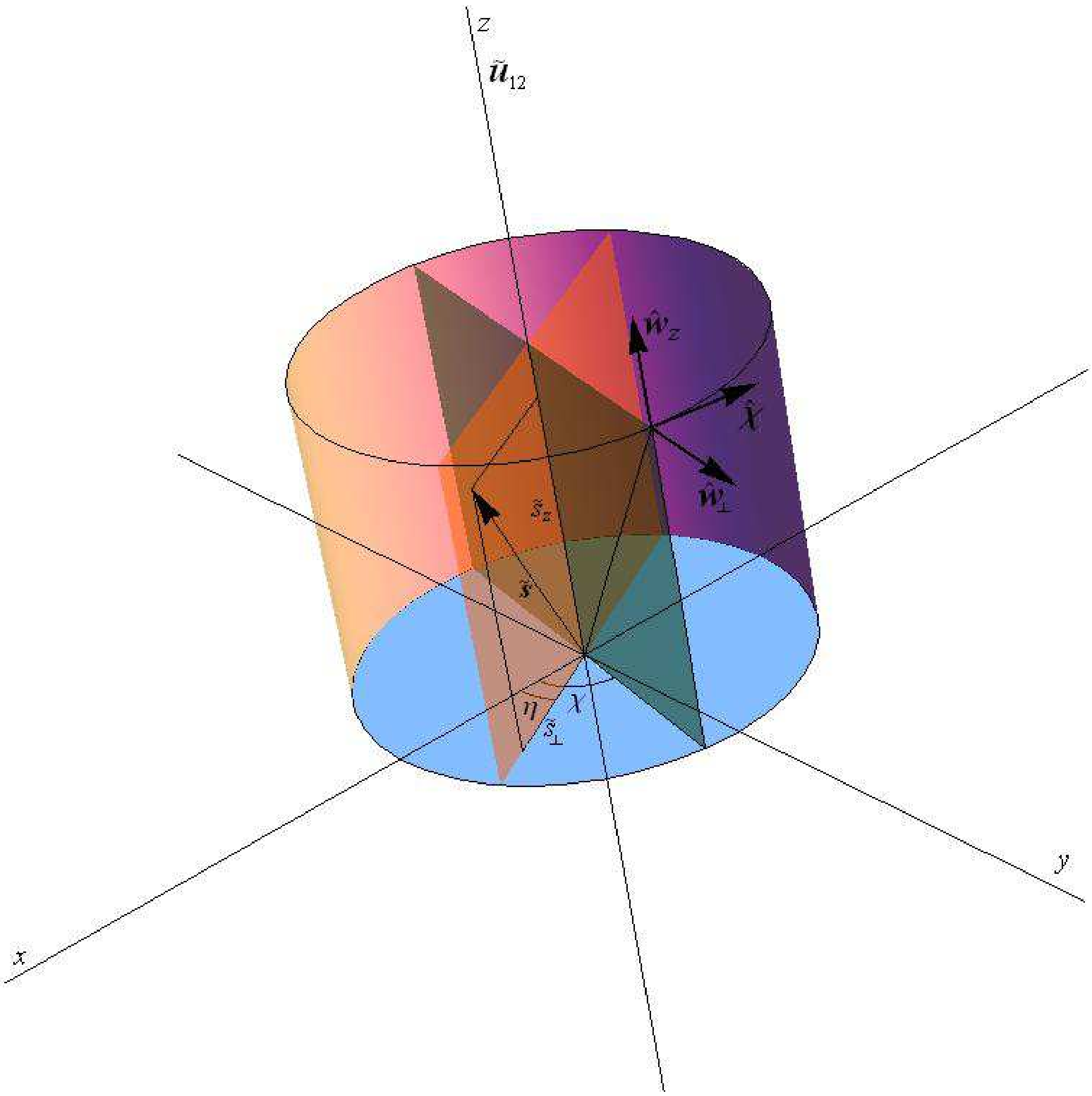


Figure C.2: Cylindrical Coordinate System $(w_\perp, \chi, w_z)$

Let $\tilde{s}_z$ and $\tilde{s}_\perp$ be the components of $\tilde{\boldsymbol{s}}$ parallel and perpendicular to $\tilde{\boldsymbol{u}}_{12}$ respectively and η be the azimuthal angle of the projection of $\tilde{\boldsymbol{s}}$ on the plane normal to $\tilde{\boldsymbol{u}}_{12}$. Hence

$$\boldsymbol{w} = w_\perp \cos\chi\,\hat{\boldsymbol{x}} + w_\perp \sin\chi\,\hat{\boldsymbol{y}} + w_z\,\hat{\boldsymbol{z}}$$
$$\tilde{\boldsymbol{s}} = \tilde{s}_\perp \cos\eta\,\hat{\boldsymbol{x}} + \tilde{s}_\perp \sin\eta\,\hat{\boldsymbol{y}} + \tilde{s}_z\,\hat{\boldsymbol{z}}$$
$$\boldsymbol{w}\cdot\tilde{\boldsymbol{s}} = w_\perp\tilde{s}_\perp\cos\chi\cos\eta + w_\perp\tilde{s}_\perp\sin\chi\sin\eta + w_z\tilde{s}_z = w_\perp\tilde{s}_\perp\cos(\chi-\eta) + w_z\tilde{s}_z.$$

Therefore

$$I_\delta = \frac{\tilde{u}_{12}}{(2\pi)^2}\int_{w_\perp=0}^{\infty}\int_{\chi=\eta}^{2\pi+\eta}\int_{w_z=-\infty}^{\infty} \mathrm{e}^{i(w_\perp\tilde{s}_\perp\cos(\chi-\eta)+w_z\tilde{s}_z)}\,w_\perp \mathrm{d}w_z\,\mathrm{d}\chi\,\mathrm{d}w_\perp$$
$$\times\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\,\mathrm{e}^{iqw_z\tilde{u}_{12}\cos^2\theta}\,J_0(qw_\perp\tilde{u}_{12}\sin\theta\cos\theta)\,\mathrm{d}\theta.$$

Let $\chi - \eta = \mu$. Hence

$$\begin{aligned}
I_\delta &= \frac{\tilde{u}_{12}}{(2\pi)^2}\int_{w_z=-\infty}^{\infty}\mathrm{d}w_z\int_{w_\perp=0}^{\infty}w_\perp \mathrm{d}w_\perp\int_{\mu=0}^{2\pi}\mathrm{e}^{i(w_\perp\tilde{s}_\perp\cos\mu+w_z\tilde{s}_z)}\,\mathrm{d}\mu \\
&\quad\times\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\,\mathrm{e}^{iqw_z\tilde{u}_{12}\cos^2\theta}\,J_0(qw_\perp\tilde{u}_{12}\sin\theta\cos\theta)\,\mathrm{d}\theta \\
&= \frac{\tilde{u}_{12}}{(2\pi)^2}\int_{w_z=-\infty}^{\infty}\mathrm{d}w_z\int_{w_\perp=0}^{\infty}w_\perp\left\{\int_{\mu=0}^{2\pi}\mathrm{e}^{i(w_\perp\tilde{s}_\perp)\cos\mu}\,\mathrm{d}\mu\right\}\mathrm{d}w_\perp \\
&\quad\times\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\,\mathrm{e}^{iw_z(\tilde{s}_z+q\tilde{u}_{12}\cos^2\theta)}\,J_0(qw_\perp\tilde{u}_{12}\sin\theta\cos\theta)\,\mathrm{d}\theta \\
&= \frac{\tilde{u}_{12}}{2\pi}\int_{w_z=-\infty}^{\infty}\mathrm{d}w_z\int_{w_\perp=0}^{\infty}w_\perp J_0(w_\perp\tilde{s}_\perp)\,\mathrm{d}w_\perp \\
&\quad\times\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\,\mathrm{e}^{iw_z(\tilde{s}_z+q\tilde{u}_{12}\cos^2\theta)}\,J_0(qw_\perp\tilde{u}_{12}\sin\theta\cos\theta)\,\mathrm{d}\theta \qquad \text{(as above)} \\
&= \tilde{u}_{12}\int_{w_\perp=0}^{\infty}w_\perp J_0(w_\perp\tilde{s}_\perp)\,\mathrm{d}w_\perp\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\,J_0(qw_\perp\tilde{u}_{12}\sin\theta\cos\theta)\,\mathrm{d}\theta \\
&\quad\times\frac{1}{2\pi}\int_{w_z=-\infty}^{\infty}\mathrm{e}^{iw_z(\tilde{s}_z+q\tilde{u}_{12}\cos^2\theta)}\,\mathrm{d}w_z \\
&= \tilde{u}_{12}\int_{w_\perp=0}^{\infty}w_\perp J_0(w_\perp\tilde{s}_\perp)\,\mathrm{d}w_\perp\int_{\theta=0}^{\pi/2}\sin\theta\cos\theta\,J_0(qw_\perp\tilde{u}_{12}\sin\theta\cos\theta)\delta(\tilde{s}_z+q\tilde{u}_{12}\cos^2\theta)\,\mathrm{d}\theta
\end{aligned}$$

because $\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{iwx}\,\mathrm{d}w$. Let $\tilde{s}_z+q\tilde{u}_{12}\cos^2\theta=t$.

$$\therefore \qquad \cos\theta=\sqrt{\frac{t-\tilde{s}_z}{q\tilde{u}_{12}}},\quad \sin\theta=\sqrt{\frac{q\tilde{u}_{12}-(t-\tilde{s}_z)}{q\tilde{u}_{12}}}\quad\text{and}\quad -2q\tilde{u}_{12}\sin\theta\cos\theta\,\mathrm{d}\theta=\mathrm{d}t.$$

Hence

$$\begin{aligned}
I_\delta &= \tilde{u}_{12}\int_{w_\perp=0}^{\infty}w_\perp J_0(w_\perp\tilde{s}_\perp)\,\mathrm{d}w_\perp \\
&\quad\times\int_{t=\tilde{s}_z+q\tilde{u}_{12}}^{\tilde{s}_z}\left(-\frac{\mathrm{d}t}{2q\tilde{u}_{12}}\right)J_0\left(qw_\perp\tilde{u}_{12}\sqrt{\frac{q\tilde{u}_{12}-(t-\tilde{s}_z)}{q\tilde{u}_{12}}}\sqrt{\frac{t-\tilde{s}_z}{q\tilde{u}_{12}}}\right)\delta(t) \\
&= \frac{1}{2q}\int_{w_\perp=0}^{\infty}w_\perp J_0(w_\perp\tilde{s}_\perp)\,\mathrm{d}w_\perp\int_{t=\tilde{s}_z+q\tilde{u}_{12}}^{\tilde{s}_z}J_0\left(qw_\perp\tilde{u}_{12}\sqrt{\frac{q\tilde{u}_{12}-(t-\tilde{s}_z)}{q\tilde{u}_{12}}}\sqrt{\frac{t-\tilde{s}_z}{q\tilde{u}_{12}}}\right)\delta(t)\,\mathrm{d}t.
\end{aligned}$$

Note that $\tilde{s}_z$ is the projection of $\tilde{\boldsymbol{s}}$ on $\tilde{\boldsymbol{u}}_{12}$, i.e.,

$$\tilde{s}_z=\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_{12}}{\tilde{u}_{12}}=\frac{(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot\tilde{\boldsymbol{u}}_{12}}{\tilde{u}_{12}}=-\frac{q(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}}{\tilde{u}_{12}}\quad\Rightarrow\quad\frac{-\tilde{s}_z}{q\tilde{u}_{12}}=\frac{(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2}{\tilde{u}_{12}^2}>0.$$

Now, to simplify the integral over t, we use the property of delta function and for this we shall change the limits of integration to $(-\infty,\infty)$ with the help of Heaviside functions. Since we know that Heaviside function is defined as

$$H(x)=\begin{cases}1 & \text{if } x\geq 0\\ 0 & \text{if } x<0\end{cases}$$

we have,

$$H\left(\frac{t-\tilde{s}_z}{q\tilde{u}_{12}}\right) = \begin{cases} 1 & \text{if } \frac{t-\tilde{s}_z}{q\tilde{u}_{12}} \geq 0 \quad \text{or } t \geq \tilde{s}_z \\ \\ 0 & \text{if } \frac{t-\tilde{s}_z}{q\tilde{u}_{12}} < 0 \quad \text{or } t < \tilde{s}_z \end{cases}$$

and

$$H\left(1-\frac{t-\tilde{s}_z}{q\tilde{u}_{12}}\right) = \begin{cases} 1 & \text{if } 1-\frac{t-\tilde{s}_z}{q\tilde{u}_{12}} \geq 0 \quad \text{or } t \leq \tilde{s}_z + q\tilde{u}_{12} \\ \\ 0 & \text{if } 1-\frac{t-\tilde{s}_z}{q\tilde{u}_{12}} < 0 \quad \text{or } t > \tilde{s}_z + q\tilde{u}_{12}. \end{cases}$$

Using these two Heaviside functions, the expression for I_δ can be written as

$$\begin{aligned} I_\delta &= \frac{1}{2q}\int_{w_\perp=0}^{\infty} w_\perp J_0(w_\perp \tilde{s}_\perp)\,\mathrm{d}w_\perp \\ &\quad \times \int_{t=-\infty}^{\infty} H\left(\frac{t-\tilde{s}_z}{q\tilde{u}_{12}}\right) H\left(1-\frac{t-\tilde{s}_z}{q\tilde{u}_{12}}\right) J_0\left(q w_\perp \tilde{u}_{12}\sqrt{\frac{q\tilde{u}_{12}-(t-\tilde{s}_z)}{q\tilde{u}_{12}}}\sqrt{\frac{t-\tilde{s}_z}{q\tilde{u}_{12}}}\right)\delta(t)\,\mathrm{d}t \\ &= \frac{1}{2q}\int_{w_\perp=0}^{\infty} w_\perp J_0(w_\perp \tilde{s}_\perp)\,\mathrm{d}w_\perp H\left(\frac{-\tilde{s}_z}{q\tilde{u}_{12}}\right) H\left(1+\frac{\tilde{s}_z}{q\tilde{u}_{12}}\right) J_0\left(q w_\perp \tilde{u}_{12}\sqrt{\frac{q\tilde{u}_{12}+\tilde{s}_z}{q\tilde{u}_{12}}}\sqrt{\frac{-\tilde{s}_z}{q\tilde{u}_{12}}}\right) \\ &= \frac{1}{2q} H\left(\frac{-\tilde{s}_z}{q\tilde{u}_{12}}\right) H\left(1+\frac{\tilde{s}_z}{q\tilde{u}_{12}}\right)\int_{w_\perp=0}^{\infty} w_\perp J_0(w_\perp \tilde{s}_\perp) J_0\left(w_\perp\sqrt{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)}\right)\,\mathrm{d}w_\perp. \end{aligned}$$

The orthogonality property of the Bessel function of first kind, which is $\int_0^\infty x J_0(ax)J_0(bx)\,\mathrm{d}x = \frac{1}{a}\delta(a-b)$, implies that

$$I_\delta = \frac{1}{2q\tilde{s}_\perp} H\left(\frac{-\tilde{s}_z}{q\tilde{u}_{12}}\right) H\left(1+\frac{\tilde{s}_z}{q\tilde{u}_{12}}\right)\delta\left(\tilde{s}_\perp - \sqrt{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)}\right). \tag{C.4}$$

Now we see that for delta function to be nonzero, we must have, $\tilde{s}_\perp - \sqrt{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)} = 0$ or $\tilde{s}_\perp^2 = -q\tilde{s}_z\tilde{u}_{12} - s_z^2$ or $-q\tilde{s}_z\tilde{u}_{12} = \tilde{s}_\perp^2 + s_z^2 = \tilde{s}^2$ and hence

$$-\frac{\tilde{s}_z}{q\tilde{u}_{12}} = -\frac{\tilde{s}_z^2}{q\tilde{s}_z\tilde{u}_{12}} = \frac{\tilde{s}_z^2}{\tilde{s}^2} \geq 0 \quad \text{and} \quad 1+\frac{\tilde{s}_z}{q\tilde{u}_{12}} = 1-\frac{\tilde{s}_z^2}{\tilde{s}^2} = \frac{\tilde{s}_\perp^2}{\tilde{s}^2} \geq 0.$$

That means, for I_δ to be nonzero, if the condition on the argument of delta function is satisfied, the conditions on the arguments of the Heaviside functions (so that the values of the Heaviside functions are nonzero) are automatically satisfied. Thus the Heaviside functions are redundant in eq. (C.4). Therefore eq. (C.4) simplifies to

$$I_\delta = \frac{1}{2q\tilde{s}_\perp}\delta\left(\tilde{s}_\perp - \sqrt{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)}\right). \tag{C.5}$$

Next we use the following property of delta function to get another form of I_δ.

$$\delta(f(x)-f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x-x_0). \tag{C.6}$$

Let $f(x) = -x^2$, $x = -\sqrt{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)}$ and $x_0 = -\tilde{s}_\perp$, so $f'(x_0) = -2x_0 = 2\tilde{s}_\perp$. For these values, eq. (C.6) gives,

$$\delta\left(-\left\{-\sqrt{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)}\right\}^2-(-\tilde{s}_\perp^2)\right)=\frac{1}{|2\tilde{s}_\perp|}\left(-\sqrt{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)}-(-\tilde{s}_\perp)\right)$$

or

$$\delta\big(-\{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)\}+\tilde{s}_\perp^2\big)=\frac{1}{2\tilde{s}_\perp}\left(\tilde{s}_\perp-\sqrt{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)}\right)$$

or

$$\delta\big(\tilde{s}_\perp^2+s_z^2+q\tilde{s}_z\tilde{u}_{12}\big)=\frac{1}{2\tilde{s}_\perp}\left(\tilde{s}_\perp-\sqrt{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)}\right)$$

or

$$\delta\big(\tilde{s}^2+q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_{12}\big)=\frac{1}{2\tilde{s}_\perp}\left(\tilde{s}_\perp-\sqrt{-\tilde{s}_z(q\tilde{u}_{12}+\tilde{s}_z)}\right). \tag{C.7}$$

From eqs. (C.5) and (C.7), we get

$$I_\delta=\frac{1}{q}\,\delta(\tilde{s}^2+q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_{12}). \tag{C.8}$$

Let θ_2' be the angle between $\tilde{\boldsymbol{s}}$ and $\tilde{\boldsymbol{u}}_2$. So

$$\cos\theta_2'=\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_2}{\tilde{s}\tilde{u}_2}.$$

But since $\tilde{\boldsymbol{s}}=\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_{12}=\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2$, we have $\tilde{\boldsymbol{u}}_2=\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_{12}=\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}-\tilde{\boldsymbol{u}}_{12}$. Thus

$$\cos\theta_2'=\frac{\tilde{\boldsymbol{s}}\cdot(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{s}}-\tilde{\boldsymbol{u}}_{12})}{\tilde{s}\tilde{u}_2}=\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}-\frac{(\tilde{s}^2+\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_{12})}{\tilde{s}\tilde{u}_2}$$

and the condition on delta function in eq. (C.8) implies that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_{12}=-\frac{1}{q}\tilde{s}^2$, hence

$$\cos\theta_2'=\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}-\frac{\left(\tilde{s}^2-\frac{1}{q}\tilde{s}^2\right)}{\tilde{s}\tilde{u}_2}=\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}+\left(\frac{1-q}{q}\right)\frac{\tilde{s}}{\tilde{u}_2}. \tag{C.9}$$

Since we know that $|\cos\theta_2'|\leq 1$, which implies that $\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}+\left(\frac{1-q}{q}\right)\frac{\tilde{s}}{\tilde{u}_2}\right|\leq 1$, hence eq. (C.9) restricts the values of $\tilde{u}_2$ to $\tilde{u}_2\geq\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}+\left(\frac{1-q}{q}\right)\tilde{s}\right|$.

Next, consider the following integral over θ_2',

$$\int_0^\pi \sin\theta_2'\,F(\cos\theta_2')I_\delta\,\mathrm{d}\theta_2'=\frac{1}{q}\int_0^\pi \sin\theta_2'\,F(\cos\theta_2')\,\delta(\tilde{s}^2+q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{d}\theta_2', \tag{C.10}$$

where F is a smooth function. Let $\tilde{s}^2+q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_{12}=p$.

$$\Rightarrow\quad p=\tilde{s}^2+q(\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}-\tilde{s}^2-\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_2)=\tilde{s}^2+q(\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}-\tilde{s}^2-\tilde{s}\tilde{u}_2\cos\theta_2')$$

$$\Rightarrow\quad \mathrm{d}p=q\tilde{s}\tilde{u}_2\sin\theta_2'\,\mathrm{d}\theta_2'\quad\text{and}\quad\cos\theta_2'=\frac{(1-q)\tilde{s}^2+q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}-p}{q\tilde{s}\tilde{u}_2}$$

hence

$$\int_0^\pi \sin\theta_2'\,F(\cos\theta_2')I_\delta\,\mathrm{d}\theta_2'=\frac{1}{q^2\tilde{s}\tilde{u}_2}\int_{(1-q)\tilde{s}^2+q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}-q\tilde{s}\tilde{u}_2}^{(1-q)\tilde{s}^2+q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}+q\tilde{s}\tilde{u}_2}F\left(\frac{(1-q)\tilde{s}^2+q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}-p}{q\tilde{s}\tilde{u}_2}\right)\delta(p)\,\mathrm{d}p. \tag{C.11}$$

Again, to simplify the integral over p, we use the property of delta function and for this we change the limits of integration to $(-\infty, \infty)$ with the help of Heaviside function as following.

$$H\left(\tilde{u}_2 - \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \frac{p}{q\tilde{s}}\right|\right) = \begin{cases} 1 & \text{if } \tilde{u}_2 - \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \frac{p}{q\tilde{s}}\right| \geq 0 \\ 0 & \text{if } \tilde{u}_2 - \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \frac{p}{q\tilde{s}}\right| < 0. \end{cases}$$

The condition $\tilde{u}_2 - \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \frac{p}{q\tilde{s}}\right| \geq 0$ can be simplified as following.

$$\tilde{u}_2 - \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \frac{p}{q\tilde{s}}\right| \geq 0 \Rightarrow \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \frac{p}{q\tilde{s}}\right| \leq \tilde{u}_2$$

or

$$-\tilde{u}_2 \leq \frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \frac{p}{q\tilde{s}} \leq \tilde{u}_2$$

or

$$\frac{p}{q\tilde{s}} \leq \frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} + \tilde{u}_2 \quad \text{and} \quad \frac{p}{q\tilde{s}} \geq \frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \tilde{u}_2$$

or

$$(1-q)\tilde{s}^2 + q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} - q\tilde{s}\tilde{u}_2 \leq p \leq (1-q)\tilde{s}^2 + q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} + q\tilde{s}\tilde{u}_2.$$

Similarly, the condition $\tilde{u}_2 - \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \frac{p}{q\tilde{s}}\right| < 0$ simplifies to

$$p < (1-q)\tilde{s}^2 + q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} - q\tilde{s}\tilde{u}_2 \quad \text{or} \quad p > (1-q)\tilde{s}^2 + q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} + q\tilde{s}\tilde{u}_2.$$

With these simplified conditions the above Heaviside function can be rewritten as,

$$\begin{aligned} &H\left(\tilde{u}_2 - \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \frac{p}{q\tilde{s}}\right|\right) \\ &= \begin{cases} 1 & \text{if} \quad (1-q)\tilde{s}^2 + q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} - q\tilde{s}\tilde{u}_2 \leq p \leq (1-q)\tilde{s}^2 + q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} + q\tilde{s}\tilde{u}_2 \\ 0 & \text{if} \quad p < (1-q)\tilde{s}^2 + q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} - q\tilde{s}\tilde{u}_2 \quad \text{or} \quad p > (1-q)\tilde{s}^2 + q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} + q\tilde{s}\tilde{u}_2. \end{cases} \end{aligned}$$

Using this Heaviside function eq. (C.11) can be written as,

$$\begin{aligned} &\int_0^\pi \sin\theta_2' \, F(\cos\theta_2') I_\delta \, \mathrm{d}\theta_2' \\ &\quad = \frac{1}{q^2\tilde{s}\tilde{u}_2}\int_{-\infty}^{\infty} F\left(\frac{(1-q)\tilde{s}^2 + q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}} - p}{q\tilde{s}\tilde{u}_2}\right) H\left(\tilde{u}_2 - \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}} - \frac{p}{q\tilde{s}}\right|\right) \delta(p)\,\mathrm{d}p \\ &\quad = \frac{1}{q^2\tilde{s}\tilde{u}_2} F\left(\frac{(1-q)\tilde{s}^2 + q\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{q\tilde{s}\tilde{u}_2}\right) H\left(\tilde{u}_2 - \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|\right) \end{aligned}$$

or

$$\int_0^\pi \sin\theta_2' \, F(\cos\theta_2') I_\delta \, \mathrm{d}\theta_2' = \frac{1}{q^2\tilde{s}\tilde{u}_2} F\left(\frac{(1-q)}{q}\frac{\tilde{s}}{\tilde{u}_2} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}\tilde{u}_2}\right) H\left(\tilde{u}_2 - \left|\frac{(1-q)}{q}\tilde{s} + \frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|\right). \quad \text{(C.12)}$$

From eq. (C.8),

$$I_\delta^{(0)} = I_\delta(q=1) = \delta(\tilde{s}^2 + \tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_{12}) \quad \text{(C.13)}$$

and the argument of delta function in eq. (C.13) can be written in another form as following:

$$\begin{aligned}\tilde{s}^2 + \tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_{12} &= (\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)^2 + (\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot(\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2)\\ &= (\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)^2 + (\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot[(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2)-(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)]\\ &= (\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)^2 + (\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2) - (\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)^2 = (\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot\tilde{\boldsymbol{t}},\end{aligned}$$

where $\tilde{\boldsymbol{t}} = \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_2$. Hence

$$I_\delta^{(0)} = \delta\big((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot\tilde{\boldsymbol{t}}\big)\,. \tag{C.14}$$

Let θ_1' be the angle between $\tilde{\boldsymbol{t}}$ and $\tilde{\boldsymbol{u}}_1$. So

$$\cos\theta_1' = \frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}_1}{\tilde{t}\tilde{u}_1}.$$

and the condition on delta function in eq. (C.14) implies that $\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}_1 = \tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}$, hence

$$\cos\theta_1' = \frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}\tilde{u}_1} \tag{C.15}$$

Since we know that $|\cos\theta_1'| \le 1$, which implies that $\left|\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}\tilde{u}_1}\right| \le 1$, hence eq. (C.15) restricts the values of $\tilde{u}_1$ to $\tilde{u}_1 \ge \left|\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}}\right|$.

Next, consider the following integral over θ_1',

$$\int_0^\pi \sin\theta_1'\, F(\cos\theta_1') I_\delta^{(0)} \mathrm{d}\theta_1' = \int_0^\pi \sin\theta_1'\, F(\cos\theta_1')\, \delta\big((\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot\tilde{\boldsymbol{t}}\big)\ \mathrm{d}\theta_1', \tag{C.16}$$

where F is a smooth function. Let $(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot\tilde{\boldsymbol{t}} = p$.

$$\Rightarrow \qquad p = \tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}} - \tilde{t}\tilde{u}_1\cos\theta_1' \quad \text{and} \quad \mathrm{d}p = \tilde{t}\tilde{u}_1 \sin\theta_1'\,\mathrm{d}\theta_1'.$$

Hence

$$\int_0^\pi \sin\theta_1'\, F(\cos\theta_1') I_\delta^{(0)} \mathrm{d}\theta_1' = \frac{1}{\tilde{t}\tilde{u}_1}\int_{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}-\tilde{t}\tilde{u}_1}^{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}+\tilde{t}\tilde{u}_1} \mathrm{d}p\, F\left(\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}-p}{\tilde{t}\tilde{u}_1}\right)\delta(p). \tag{C.17}$$

Again, to simplify the integral over p, we use the property of delta function and for this we change the limits of integration to $(-\infty,\infty)$ with the help of Heaviside function as following.

$$H\left(\tilde{u}_1 - \left|\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}-p}{\tilde{t}}\right|\right) = \begin{cases} 1 & \text{if } \left(\tilde{u}_1 \ge \left|\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}-p}{\tilde{t}}\right| \Rightarrow \tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}} - \tilde{t}\tilde{u}_1 \le p \le \tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}} + \tilde{t}\tilde{u}_1\right) \\ 0 & \text{if } \left(\tilde{u}_1 < \left|\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}-p}{\tilde{t}\tilde{u}_1}\right| \Rightarrow p < \tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}} - \tilde{t}\tilde{u}_1 \text{ or } p > \tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}} + \tilde{t}\tilde{u}_1\right). \end{cases}$$

Using this Heaviside function eq. (C.17) can be written as,

$$\begin{aligned}\int_0^\pi \sin\theta_1'\, F(\cos\theta_1') I_\delta^{(0)} \mathrm{d}\theta_1' &= \frac{1}{\tilde{t}\tilde{u}_1}\int_{-\infty}^{\infty} \mathrm{d}p\, F\left(\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}-p}{\tilde{t}\tilde{u}_1}\right) H\left(\tilde{u}_1 - \left|\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}-p}{\tilde{t}}\right|\right)\delta(p)\\ &= \frac{1}{\tilde{t}\tilde{u}_1} F\left(\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}\tilde{u}_1}\right) H\left(\tilde{u}_1 - \left|\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}}\right|\right).\end{aligned} \tag{C.18}$$

Appendix D

Action of operator $\tilde{\mathscr{D}}$

Eq. (3.18) implies that

$$\tilde{\mathscr{D}}\Phi_K = \tilde{\mathscr{D}}\left\{2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\right\}$$

and since $\tilde{\mathscr{D}}$ is a differential operator, one can write

$$\begin{aligned}
\tilde{\mathscr{D}}\Phi_K = {}& 2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}K + 2K\frac{2\Theta}{3g}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}\left\{\hat{\Phi}_v(\tilde{u})\right\} \\
&+ 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\} + 2K\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}\left\{\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\right\} \\
&+ 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}\left\{\frac{\partial V_i}{\partial r_j}\right\} + \frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\tilde{\mathscr{D}}K \\
&+ K\frac{2\Theta}{3g}\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\tilde{\mathscr{D}}\left\{\hat{\Phi}_c(\tilde{u})\right\} + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\tilde{\mathscr{D}}\left(\tilde{u}^2-\frac{5}{2}\right) \\
&+ K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\tilde{\mathscr{D}}\tilde{u}_i + K\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\tilde{\mathscr{D}}\left\{\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\right\} \\
= {}& 2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}K + 2K\frac{2\Theta}{3g}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\hat{\Phi}'_v(\tilde{u})\tilde{\mathscr{D}}(\tilde{u}^2) \\
&+ 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\} + 2K\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial V_i}{\partial r_j}\frac{1}{g}\tilde{\mathscr{D}}\left\{\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\} \\
&+ 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}\left\{\frac{\partial V_i}{\partial r_j}\right\} + \frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\tilde{\mathscr{D}}K \\
&+ K\frac{2\Theta}{3g}\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\hat{\Phi}'_c(\tilde{u})\tilde{\mathscr{D}}(\tilde{u}^2) + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\tilde{\mathscr{D}}(\tilde{u}^2) \\
&+ K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\tilde{\mathscr{D}}\tilde{u}_i + K\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{2}{3g}\tilde{\mathscr{D}}\left\{\frac{\partial\Theta}{\partial r_i}\right\}
\end{aligned}\tag{D.1}$$

where prime denotes differentiation with respect to $\tilde{u}^2$. Now using eq. (2.16), the action of $\tilde{\mathscr{D}}$ on various variables can be obtained as following.

$$\begin{aligned}
\tilde{\mathscr{D}}K = \tilde{\mathscr{D}}\left(\frac{\ell}{L}\right) &= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\left(\frac{1}{\pi nd^2}\frac{3g}{2\Theta}\right) \\
&= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{3g}{2\pi d^2}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\left(\frac{1}{n\Theta}\right) = -\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{3g}{2\pi d^2}\left(\frac{1}{n\Theta}\right)^2\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)(n\Theta) \\
&= -\left(\frac{1}{\pi nd^2}\frac{3g}{2\Theta}\right)\left[\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\ln(n\Theta)\right]
\end{aligned}$$

or

$$\tilde{\mathscr{D}}K = -K\tilde{\mathscr{D}}\ln(n\Theta) = -K(\tilde{\mathscr{D}}\ln n + \tilde{\mathscr{D}}\ln\Theta). \tag{D.2}$$

$$\begin{aligned}
\tilde{\mathscr{D}}(\tilde{u}^2) &= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\left\{\frac{3}{2\Theta}(\boldsymbol{v}-\boldsymbol{V})^2\right\} \\
&= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left[\frac{3}{2\Theta}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)(v_i-V_i)^2+(\boldsymbol{v}-\boldsymbol{V})^2\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\frac{3}{2\Theta}\right] \\
&= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\times\frac{3}{2\Theta}2(v_i-V_i)\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)(-V_i) \\
&\quad+\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\times(\boldsymbol{v}-\boldsymbol{V})^2\left(-\frac{3}{2\Theta^2}\right)\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\Theta \\
&= -2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(v_i-V_i)\right\}\left\{\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)V_i\right\} \\
&\quad-\left\{\frac{3}{2\Theta}(\boldsymbol{v}-\boldsymbol{V})^2\right\}\left\{\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\ln\Theta\right\} \\
&= -2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\,\tilde{\mathscr{D}}V_i-\tilde{u}^2\tilde{\mathscr{D}}\ln\Theta.
\end{aligned} \tag{D.3}$$

$$\begin{aligned}
\tilde{\mathscr{D}}\tilde{u}_j &= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(v_j-V_j)\right\} \\
&= \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left[\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)(-V_j)\right] \\
&\quad+\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}(v_j-V_j)\frac{1}{2}\left(\frac{3}{2\Theta}\right)^{-\frac{1}{2}}\times\frac{3}{2}\left(-\frac{1}{\Theta^2}\right)\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\Theta \\
&= -\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}V_j-\frac{1}{2}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(v_j-V_j)\right\}\left[\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\ln\Theta\right] \\
&= -\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}V_j-\frac{1}{2}\tilde{u}_j\tilde{\mathscr{D}}\ln\Theta.
\end{aligned} \tag{D.4}$$

$$\tilde{\mathscr{D}}\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\} = \tilde{\mathscr{D}}\left\{\frac{\tilde{u}_i\tilde{u}_j+\tilde{u}_j\tilde{u}_i}{2}-\frac{1}{3}\tilde{u}_k\tilde{u}_k\delta_{ij}\right\} = \tilde{u}_i\,\tilde{\mathscr{D}}\tilde{u}_j+\tilde{u}_j\,\tilde{\mathscr{D}}\tilde{u}_i-\frac{2}{3}\,\delta_{ij}\tilde{u}_k\,\tilde{\mathscr{D}}\tilde{u}_k \tag{D.5}$$

$$\begin{aligned}
\tilde{\mathscr{D}}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} &= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \\
&= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\times\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{-\frac{1}{2}}\frac{2}{3}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\Theta \\
&= \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\times\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{\Theta}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\Theta \\
&= \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\ln\Theta = \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}\ln\Theta.
\end{aligned} \tag{D.6}$$

$$\tilde{\mathscr{D}}\left(\frac{\partial V_i}{\partial r_j}\right) = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\frac{\partial V_i}{\partial r_j} = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_j}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)V_i$$
$$= \frac{\partial}{\partial r_j}\left\{\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)V_i\right\} - \frac{1}{g}\left\{\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)V_i\right\}\frac{\partial}{\partial r_j}\left\{K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\},$$

where

$$\frac{\partial}{\partial r_j}\left\{K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\} = \frac{\partial}{\partial r_j}\left\{\frac{1}{\pi n d^2}\frac{3g}{2\Theta}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\} = \frac{g}{\pi d^2}\frac{\partial}{\partial r_j}\left\{\frac{1}{n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\right\}$$
$$= \frac{g}{\pi d^2}\left\{-\frac{1}{n^2}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial n}{\partial r_j} + \frac{1}{n}\frac{1}{2}\left(\frac{3}{2\Theta}\right)^{-\frac{1}{2}}\left(-\frac{3}{2\Theta^2}\right)\frac{\partial \Theta}{\partial r_j}\right\}$$
$$= -\frac{g}{\pi d^2}\frac{1}{n}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(\frac{1}{n}\frac{\partial n}{\partial r_j} + \frac{1}{2}\frac{1}{\Theta}\frac{\partial \Theta}{\partial r_j}\right)$$
$$= -\left(\frac{1}{\pi n d^2}\frac{3g}{2\Theta}\right)\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial \ln n}{\partial r_j} + \frac{1}{2}\frac{\partial \ln \Theta}{\partial r_j}\right)$$
$$= -K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_j}\left(\ln n + \frac{1}{2}\ln \Theta\right).$$

Hence

$$\tilde{\mathscr{D}}\left(\frac{\partial V_i}{\partial r_j}\right) = \frac{\partial}{\partial r_j}\left(\tilde{\mathscr{D}}V_i\right) - \frac{1}{g}\left\{\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)V_i\right\}\left\{-K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_j}\left(\ln n + \frac{1}{2}\ln \Theta\right)\right\}$$
$$= \frac{\partial}{\partial r_j}\left(\tilde{\mathscr{D}}V_i\right) + \left\{\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)V_i\right\}\frac{\partial}{\partial r_j}\left(\ln n + \frac{1}{2}\ln \Theta\right)$$
$$= \frac{\partial}{\partial r_j}\left(\tilde{\mathscr{D}}V_i\right) + \left(\tilde{\mathscr{D}}V_i\right)\frac{\partial}{\partial r_j}\left(\ln n + \frac{1}{2}\ln \Theta\right). \tag{D.7}$$

$$\tilde{\mathscr{D}}\left(\frac{\partial \Theta}{\partial r_i}\right) = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\frac{\partial \Theta}{\partial r_i} = \frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_i}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\Theta$$
$$= \frac{\partial}{\partial r_i}\left\{\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\Theta\right\} - \frac{1}{g}\left\{\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\Theta\right\}\frac{\partial}{\partial r_i}\left\{K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\}$$
$$= \frac{\partial}{\partial r_i}\left(\tilde{\mathscr{D}}\Theta\right) - \frac{1}{g}\left\{\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\Theta\right\}\left\{-K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln \Theta\right)\right\}$$
$$= \frac{\partial}{\partial r_i}\left(\tilde{\mathscr{D}}\Theta\right) + \left\{\frac{K}{g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial t}+v_k\frac{\partial}{\partial r_k}\right)\Theta\right\}\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln \Theta\right)$$
$$= \frac{\partial}{\partial r_i}\left(\tilde{\mathscr{D}}\Theta\right) + \left(\tilde{\mathscr{D}}\Theta\right)\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln \Theta\right).$$

Since $\tilde{\mathscr{D}}$ is a differential operator, one can write $\tilde{\mathscr{D}}\ln\Theta = \frac{1}{\Theta}\tilde{\mathscr{D}}\Theta$ and hence

$$\tilde{\mathscr{D}}\left(\frac{\partial \Theta}{\partial r_i}\right) = \frac{\partial}{\partial r_i}\left(\Theta\tilde{\mathscr{D}}\ln\Theta\right) + \left(\Theta\tilde{\mathscr{D}}\ln\Theta\right)\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln \Theta\right). \tag{D.8}$$

Appendix E

The term: $\tilde{\mathscr{D}}_K \Phi_K + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{\boldsymbol{g}} \cdot \boldsymbol{\nabla}_v \Phi_K$

Using eq. (D.1), we have

$$
\begin{aligned}
\tilde{\mathscr{D}}_K \Phi_K =& \, 2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}_K K + 2K\frac{2\Theta}{3g}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\hat{\Phi}'_v(\tilde{u})\tilde{\mathscr{D}}_K(\tilde{u}^2) \\
&+ 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}_K\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\} + 2K\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial V_i}{\partial r_j}\frac{1}{g}\tilde{\mathscr{D}}_K\left\{\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\} \\
&+ 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}_K\left\{\frac{\partial V_i}{\partial r_j}\right\} + \frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i}\tilde{\mathscr{D}}_K K \\
&+ K\frac{2\Theta}{3g}\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i}\hat{\Phi}'_c(\tilde{u})\tilde{\mathscr{D}}_K(\tilde{u}^2) + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i}\tilde{\mathscr{D}}_K(\tilde{u}^2) \\
&+ K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\frac{\partial \ln\Theta}{\partial r_i}\tilde{\mathscr{D}}_K\tilde{u}_i + K\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{2}{3g}\tilde{\mathscr{D}}_K\left\{\frac{\partial\Theta}{\partial r_i}\right\} \\
=& \, 2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}_K K + \frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i}\tilde{\mathscr{D}}_K K \\
&+ 2K\frac{2\Theta}{3g}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\hat{\Phi}'_v(\tilde{u})\tilde{\mathscr{D}}_K(\tilde{u}^2) + K\frac{2\Theta}{3g}\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i}\hat{\Phi}'_c(\tilde{u})\tilde{\mathscr{D}}_K(\tilde{u}^2) \\
&+ K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i}\tilde{\mathscr{D}}_K(\tilde{u}^2) + K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\frac{\partial \ln\Theta}{\partial r_i}\tilde{\mathscr{D}}_K\tilde{u}_i \\
&+ 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}_K\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\} + 2K\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial V_i}{\partial r_j}\frac{1}{g}\tilde{\mathscr{D}}_K\left\{\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\} \\
&+ 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}_K\left\{\frac{\partial V_i}{\partial r_j}\right\} + K\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\tilde{u}_i\frac{2}{3g}\tilde{\mathscr{D}}_K\left\{\frac{\partial\Theta}{\partial r_i}\right\}. \qquad \text{(E.1)}
\end{aligned}
$$

The action of $\tilde{\mathscr{D}}_K$ on various variables can be obtained with the help of eqs. (D.2)-(D.8). Using eq. (D.2),

$$
\tilde{\mathscr{D}}_K K = -K\tilde{\mathscr{D}}_K \ln(n\Theta) = -K\left(\tilde{\mathscr{D}}_K \ln n + \tilde{\mathscr{D}}_K \ln\Theta\right).
$$

Therefore, using eqs. (3.2) and (3.4),

$$
\begin{aligned}
\tilde{\mathscr{D}}_K K &= -K\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial \ln n}{\partial r_i} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\} + K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}\right] \\
&= -K^2\frac{2\Theta}{3g}\left[\left\{\tilde{u}_i\frac{\partial \ln n}{\partial r_i} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\} + \left\{\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}\right]. \qquad \text{(E.2)}
\end{aligned}
$$

Using eq. (D.3),

$$\tilde{\mathscr{D}}_K(\tilde{u}^2) = -2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{u}_i\, \tilde{\mathscr{D}}_K V_i - \tilde{u}^2 \tilde{\mathscr{D}}_K \ln\Theta.$$

Therefore, using eqs. (3.3) and (3.4),

$$\begin{aligned}
\tilde{\mathscr{D}}_K(\tilde{u}^2) &= -2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \tilde{u}_i \left[K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial V_i}{\partial r_j} - \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial \ln(n\Theta)}{\partial r_i}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\right] \\
&\quad - \tilde{u}^2\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}\right] \\
&= -K\frac{2\Theta}{3g}\left\{2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{u}_j\frac{\partial V_i}{\partial r_j} - \tilde{u}_i\frac{\partial \ln(n\Theta)}{\partial r_i}\right\} - 2K\tilde{g}_i\tilde{u}_i \\
&\quad - K\frac{2\Theta}{3g}\left\{\tilde{u}^2\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}^2\frac{\partial V_i}{\partial r_i}\right\} \\
&= K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial \ln(n\Theta)}{\partial r_i} - \tilde{u}^2\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} - 2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\tilde{u}_j\frac{\partial V_i}{\partial r_j} + \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}^2\frac{\partial V_i}{\partial r_i}\right\} - 2K\tilde{g}_i\tilde{u}_i.
\end{aligned} \tag{E.3}$$

Using eq. (D.4),

$$\tilde{\mathscr{D}}_K\tilde{u}_j = -\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}_K V_j - \frac{1}{2}\tilde{u}_j\tilde{\mathscr{D}}_K\ln\Theta.$$

Therefore, using eqs. (3.3) and (3.4),

$$\begin{aligned}
\tilde{\mathscr{D}}_K\tilde{u}_j &= -\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial V_j}{\partial r_i} - \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial \ln(n\Theta)}{\partial r_j}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_j\right] \\
&\quad - \frac{1}{2}\tilde{u}_j\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}\right] \\
&= K\frac{2\Theta}{3g}\left\{\frac{1}{2}\frac{\partial \ln(n\Theta)}{\partial r_j} - \frac{1}{2}\tilde{u}_i\tilde{u}_j\frac{\partial \ln\Theta}{\partial r_i} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\frac{\partial V_j}{\partial r_i} + \frac{1}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_j\frac{\partial V_i}{\partial r_i}\right\} - K\tilde{g}_j.
\end{aligned} \tag{E.4}$$

Using eq. (D.5),

$$\tilde{\mathscr{D}}_K(\overline{\tilde{u}_i\tilde{u}_j}) = \tilde{u}_i\,\tilde{\mathscr{D}}_K\tilde{u}_j + \tilde{u}_j\,\tilde{\mathscr{D}}_K\tilde{u}_i - \frac{2}{3}\delta_{ij}\tilde{u}_k\,\tilde{\mathscr{D}}_K\tilde{u}_k.$$

Hence, using eq. (E.4),

$$\begin{aligned}
&\tilde{\mathscr{D}}_K\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\} \\
&= \tilde{u}_i\left[K\frac{2\Theta}{3g}\left\{\frac{1}{2}\frac{\partial \ln(n\Theta)}{\partial r_j} - \frac{1}{2}\tilde{u}_k\tilde{u}_j\frac{\partial \ln\Theta}{\partial r_k} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_k\frac{\partial V_j}{\partial r_k} + \frac{1}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_j\frac{\partial V_k}{\partial r_k}\right\} - K\tilde{g}_j\right] \\
&\quad + \tilde{u}_j\left[K\frac{2\Theta}{3g}\left\{\frac{1}{2}\frac{\partial \ln(n\Theta)}{\partial r_k} - \frac{1}{2}\tilde{u}_k\tilde{u}_i\frac{\partial \ln\Theta}{\partial r_k} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_k\frac{\partial V_i}{\partial r_k} + \frac{1}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\frac{\partial V_k}{\partial r_k}\right\} - K\tilde{g}_i\right] \\
&\quad - \frac{2}{3}\delta_{ij}\tilde{u}_k\left[K\frac{2\Theta}{3g}\left\{\frac{1}{2}\frac{\partial \ln(n\Theta)}{\partial r_k} - \frac{1}{2}\tilde{u}_l\tilde{u}_k\frac{\partial \ln\Theta}{\partial r_l} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_l\frac{\partial V_k}{\partial r_l} + \frac{1}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_k\frac{\partial V_l}{\partial r_l}\right\} - K\tilde{g}_k\right]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \tilde{\mathscr{D}}_K\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\} = K\frac{2\Theta}{3g}\Bigg[&\left\{\frac{1}{2}\left(\tilde{u}_i\frac{\partial\ln(n\Theta)}{\partial r_j} + \tilde{u}_j\frac{\partial\ln(n\Theta)}{\partial r_k}\right) - \frac{1}{3}\delta_{ij}\tilde{u}_k\frac{\partial\ln(n\Theta)}{\partial r_k}\right\} \\
&- \frac{1}{2}\left\{\tilde{u}_i\tilde{u}_k\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_k} + \tilde{u}_j\tilde{u}_k\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_k} - \frac{2}{3}\delta_{ij}\tilde{u}_k\tilde{u}_l\tilde{u}_k\frac{\partial\ln\Theta}{\partial r_l}\right\} \\
&- \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\tilde{u}_i\tilde{u}_k\frac{\partial V_j}{\partial r_k} + \tilde{u}_j\tilde{u}_k\frac{\partial V_k}{\partial r_k} - \frac{2}{3}\delta_{ij}\tilde{u}_k\tilde{u}_l\frac{\partial V_k}{\partial r_l}\right\} \\
&+ \frac{1}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\tilde{u}_i\tilde{u}_j\frac{\partial V_k}{\partial r_k} + \tilde{u}_j\tilde{u}_i\frac{\partial V_k}{\partial r_k} - \frac{2}{3}\delta_{ij}\tilde{u}_k\tilde{u}_k\frac{\partial V_l}{\partial r_l}\right\}\Bigg] \\
&- 2K\left(\frac{\tilde{u}_i\tilde{g}_j + \tilde{u}_j\tilde{g}_i}{2} - \frac{1}{3}\delta_{ij}\tilde{u}_k\tilde{g}_k\right) \\
= K\frac{2\Theta}{3g}\Bigg[&\overline{\tilde{u}_i\frac{\partial\ln(n\Theta)}{\partial r_j}} - \tilde{u}_k\frac{\partial\ln\Theta}{\partial r_k}\left\{\frac{\tilde{u}_i\tilde{u}_j + \tilde{u}_j\tilde{u}_i}{2} - \frac{1}{3}\delta_{ij}\tilde{u}_l\tilde{u}_l\right\} \\
&- \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\tilde{u}_i\tilde{u}_k\frac{\partial V_j}{\partial r_k} + \tilde{u}_j\tilde{u}_k\frac{\partial V_k}{\partial r_k} - \frac{2}{3}\delta_{ij}\tilde{u}_k\tilde{u}_l\frac{\partial V_k}{\partial r_l}\right\} \\
&+ \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\left\{\frac{\tilde{u}_i\tilde{u}_j + \tilde{u}_j\tilde{u}_i}{2} - \frac{1}{3}\delta_{ij}\tilde{u}_l\tilde{u}_l\right\}\Bigg] - 2K\overline{\tilde{u}_i\tilde{g}_j} \\
= K\frac{2\Theta}{3g}&\left\{\overline{\tilde{u}_i\frac{\partial\ln(n\Theta)}{\partial r_j}} - \overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\frac{\partial\ln\Theta}{\partial r_k} + \frac{2}{3}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\right\} \\
&- K\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(\tilde{u}_i\tilde{u}_k\frac{\partial V_j}{\partial r_k} + \tilde{u}_j\tilde{u}_k\frac{\partial V_i}{\partial r_k} - \frac{2}{3}\delta_{ij}\tilde{u}_k\tilde{u}_l\frac{\partial V_k}{\partial r_l}\right) - 2K\overline{\tilde{u}_i\tilde{g}_j}\,. \qquad \text{(E.5)}
\end{aligned}$$

Using eq. (D.6),

$$\tilde{\mathscr{D}}_K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}_K\ln\Theta.$$

Therefore, using eq. (3.4),

$$\begin{aligned}
\tilde{\mathscr{D}}_K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} &= \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}\right] \\
&= \frac{1}{2}K\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left\{\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_i}\right\}. \qquad \text{(E.6)}
\end{aligned}$$

Using eq. (D.7),

$$\tilde{\mathscr{D}}_K\left(\frac{\partial V_i}{\partial r_j}\right) = \frac{\partial}{\partial r_j}\left(\tilde{\mathscr{D}}_K V_i\right) + \left(\tilde{\mathscr{D}}_K V_i\right)\frac{\partial}{\partial r_j}\left(\ln n + \frac{1}{2}\ln\Theta\right).$$

Therefore, using eq. (3.3),

$$\begin{aligned}
\tilde{\mathscr{D}}_K\left(\frac{\partial V_i}{\partial r_j}\right) = \frac{\partial}{\partial r_j}&\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_k\frac{\partial V_i}{\partial r_k} - \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\ln(n\Theta)}{\partial r_i}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\right] \\
&+ \left[K\frac{2\Theta}{3g}\left\{\tilde{u}_k\frac{\partial V_i}{\partial r_k} - \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\ln(n\Theta)}{\partial r_i}\right\} + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\right]\frac{\partial}{\partial r_j}\left(\ln n + \frac{1}{2}\ln\Theta\right)
\end{aligned}$$

or

$$\tilde{\mathscr{D}}_K\left(\frac{\partial V_i}{\partial r_j}\right)=K\frac{2\Theta}{3g}\left[\tilde{u}_k\frac{\partial^2 V_i}{\partial r_j\partial r_k}+\frac{\partial \tilde{u}_k}{\partial r_j}\frac{\partial V_i}{\partial r_k}-\frac{1}{2}\left\{\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial^2\ln(n\Theta)}{\partial r_j\partial r_i}+\frac{\partial\ln(n\Theta)}{\partial r_i}\frac{\partial}{\partial r_j}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\}\right]$$
$$+\left\{\tilde{u}_k\frac{\partial V_i}{\partial r_k}-\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\ln(n\Theta)}{\partial r_i}\right\}\frac{\partial}{\partial r_j}\left(K\frac{2\Theta}{3g}\right)+\tilde{g}_i\frac{\partial}{\partial r_j}\left\{K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\}$$
$$+K\frac{2\Theta}{3g}\left\{\tilde{u}_k\frac{\partial V_i}{\partial r_k}-\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\ln(n\Theta)}{\partial r_i}\right\}\frac{\partial}{\partial r_j}\left(\ln n+\frac{1}{2}\ln\Theta\right)$$
$$+K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\frac{\partial}{\partial r_j}\left(\ln n+\frac{1}{2}\ln\Theta\right).$$

Now, let us simplify the following terms in the above expression, separately.

$$\frac{\partial\tilde{u}_k}{\partial r_j}=\frac{\partial}{\partial r_j}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(v_k-V_k)\right\}=\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_j}(-V_k)+(v_k-V_k)\frac{1}{2}\left(\frac{3}{2\Theta}\right)^{-\frac{1}{2}}\left(-\frac{3}{2\Theta^2}\right)\frac{\partial\Theta}{\partial r_j}$$
$$=-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_j}-\frac{1}{2}\left\{\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}(v_k-V_k)\right\}\frac{\partial\ln\Theta}{\partial r_j}=-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_j}-\frac{1}{2}\tilde{u}_k\frac{\partial\ln\Theta}{\partial r_j},$$

$$\frac{\partial}{\partial r_j}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}=\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{-\frac{1}{2}}\frac{2}{3}\frac{\partial\Theta}{\partial r_j}=\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{1}{\Theta}\frac{\partial\Theta}{\partial r_j}=\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\ln\Theta}{\partial r_j},$$

and

$$\frac{\partial}{\partial r_j}\left(K\frac{2\Theta}{3g}\right)=\frac{\partial}{\partial r_j}\left(\frac{1}{\pi n d^2}\right)=\frac{1}{\pi d^2}\left(-\frac{1}{n^2}\right)\frac{\partial n}{\partial r_j}=-\frac{1}{\pi n d^2}\frac{\partial\ln n}{\partial r_j}=-K\frac{2\Theta}{3g}\frac{\partial\ln n}{\partial r_j}.$$

Hence

$$\tilde{\mathscr{D}}_K\left(\frac{\partial V_i}{\partial r_j}\right)=K\frac{2\Theta}{3g}\left[\tilde{u}_k\frac{\partial^2 V_i}{\partial r_j\partial r_k}+\left\{-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_j}-\frac{1}{2}\tilde{u}_k\frac{\partial\ln\Theta}{\partial r_j}\right\}\frac{\partial V_i}{\partial r_k}\right.$$
$$\left.-\frac{1}{2}\left\{\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial^2\ln(n\Theta)}{\partial r_j\partial r_i}+\frac{\partial\ln(n\Theta)}{\partial r_i}\left(\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\ln\Theta}{\partial r_j}\right)\right\}\right]$$
$$+\left\{\tilde{u}_k\frac{\partial V_i}{\partial r_k}-\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\ln(n\Theta)}{\partial r_i}\right\}\left(-K\frac{2\Theta}{3g}\frac{\partial\ln n}{\partial r_j}\right)$$
$$+\tilde{g}_i\left\{-K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_j}\left(\ln n+\frac{1}{2}\ln\Theta\right)\right\}$$
$$+K\frac{2\Theta}{3g}\left\{\tilde{u}_k\frac{\partial V_i}{\partial r_k}-\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\ln(n\Theta)}{\partial r_i}\right\}\frac{\partial}{\partial r_j}\left(\ln n+\frac{1}{2}\ln\Theta\right)$$
$$+K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{g}_i\frac{\partial}{\partial r_j}\left(\ln n+\frac{1}{2}\ln\Theta\right)$$

or

$$\begin{aligned}
\tilde{\mathscr{D}}_K \left(\frac{\partial V_i}{\partial r_j}\right) &= K\frac{2\Theta}{3g}\left[\tilde{u}_k \frac{\partial^2 V_i}{\partial r_j \partial r_k} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j} - \frac{1}{2}\tilde{u}_k \frac{\partial V_i}{\partial r_k}\frac{\partial \ln \Theta}{\partial r_j}\right. \\
&\quad \left. -\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \frac{\partial^2 \ln(n\Theta)}{\partial r_i \partial r_j} - \frac{1}{4}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \frac{\partial \ln(n\Theta)}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j}\right] \\
&\quad + K\frac{2\Theta}{3g}\left\{\frac{1}{2}\tilde{u}_k \frac{\partial V_i}{\partial r_k}\frac{\partial \ln \Theta}{\partial r_j} - \frac{1}{4}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \frac{\partial \ln(n\Theta)}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j}\right\} \\
&= K\frac{2\Theta}{3g}\left\{\tilde{u}_k \frac{\partial^2 V_i}{\partial r_j \partial r_k} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j} - \frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \frac{\partial^2 \ln(n\Theta)}{\partial r_i \partial r_j}\right. \\
&\quad \left. -\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \frac{\partial \ln(n\Theta)}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j}\right\}. \qquad \text{(E.7)}
\end{aligned}$$

Using eq. (D.8),

$$\tilde{\mathscr{D}}_K \left(\frac{\partial \Theta}{\partial r_i}\right) = \frac{\partial}{\partial r_i}\left(\Theta \tilde{\mathscr{D}}_K \ln \Theta\right) + \left(\Theta \tilde{\mathscr{D}}_K \ln \Theta\right)\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln \Theta\right).$$

Therefore, using eq. (3.4),

$$\begin{aligned}
\tilde{\mathscr{D}}_K \left(\frac{\partial \Theta}{\partial r_i}\right) &= \frac{\partial}{\partial r_i}\left[\Theta\left\{K\frac{2\Theta}{3g}\left(\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right)\right\}\right] \\
&\quad + \left[\Theta\left\{K\frac{2\Theta}{3g}\left(\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right)\right\}\right]\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln \Theta\right) \\
&= \left\{\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right\}\left\{K\frac{2\Theta}{3g}\frac{\partial \Theta}{\partial r_i} + \Theta\frac{\partial}{\partial r_i}\left(K\frac{2\Theta}{3g}\right)\right\} \\
&\quad + \Theta K\frac{2\Theta}{3g}\left[\frac{\partial \tilde{u}_j}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j} + \tilde{u}_j \frac{\partial^2 \ln \Theta}{\partial r_i \partial r_j} - \frac{2}{3}\frac{\partial V_j}{\partial r_j}\frac{\partial}{\partial r_i}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial^2 V_j}{\partial r_i \partial r_j}\right] \\
&\quad + \Theta K\frac{2\Theta}{3g}\left\{\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right\}\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln \Theta\right) \\
&= \left\{\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right\}\left\{\Theta K\frac{2\Theta}{3g}\frac{\partial \ln \Theta}{\partial r_i} + \Theta\left(-K\frac{2\Theta}{3g}\frac{\partial \ln n}{\partial r_i}\right)\right\} \\
&\quad + \Theta K\frac{2\Theta}{3g}\left[\left\{-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_i} - \frac{1}{2}\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_i}\right\}\frac{\partial \ln \Theta}{\partial r_j} + \tilde{u}_j \frac{\partial^2 \ln \Theta}{\partial r_i \partial r_j}\right. \\
&\quad \left. -\frac{2}{3}\frac{\partial V_j}{\partial r_j}\left\{-\frac{1}{2}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{1}{\Theta}\frac{\partial \Theta}{\partial r_i}\right\} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial^2 V_j}{\partial r_i \partial r_j}\right] \\
&\quad + \Theta K\frac{2\Theta}{3g}\left\{\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right\}\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln \Theta\right) \quad \text{(as above)}
\end{aligned}$$

or

$$\tilde{\mathscr{D}}_K\left(\frac{\partial\Theta}{\partial r_i}\right) = \Theta K\frac{2\Theta}{3g}\left[\left\{\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}\frac{\partial}{\partial r_i}(-\ln n + \ln\Theta) - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\right.$$
$$-\frac{1}{2}\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j} + \tilde{u}_j\frac{\partial^2\ln\Theta}{\partial r_i\partial r_j} + \frac{1}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_i} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial^2 V_j}{\partial r_i\partial r_j}$$
$$\left.+\left\{\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}\frac{\partial}{\partial r_i}\left(\ln n + \frac{1}{2}\ln\Theta\right)\right]$$
$$= \Theta K\frac{2\Theta}{3g}\left[\left\{\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}\frac{\partial\ln\Theta}{\partial r_i} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\right.$$
$$+\tilde{u}_j\frac{\partial^2\ln\Theta}{\partial r_i\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial^2 V_j}{\partial r_i\partial r_j} - \frac{1}{2}\left\{\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}\frac{\partial\ln\Theta}{\partial r_i}$$
$$\left.+\frac{1}{2}\left\{\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}\frac{\partial\ln\Theta}{\partial r_i}\right]$$
$$= \Theta K\frac{2\Theta}{3g}\left[\left\{\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}\frac{\partial\ln\Theta}{\partial r_i} + \tilde{u}_j\frac{\partial^2\ln\Theta}{\partial r_i\partial r_j}\right.$$
$$\left.-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial^2 V_j}{\partial r_i\partial r_j}\right]. \tag{E.8}$$

Now, we shall simplify each term on the right-hand side of eq. (E.1), with the help of eqs. (E.2)-(E.8) and (F.3), separately.

- **1st term in eq.** (E.1):

$$2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}_K K$$
$$= 2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\frac{\partial V_i}{\partial r_j}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}_K K$$
$$= 2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\frac{\partial V_i}{\partial r_j}}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}$$
$$\times\left[-K^2\frac{2\Theta}{3g}\left\{\left(\tilde{u}_k\frac{\partial\ln n}{\partial r_k} - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\right) + \left(\tilde{u}_k\frac{\partial\ln\Theta}{\partial r_k} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\right)\right\}\right]$$
$$= -2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial\ln n}{\partial r_k} - \tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\right\}$$
$$-2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial\ln\Theta}{\partial r_k} - \frac{2}{3}\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\right\}. \tag{E.9a}$$

- **2^nd term in eq. (E.1):**

$$
\begin{aligned}
&\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln \Theta}{\partial r_i}\tilde{\mathscr{D}}_K K \\
&= \frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln \Theta}{\partial r_i} \\
&\quad\times\left[-K^2\frac{2\Theta}{3g}\left\{\left(\tilde{u}_j\frac{\partial \ln n}{\partial r_j}-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right)+\left(\tilde{u}_j\frac{\partial \ln \Theta}{\partial r_j}-\frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right)\right\}\right] \\
&= -\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{\tilde{u}_i\tilde{u}_j\frac{\partial \ln n}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i}-\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i}\right\} \\
&\quad-\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial \ln \Theta}{\partial r_i}\left\{\tilde{u}_i\tilde{u}_j\frac{\partial \ln \Theta}{\partial r_j}-\frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}.
\end{aligned}
\tag{E.9b}
$$

- **3^rd term in eq. (E.1):**

$$
\begin{aligned}
&2K\frac{2\Theta}{3g}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\hat{\Phi}_v'(\tilde{u})\tilde{\mathscr{D}}_K(\tilde{u}^2) \\
&= 2K\frac{2\Theta}{3g}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\hat{\Phi}_v'(\tilde{u}) \\
&\quad\times\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_k\frac{\partial \ln(n\Theta)}{\partial r_k}-\tilde{u}^2\tilde{u}_k\frac{\partial \ln \Theta}{\partial r_k}-2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_k\tilde{u}_l\frac{\partial V_k}{\partial r_l}+\frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}^2\frac{\partial V_k}{\partial r_k}\right\}-2K\tilde{g}_k\tilde{u}_k\right] \\
&= -2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v'(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\left\{\tilde{u}^2\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial \ln \Theta}{\partial r_k}-\frac{2}{3}\tilde{u}^2\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\right\} \\
&\quad-2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v'(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\left\{2\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_l}-\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial \ln(n\Theta)}{\partial r_k}\right\} \\
&\quad-4K^2\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\hat{\Phi}_v'(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{g}_k.
\end{aligned}
\tag{E.9c}
$$

- **4^th term in eq. (E.1):**

$$
\begin{aligned}
&K\frac{2\Theta}{3g}\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln \Theta}{\partial r_i}\hat{\Phi}_c'(\tilde{u})\tilde{\mathscr{D}}_K(\tilde{u}^2) \\
&= K\frac{2\Theta}{3g}\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{\partial \ln \Theta}{\partial r_i}\hat{\Phi}_c'(\tilde{u})\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_j\frac{\partial \ln(n\Theta)}{\partial r_j}-\tilde{u}^2\tilde{u}_j\frac{\partial \ln \Theta}{\partial r_j}\right.\right. \\
&\quad\left.\left.-2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_j\tilde{u}_k\frac{\partial V_j}{\partial r_k}+\frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}^2\frac{\partial V_j}{\partial r_j}\right\}-2K\tilde{g}_j\tilde{u}_j\right] \\
&= -\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial \ln \Theta}{\partial r_i}\left\{\tilde{u}^2\tilde{u}_i\tilde{u}_j\frac{\partial \ln \Theta}{\partial r_j}-\frac{2}{3}\tilde{u}^2\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\} \\
&\quad-\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{2\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_k}\frac{\partial \ln \Theta}{\partial r_i}-\tilde{u}_i\tilde{u}_j\frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_j}\right\} \\
&\quad-2K^2\frac{2\Theta}{3g}\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial \ln \Theta}{\partial r_i}\hat{\Phi}_c'(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{g}_j.
\end{aligned}
\tag{E.9d}
$$

- **5^th^ term in eq. (E.1):**

$$
\begin{aligned}
&K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\tilde{\mathscr{D}}_K(\tilde{u}^2)\\
&=K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_i}\Bigg[K\frac{2\Theta}{3g}\Bigg\{\tilde{u}_j\frac{\partial\ln(n\Theta)}{\partial r_j}-\tilde{u}^2\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j}\\
&\quad-2\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_j\tilde{u}_k\frac{\partial V_j}{\partial r_k}+\frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}^2\frac{\partial V_j}{\partial r_j}\Bigg\}-2K\tilde{g}_j\tilde{u}_j\Bigg]\\
&=-\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\frac{\partial\ln\Theta}{\partial r_i}\left\{\tilde{u}^2\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j}-\frac{2}{3}\tilde{u}^2\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}\\
&\quad-\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left\{2\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_k}\frac{\partial\ln\Theta}{\partial r_i}-\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln(n\Theta)}{\partial r_j}\right\}\\
&\quad-2K^2\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\hat{\Phi}_c(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{g}_j.
\end{aligned}
\tag{E.9e}
$$

- **6^th^ term in eq. (E.1):**

$$
\begin{aligned}
&K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\tilde{\mathscr{D}}_K\tilde{u}_i\\
&=K\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\\
&\quad\times\left[K\frac{2\Theta}{3g}\left\{\frac{1}{2}\frac{\partial\ln(n\Theta)}{\partial r_i}-\frac{1}{2}\tilde{u}_j\tilde{u}_i\frac{\partial\ln\Theta}{\partial r_j}-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_j\frac{\partial V_i}{\partial r_j}+\frac{1}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{u}_i\frac{\partial V_j}{\partial r_j}\right\}-K\tilde{g}_i\right]\\
&=-\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_i}-\frac{1}{2}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln(n\Theta)}{\partial r_i}\right\}\\
&\quad-\frac{1}{2}\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\left\{\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j}-\frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}\\
&\quad-K^2\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\tilde{g}_i.
\end{aligned}
\tag{E.9f}
$$

- **7^th^ term in eq. (E.1):**

$$
\begin{aligned}
&2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}_K\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\}\\
&=2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\Bigg[K\frac{2\Theta}{3g}\Bigg\{\overline{\tilde{u}_i\frac{\partial\ln(n\Theta)}{\partial r_j}}-\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\frac{\partial\ln\Theta}{\partial r_k}+\frac{2}{3}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\Bigg\}\\
&\quad-K\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(\tilde{u}_i\tilde{u}_k\frac{\partial V_j}{\partial r_k}+\tilde{u}_j\tilde{u}_k\frac{\partial V_i}{\partial r_k}-\frac{2}{3}\delta_{ij}\tilde{u}_k\tilde{u}_l\frac{\partial V_k}{\partial r_l}\right)-2K\overline{\tilde{u}_i\tilde{g}_j}\Bigg]\\
&=2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\Bigg[\frac{\partial V_i}{\partial r_j}\Bigg\{\overline{\tilde{u}_i\frac{\partial\ln(n\Theta)}{\partial r_j}}-\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\frac{\partial\ln\Theta}{\partial r_k}+\frac{2}{3}\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\Bigg\}\\
&\quad-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left(\tilde{u}_i\tilde{u}_k\frac{\partial V_i}{\partial r_j}\frac{\partial V_j}{\partial r_k}+\tilde{u}_j\tilde{u}_k\frac{\partial V_i}{\partial r_j}\frac{\partial V_i}{\partial r_k}-\frac{2}{3}\tilde{u}_k\tilde{u}_l\frac{\partial V_i}{\partial r_i}\frac{\partial V_k}{\partial r_l}\right)-2K\frac{\partial V_i}{\partial r_j}\overline{\tilde{u}_i\tilde{g}_j}\Bigg].
\end{aligned}
$$

Now, using the simplification

$$\begin{aligned}
&\tilde{u}_i\tilde{u}_k \frac{\partial V_i}{\partial r_j}\frac{\partial V_j}{\partial r_k} + \tilde{u}_j\tilde{u}_k \frac{\partial V_i}{\partial r_j}\frac{\partial V_i}{\partial r_k} - \frac{2}{3}\tilde{u}_k\tilde{u}_l \frac{\partial V_i}{\partial r_i}\frac{\partial V_k}{\partial r_l} \\
&= \tilde{u}_i\tilde{u}_j \frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j} + \tilde{u}_j\tilde{u}_i \frac{\partial V_k}{\partial r_j}\frac{\partial V_k}{\partial r_i} - \frac{2}{3}(\delta_{ik}\tilde{u}_i)\tilde{u}_j \frac{\partial V_l}{\partial r_l}\frac{\partial V_k}{\partial r_j} \\
&= 2\tilde{u}_i\tilde{u}_j \left\{ \frac{1}{2}\left(\frac{\partial V_i}{\partial r_k} + \frac{\partial V_k}{\partial r_i} \right) - \frac{1}{3}\delta_{ik}\frac{\partial V_l}{\partial r_l} \right\} \frac{\partial V_k}{\partial r_j} = 2\tilde{u}_i\tilde{u}_j \overline{\frac{\partial V_i}{\partial r_k}} \frac{\partial V_k}{\partial r_j},
\end{aligned}$$

and eqs. (F.3) and (F.2), we get

$$\begin{aligned}
&2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j}\tilde{\mathscr{D}}_K\left\{\overline{\tilde{u}_i\tilde{u}_j}\right\} \\
&= 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \left[\overline{\frac{\partial V_i}{\partial r_j}} \left\{ \tilde{u}_i \frac{\partial \ln(n\Theta)}{\partial r_j} - \tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln\Theta}{\partial r_k} + \frac{2}{3}\tilde{u}_i\tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k} \right\} \right. \\
&\quad \left. - \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \times 2\tilde{u}_i\tilde{u}_j \overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_j} - 2K\frac{\partial V_i}{\partial r_j}\overline{\tilde{u}_i\tilde{g}_j} \right] \\
&= -2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \left\{ 2\tilde{u}_i\tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_j} - \tilde{u}_i \overline{\frac{\partial V_i}{\partial r_k}} \frac{\partial \ln(n\Theta)}{\partial r_k} \right\} \\
&\quad - 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} \left\{ \tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln\Theta}{\partial r_k} - \frac{2}{3}\tilde{u}_i\tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k} \right\} \\
&\quad - 4K^2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j}\overline{\tilde{u}_i\tilde{g}_j}\,. \qquad \text{(E.9g)}
\end{aligned}$$

- **8th term in eq. (E.1):**

$$\begin{aligned}
&2K\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial V_i}{\partial r_j}\frac{1}{g}\tilde{\mathscr{D}}_K\left\{\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\} \\
&= 2K\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\frac{\partial V_i}{\partial r_j}}\frac{1}{g}\tilde{\mathscr{D}}_K\left\{\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\right\} \\
&= 2K\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\frac{\partial V_i}{\partial r_j}}\frac{1}{g}\left[\frac{1}{2}K\frac{2\Theta}{3g}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\left\{\tilde{u}_k\frac{\partial \ln\Theta}{\partial r_k} - \frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\right\}\right] \\
&= \left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial \ln\Theta}{\partial r_k} - \frac{2}{3}\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\right\}. \qquad \text{(E.9h)}
\end{aligned}$$

- **9th term in eq. (E.1):**

$$
\begin{aligned}
&2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}_K\left\{\frac{\partial V_i}{\partial r_j}\right\} \\
&= 2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left[K\frac{2\Theta}{3g}\left\{\tilde{u}_k\frac{\partial^2 V_i}{\partial r_j\partial r_k}-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}\right.\right. \\
&\left.\left.-\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial^2\ln(n\Theta)}{\partial r_i\partial r_j}-\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\frac{\partial\ln(n\Theta)}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\right\}\right] \\
&= 2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\overline{\tilde{u}_i\tilde{u}_j\tilde{u}_k}\frac{\partial}{\partial r_k}\frac{\partial V_i}{\partial r_j}-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}\right. \\
&\left.-\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial^2\ln(n\Theta)}{\partial r_i\partial r_j}-\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial\ln(n\Theta)}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}\right\}.
\end{aligned}
$$

Using eqs. (F.4)-(F.6)),

$$
\begin{aligned}
&2K\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\tilde{\mathscr{D}}_K\left\{\frac{\partial V_i}{\partial r_j}\right\} \\
&= 2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}}-\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}}\right. \\
&\left.-\tilde{u}_i\tilde{u}_j\frac{1}{2}\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial^2\ln(n\Theta)}{\partial r_i\partial r_j}}-\frac{1}{2}\tilde{u}_i\tilde{u}_j\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\overline{\frac{\partial\ln(n\Theta)}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}}\right\} \\
&= -2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}}-\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}}\right\} \\
&-\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left\{\tilde{u}_i\tilde{u}_j\overline{\frac{\partial\ln(n\Theta)}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}}+\tilde{u}_i\tilde{u}_j\overline{\frac{\partial^2\ln(n\Theta)}{\partial r_i\partial r_j}}\right\}.
\end{aligned}
\tag{E.9i}
$$

- **10th term in eq. (E.1):**

$$
\begin{aligned}
&K\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{2}{3g}\tilde{\mathscr{D}}_K\left\{\frac{\partial\Theta}{\partial r_i}\right\} \\
&= K\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}_i\frac{2}{3g}\left[\Theta K\frac{2\Theta}{3g}\left\{\left(\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j}-\frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right)\frac{\partial\ln\Theta}{\partial r_i}+\tilde{u}_j\frac{\partial^2\ln\Theta}{\partial r_i\partial r_j}\right.\right. \\
&\left.\left.-\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}-\frac{2}{3}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial^2 V_j}{\partial r_i\partial r_j}\right\}\right] \\
&= \left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{\tilde{u}_i\tilde{u}_j\frac{\partial^2\ln\Theta}{\partial r_i\partial r_j}-\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}-\frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial^2 V_j}{\partial r_i\partial r_j}\right\} \\
&+\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j}-\frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}.
\end{aligned}
\tag{E.9j}
$$

Adding eqs. (E.9a)-(E.9j), we get

$$
\begin{aligned}
\tilde{\mathscr{D}}_K \Phi_K = & \underbrace{-2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u}) \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} \left\{ \tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln n}{\partial r_k} - \tilde{u}_i\tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k} \right\}}_{1} \\
& \underbrace{-2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u}) \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} \left\{ \tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln \Theta}{\partial r_k} - \frac{2}{3}\tilde{u}_i\tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k} \right\}}_{8} \\
& \underbrace{-\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \left\{ \tilde{u}_i\tilde{u}_j \frac{\partial \ln n}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i} - \tilde{u}_i \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i} \right\}}_{2} \\
& \underbrace{-\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \frac{\partial \ln \Theta}{\partial r_i} \left\{ \tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\tilde{u}_i \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j} \right\}} \\
& \underbrace{-2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_v(\tilde{u}) \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} \left\{ \tilde{u}^2\tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln \Theta}{\partial r_k} - \frac{2}{3}\tilde{u}^2\tilde{u}_i\tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k} \right\}}_{9} \\
& \underbrace{-2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_v(\tilde{u}) \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} \left\{ 2\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_l} - \tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln (n\Theta)}{\partial r_k} \right\}}_{3} \\
& \underbrace{-4K^2\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j} \hat{\Phi}'_v(\tilde{u}) \overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{g}_k} \\
& \underbrace{-\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \frac{\partial \ln \Theta}{\partial r_i} \left\{ \tilde{u}^2\tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\tilde{u}^2\tilde{u}_i \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j} \right\}}_{10} \\
& \underbrace{-\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \left\{ 2\tilde{u}_i\tilde{u}_j\tilde{u}_k \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_k}\frac{\partial \ln \Theta}{\partial r_i} - \tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln (n\Theta)}{\partial r_j} \right\}}_{6} \\
& \underbrace{-2K^2\frac{2\Theta}{3g}\left(\tilde{u}^2 - \frac{5}{2}\right) \frac{\partial \ln \Theta}{\partial r_i} \hat{\Phi}'_c(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{g}_j} \\
& \underbrace{-\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u}) \frac{\partial \ln \Theta}{\partial r_i} \left\{ \tilde{u}^2\tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\tilde{u}^2\tilde{u}_i \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j} \right\}}_{11} \\
& \underbrace{-\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u}) \left\{ 2\tilde{u}_i\tilde{u}_j\tilde{u}_k \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_k}\frac{\partial \ln \Theta}{\partial r_i} - \tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln (n\Theta)}{\partial r_j} \right\}}_{5} \\
& \underbrace{-2K^2\frac{2\Theta}{3g}\frac{\partial \ln \Theta}{\partial r_i} \hat{\Phi}_c(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{g}_j} \\
& \underbrace{-\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u}) \left(\tilde{u}^2 - \frac{5}{2}\right) \left\{ \tilde{u}_j \left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i} - \frac{1}{2}\frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln (n\Theta)}{\partial r_i} \right\}}_{7}
\end{aligned}
$$

$$\underbrace{-\frac{1}{2}\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\left\{\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j}-\frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}}_{12}$$

$$\underbrace{-K^2\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\tilde{g}_i}$$

$$\underbrace{-2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{2\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_j}-\tilde{u}_i\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial\ln(n\Theta)}{\partial r_k}\right\}}_{4}$$

$$\underbrace{-2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial\ln\Theta}{\partial r_k}-\frac{2}{3}\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\right\}}_{8}$$

$$\underbrace{-4K^2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\overline{\tilde{u}_i\tilde{g}_j}}$$

$$\underbrace{+\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial\ln\Theta}{\partial r_k}-\frac{2}{3}\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_k}{\partial r_k}\right\}}_{8}$$

$$\underbrace{-2\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\left\{\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}}-\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}}\right\}}_{13}$$

$$\underbrace{-\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_v(\tilde{u})\left\{\tilde{u}_i\tilde{u}_j\overline{\frac{\partial\ln(n\Theta)}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}}+\tilde{u}_i\tilde{u}_j\overline{\frac{\partial^2\ln(n\Theta)}{\partial r_i\partial r_j}}\right\}}_{14}$$

$$\underbrace{+\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{\tilde{u}_i\tilde{u}_j\frac{\partial^2\ln\Theta}{\partial r_i\partial r_j}-\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}-\frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial^2 V_j}{\partial r_i\partial r_j}\right\}}_{15}$$

$$\underbrace{+\left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_j}-\frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\right\}}.$$

In the above equation, the numbers below the underbraces represent the order, in which the terms will appear in the next step, e.g., all the terms in the blue underbraces contribute the 8^{th} term in the next step. The terms in the red underbraces cancel each other and the terms in green underbraces simplify to the following (see eq. (4.140)):

$$\underbrace{-4K^2\frac{2\Theta}{3g}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\hat{\Phi}'_v(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{g}_k}\underbrace{-2K^2\frac{2\Theta}{3g}\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\hat{\Phi}'_c(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{g}_j}$$

$$\underbrace{-2K^2\frac{2\Theta}{3g}\frac{\partial\ln\Theta}{\partial r_i}\hat{\Phi}_c(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{g}_j}\underbrace{-K^2\frac{2\Theta}{3g}\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\frac{\partial\ln\Theta}{\partial r_i}\tilde{g}_i}$$

$$\underbrace{-4K^2\frac{2\Theta}{3g}\hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\overline{\tilde{u}_i\tilde{g}_j}}=-K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}}\tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v\Phi_K.$$

Hence

$$
\begin{aligned}
&\tilde{\mathscr{D}}_K \Phi_K + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{\boldsymbol{g}} \cdot \boldsymbol{\nabla}_v \Phi_K \\
&= -2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} \left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln n}{\partial r_k} - \tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k}\right\} \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\left\{\tilde{u}_i\tilde{u}_j \frac{\partial \ln n}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j} - \tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_i}\right\} \\
&\quad - 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} \left\{2\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_l} - \tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln(n\Theta)}{\partial r_k}\right\} \\
&\quad - 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \left\{2\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_j} - \tilde{u}_i \overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial \ln(n\Theta)}{\partial r_k}\right\} \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\left\{2\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_k} - \tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_j}\right\} \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\left\{2\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_i}{\partial r_j}\frac{\partial \ln \Theta}{\partial r_k} - \tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_j}\right\} \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\left\{\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j} - \frac{1}{2}\frac{\partial \ln \Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_i}\right\} \\
&\quad - 3\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} \left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln \Theta}{\partial r_k} - \frac{2}{3}\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k}\right\} \\
&\quad - 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_j}} \left\{\tilde{u}^2\tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial \ln \Theta}{\partial r_k} - \frac{2}{3}\tilde{u}^2\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_k}{\partial r_k}\right\} \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}'_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\frac{\partial \ln \Theta}{\partial r_i}\left\{\tilde{u}^2\tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\tilde{u}^2\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right\} \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\frac{\partial \ln \Theta}{\partial r_i}\left\{\tilde{u}^2\tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\tilde{u}^2\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right\} \\
&\quad - \frac{1}{2}\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\frac{\partial \ln \Theta}{\partial r_i}\left\{\tilde{u}_i\tilde{u}_j \frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_j}\right\} \\
&\quad - 2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \left\{\tilde{u}_i\tilde{u}_j\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}} - \tilde{u}_i\tilde{u}_j\tilde{u}_k \frac{\partial}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}}\right\} \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left\{\tilde{u}_i\tilde{u}_j \overline{\frac{\partial \ln(n\Theta)}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j}} + \tilde{u}_i\tilde{u}_j \overline{\frac{\partial^2 \ln(n\Theta)}{\partial r_i \partial r_j}}\right\} \\
&\quad + \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2 - \frac{5}{2}\right)\left\{\tilde{u}_i\tilde{u}_j \frac{\partial^2 \ln \Theta}{\partial r_i \partial r_j} - \tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial V_j}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j} - \frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}} \frac{\partial^2 V_j}{\partial r_i \partial r_j}\right\}.
\end{aligned}
\tag{E.10}
$$

Eqn. (E.10) can be written in another form, to compare it with the equations in Appendix E of Sela & Goldhirsch (1998).

$$
\begin{aligned}
&\tilde{\mathscr{D}}_K \Phi_K + K\left(\frac{2\Theta}{3}\right)^{\frac{1}{2}} \tilde{\boldsymbol{g}}\cdot\boldsymbol{\nabla}_v \Phi_K \\
&= -2\left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_v(\tilde{u})\left\{\tilde{u}_i\tilde{u}_j\tilde{u}_k\frac{\partial \ln n}{\partial r_k}\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}} - \tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\frac{\partial V_k}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}}\right\} \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2 \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{\tilde{u}_i\tilde{u}_j\frac{\partial \ln n}{\partial r_i}\frac{\partial \ln \Theta}{\partial r_j} - \tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\frac{\partial \ln\Theta}{\partial r_i}\right\} \\
&\quad - 2\left(K\frac{2\Theta}{3g}\right)^2\left[2\hat{\Phi}_v'(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_l} - \Phi_v'(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial \ln(n\Theta)}{\partial r_k}\right. \\
&\quad \left. +2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial V_k}{\partial r_j} - \hat{\Phi}_v(\tilde{u})\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial\ln(n\Theta)}{\partial r_k}\right] \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2\left[\left\{\hat{\Phi}_c(\tilde{u}) + \hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\right\}\right. \\
&\quad \times\left\{2\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_i}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_k} - \tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln(n\Theta)}{\partial r_j}\right\} \\
&\quad \left. + \hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j} - \frac{1}{2}\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln(n\Theta)}{\partial r_i}\right\}\right] \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2\left[\left\{3\hat{\Phi}_v(\tilde{u}) + 2\hat{\Phi}_v'(\tilde{u})\tilde{u}^2\right\}\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial\ln\Theta}{\partial r_k}\right. \\
&\quad \left. - 2\left\{\hat{\Phi}_v(\tilde{u}) + \frac{2}{3}\hat{\Phi}_v'(\tilde{u})\tilde{u}^2\right\}\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_j}}\frac{\partial V_k}{\partial r_k}\right] \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2\left[\left\{\hat{\Phi}_c'(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\tilde{u}^2 + \hat{\Phi}_c(\tilde{u})\left(\frac{3}{2}\tilde{u}^2-\frac{5}{4}\right)\right\}\right. \\
&\quad \left. \times\left\{\tilde{u}_i\tilde{u}_j\frac{\partial\ln\Theta}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j} - \frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_j}\frac{\partial\ln\Theta}{\partial r_i}\right\}\right] \\
&\quad - \left(K\frac{2\Theta}{3g}\right)^2\left[2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\frac{3}{2\Theta}\overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}} - 2\hat{\Phi}_v(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}}\right. \\
&\quad \left. +\Phi_v(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\frac{\partial\ln(n\Theta)}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j}} + \Phi_v(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\frac{\partial^2\ln(n\Theta)}{\partial r_i\partial r_j}}\right] \\
&\quad + \left(K\frac{2\Theta}{3g}\right)^2\hat{\Phi}_c(\tilde{u})\left(\tilde{u}^2-\frac{5}{2}\right)\left\{\tilde{u}_i\tilde{u}_j\frac{\partial^2\ln\Theta}{\partial r_i\partial r_j} - \tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial V_j}{\partial r_i}\frac{\partial\ln\Theta}{\partial r_j} - \frac{2}{3}\tilde{u}_i\left(\frac{3}{2\Theta}\right)^{\frac{1}{2}}\frac{\partial^2 V_j}{\partial r_i\partial r_j}\right\}.
\end{aligned}
\tag{E.11}
$$

Appendix F

Few Claims and their Proofs

Claim 1.

$$\overline{A_{ij}}\, B_{ij} = A_{ij}\, \overline{B_{ij}} \tag{F.1}$$

where A_{ij} and B_{ij} are the components of the second order tensors.

Proof.

$$\begin{aligned}
\overline{A_{ij}}\, B_{ij} &= \left(\frac{A_{ij}+A_{ji}}{2} - \frac{1}{3}A_{kk}\delta_{ij}\right) B_{ij} \\
&= \frac{1}{2}\Big(A_{ij}B_{ij} + A_{ji}B_{ij}\Big) - \frac{1}{3}A_{kk}B_{ii} \\
&= \frac{1}{2}\Big(A_{ij}B_{ij} + A_{ij}B_{ji}\Big) - \frac{1}{3}(A_{ij}\delta_{ij})B_{kk} \\
&= A_{ij}\left\{\frac{B_{ij}+B_{ji}}{2} - \frac{1}{3}\delta_{ij}B_{kk}\right\} \\
&= A_{ij}\,\overline{B_{ij}}\,.
\end{aligned}$$

■

As a consequence of eq. (F.1), we have the following:

$$\overline{\tilde{u}_i \frac{\partial \ln(n\Theta)}{\partial r_j}}\frac{\partial V_i}{\partial r_j} = \tilde{u}_i \frac{\partial \ln(n\Theta)}{\partial r_j}\overline{\frac{\partial V_i}{\partial r_j}}, \tag{F.2}$$

$$\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial V_i}{\partial r_j} = \tilde{u}_i\tilde{u}_j\overline{\frac{\partial V_i}{\partial r_j}}, \tag{F.3}$$

$$\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j} = \tilde{u}_i\tilde{u}_j\overline{\frac{\partial V_i}{\partial r_k}\frac{\partial V_k}{\partial r_j}}, \tag{F.4}$$

$$\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial}{\partial r_k}\frac{\partial V_i}{\partial r_j} = \tilde{u}_i\tilde{u}_j\frac{\partial}{\partial r_k}\overline{\frac{\partial V_i}{\partial r_j}}, \tag{F.5}$$

$$\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial^2 \ln(n\Theta)}{\partial r_i \partial r_j} = \tilde{u}_i\tilde{u}_j\overline{\frac{\partial^2 \ln(n\Theta)}{\partial r_i \partial r_j}}, \tag{F.6}$$

and

$$\overline{\tilde{u}_i\tilde{u}_j}\frac{\partial \ln\Theta}{\partial r_j}\frac{\partial \ln(n\Theta)}{\partial r_i} = \tilde{u}_i\tilde{u}_j\overline{\frac{\partial \ln\Theta}{\partial r_j}\frac{\partial \ln(n\Theta)}{\partial r_i}} = \tilde{u}_i\tilde{u}_j\overline{\frac{\partial \ln\Theta}{\partial r_i}\frac{\partial \ln(n\Theta)}{\partial r_j}}. \tag{F.7}$$

The second equality in the above equation can be obtained by interchanging the dummy indices i and j in the middle term.

Claim 2.

$$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}<0} \mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^m = \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^m \tag{F.8}$$

Proof. Let us transform the above integrations over $\hat{\boldsymbol{k}}$ to the spherical polar coordinate system $(\hat{k}, \theta, \phi)$, whose z-axis coincides with $\tilde{\boldsymbol{u}}_{12}$ (see Appendix C for the figure and the following equations). Hence

$$\hat{\boldsymbol{k}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}},$$
$$\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} = \tilde{u}_{12}\cos\theta.$$

First, consider LHS of eq. (F.8),

$$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}<0} \mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^m = \int_{\hat{k}=0}^{1}\int_{\theta=\pi/2}^{\pi}\int_{\phi=0}^{2\pi}(-\tilde{u}_{12}\cos\theta)^m\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}\hat{k}.$$

Let $-\cos\theta = t \Rightarrow \sin\theta\,\mathrm{d}\theta = \mathrm{d}t$, hence

$$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}<0} \mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^m = \int_{\hat{k}=0}^{1}\mathrm{d}\hat{k}\int_{t=0}^{1}(\tilde{u}_{12}t)^m\mathrm{d}t\int_{\phi=0}^{2\pi}\mathrm{d}\phi = 2\pi\int_0^1(\tilde{u}_{12}t)^m\mathrm{d}t. \qquad (*)$$

Similarly, consider RHS of eq. (F.8),

$$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^m = \int_{\hat{k}=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(\tilde{u}_{12}\cos\theta)^m\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}\hat{k}.$$

Let $\cos\theta = p \Rightarrow -\sin\theta\,\mathrm{d}\theta = \mathrm{d}p$, hence

$$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^m = \int_{\hat{k}=0}^{1}\mathrm{d}\hat{k}\int_{p=1}^{0}(\tilde{u}_{12}p)^m(-\mathrm{d}p)\int_{\phi=0}^{2\pi}\mathrm{d}\phi = 2\pi\int_0^1(\tilde{u}_{12}p)^m\mathrm{d}p. \qquad (**)$$

From eqs. (*) and (**), we see that

$$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}<0} \mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^m = \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^m. \qquad \blacksquare$$

Claim 3.

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,f(\tilde{u})\tilde{u}_i\tilde{u}_j = \frac{4\pi}{3}\delta_{ij}\int_0^\infty \mathrm{d}\tilde{u}\,f(\tilde{u})\tilde{u}^4 \qquad \text{(F.9a)}$$

and therefore

$$a_j\int \mathrm{d}\tilde{\boldsymbol{u}}\,f(\tilde{u})\tilde{u}_i\tilde{u}_j = \frac{4\pi}{3}a_i\int_0^\infty \mathrm{d}\tilde{u}\,f(\tilde{u})\tilde{u}^4, \qquad \text{(F.9b)}$$
$$A_{ij}\int \mathrm{d}\tilde{\boldsymbol{u}}\,f(\tilde{u})\tilde{u}_i\tilde{u}_j = \frac{4\pi}{3}A_{ii}\int_0^\infty \mathrm{d}\tilde{u}\,f(\tilde{u})\tilde{u}^4 \qquad \text{(F.9c)}$$

where a_j and A_{ij} are the components of a vector and a second order tensor, respectively.

Proof. Consider the left-hand side of eq. (F.9a). For $i \neq j$ the integral vanishes because in this case integrand is an odd function in components of $\tilde{\boldsymbol{u}}$, and because of symmetry,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\,f(\tilde{u})\tilde{u}_1^2 = \int \mathrm{d}\tilde{\boldsymbol{u}}\,f(\tilde{u})\tilde{u}_2^2 = \int \mathrm{d}\tilde{\boldsymbol{u}}\,f(\tilde{u})\tilde{u}_3^2,$$

therefore one can write

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j = \delta_{ij} \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^2.$$

Let us transform the integration over $\tilde{\boldsymbol{u}}$ to the spherical polar coordinate system $(\tilde{u}, \theta, \phi)$. This transformation implies that

$$\left.\begin{aligned} \tilde{u}_1 &= \tilde{u}\sin\theta\cos\phi \\ \tilde{u}_2 &= \tilde{u}\sin\theta\sin\phi \\ \tilde{u}_3 &= \tilde{u}\cos\theta. \end{aligned}\right\} \tag{***}$$

Hence

$$\begin{aligned} \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j &= \delta_{ij} \int_{\tilde{u}=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \mathrm{d}\tilde{u}\,\mathrm{d}\theta\,\mathrm{d}\phi\,\tilde{u}^2\sin\theta\, f(\tilde{u})(\tilde{u}^2\sin^2\theta\cos^2\phi) \\ &= \delta_{ij}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^4 \int_{\theta=0}^{\pi}\mathrm{d}\theta\,\sin^3\theta \int_{\phi=0}^{2\pi}\mathrm{d}\phi\,\cos^2\phi \\ &= \delta_{ij}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^4 \times \frac{4}{3}\times\pi = \frac{4\pi}{3}\delta_{ij}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^4. \end{aligned}$$

Therefore

$$a_j \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j = a_j\frac{4\pi}{3}\delta_{ij}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^4 = \frac{4\pi}{3}a_i\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^4$$

and

$$A_{ij} \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j = A_{ij}\frac{4\pi}{3}\delta_{ij}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^4 = \frac{4\pi}{3}A_{ii}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^4. \qquad \blacksquare$$

Claim 4.

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\,\overline{\tilde{u}_i\tilde{u}_j} = 0 \tag{F.10}$$

Proof.

$$\begin{aligned} \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\,\overline{\tilde{u}_i\tilde{u}_j} &= \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\left(\frac{\tilde{u}_i\tilde{u}_j+\tilde{u}_j\tilde{u}_i}{2} - \frac{1}{3}\delta_{ij}\tilde{u}_k^2\right) = \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j - \frac{1}{3}\delta_{ij}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}^2 \\ &= \frac{4\pi}{3}\delta_{ij}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^4 - \frac{1}{3}\delta_{ij}\times 4\pi\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^4 = 0. \qquad \blacksquare \end{aligned}$$

Claim 5.

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \tag{F.11}$$

Proof. The integral on the left-hand side of eq. (F.11) will be nonzero only if either $(i = j = k = l)$ or $(i = j \;\&\; k = l)$ or $(i = k \;\&\; j = l)$ or $(i = l \;\&\; j = k)$; in any other case the integrand is an odd function in components of $\tilde{\boldsymbol{u}}$, which results into vanishing integral. The combination of delta functions, $(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$ takes care of all these conditions. Also, because of symmetry,

$$\begin{aligned} \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^4 = \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_2^4 = \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_3^4, \\ \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^2\tilde{u}_2^2 = \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_2^2\tilde{u}_3^2 = \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_3^2\tilde{u}_1^2. \end{aligned}$$

Note that for $i = j = k = l$ case, $\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} = 3$, hence in this case

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l = \frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^4.$$

Let us transform the integration over $\tilde{\boldsymbol{u}}$ to the spherical polar coordinate system $(\tilde{u}, \theta, \phi)$. Using eq. (***), we get

$$\begin{aligned}
&\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l \\
&= \frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int_{\tilde{u}=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \mathrm{d}\tilde{u}\,\mathrm{d}\theta\,\mathrm{d}\phi\, \tilde{u}^2 \sin\theta\, f(\tilde{u})(\tilde{u}^4 \sin^4\theta \cos^4\phi) \\
&= \frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \int_{\theta=0}^{\pi} \mathrm{d}\theta\, \sin^5\theta \int_{\phi=0}^{2\pi} \mathrm{d}\phi\, \cos^4\phi \\
&= \frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \times \frac{16}{15} \times \frac{3\pi}{4} \\
&= \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int_0^{\infty} \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6.
\end{aligned}$$

Next consider the case: ($i = j$ & $k = l$) or ($i = k$ & $j = l$) or ($i = l$ & $j = k$) but all of i, j, k and l are not equal. In this case $\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} = 1$, hence in this case

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l = (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^2\tilde{u}_2^2.$$

Let us transform the integration over $\tilde{\boldsymbol{u}}$ to the spherical polar coordinate system $(\tilde{u}, \theta, \phi)$. Using eq. (***), we get

$$\begin{aligned}
&\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l \\
&= (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int_{\tilde{u}=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \mathrm{d}\tilde{u}\,\mathrm{d}\theta\,\mathrm{d}\phi\, \tilde{u}^2 \sin\theta\, f(\tilde{u})(\tilde{u}^4 \sin^4\theta \cos^2\phi \sin^2\phi) \\
&= (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \int_{\theta=0}^{\pi} \mathrm{d}\theta\, \sin^5\theta \int_{\phi=0}^{2\pi} \mathrm{d}\phi\, \sin^2\phi \cos^2\phi \\
&= (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int_{\tilde{u}=0}^{\infty} \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \times \frac{16}{15} \times \frac{\pi}{4} = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int_0^{\infty} \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6.
\end{aligned}$$

Therefore, in any case

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \int_0^{\infty} \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6. \qquad \blacksquare$$

Claim 6.

$$\begin{aligned}
&A_{kl} \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\overline{\overline{\tilde{u}_i\tilde{u}_j}}\tilde{u}_k\tilde{u}_l = A_{kl} \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\overline{\tilde{u}_k\tilde{u}_l}} = \overline{\overline{A_{kl}}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\overline{\overline{\tilde{u}_i\tilde{u}_j}}\tilde{u}_k\tilde{u}_l \\
&= \overline{\overline{A_{kl}}} \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\overline{\tilde{u}_k\tilde{u}_l}} = \frac{8\pi}{15}\overline{\overline{A_{ij}}} \int_0^{\infty} \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6, \qquad \text{(F.12)}
\end{aligned}$$

where A_{ij} are the components of a second order tensor.

Proof. Since

$$A_{kl}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l = A_{kl}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l - \frac{1}{3}\delta_{ij}A_{kl}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}^2\tilde{u}_k\tilde{u}_l,$$

using eqs. (F.9c) and (F.11),

$$\begin{aligned}
&A_{kl}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l \\
&= A_{kl}\frac{4\pi}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 - \frac{1}{3}\delta_{ij}A_{kk}\frac{4\pi}{3}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{4\pi}{15}\left(\delta_{ij}A_{kk}+A_{ij}+A_{ji}-\frac{5}{3}\delta_{ij}A_{kk}\right)\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{4\pi}{15}\left(A_{ij}+A_{ji}-\frac{2}{3}\delta_{ij}A_{kk}\right)\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{8\pi}{15}\left(\frac{A_{ij}+A_{ji}}{2}-\frac{1}{3}\delta_{ij}A_{kk}\right)\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{8\pi}{15}\overline{A_{ij}}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6.
\end{aligned}$$

Similarly,

$$\begin{aligned}
A_{kl}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\tilde{u}_k\tilde{u}_l} &= A_{kl}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l - \frac{1}{3}\delta_{kl}A_{kl}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}^2\tilde{u}_i\tilde{u}_j \\
&= A_{kl}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l - \frac{1}{3}A_{kk}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}^2\tilde{u}_i\tilde{u}_j.
\end{aligned}$$

Using eqs. (F.9a) and (F.11),

$$\begin{aligned}
&A_{kl}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\tilde{u}_k\tilde{u}_l} \\
&= A_{kl}\frac{4\pi}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 - \frac{1}{3}A_{kk}\frac{4\pi}{3}\delta_{ij}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{8\pi}{15}\overline{A_{ij}}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6. \quad \text{(same as above)}
\end{aligned}$$

Now using the above results and the fact that $\overline{(\overline{A_{ij}})} = \overline{A_{ij}}$,

$$\overline{A_{kl}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l = \overline{A_{kl}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\overline{\tilde{u}_k\tilde{u}_l} = \frac{8\pi}{15}\overline{A_{ij}}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6.$$ ■

Claim 7.

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n = \frac{4\pi}{105}T_{ijklmn}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8, \tag{F.13}$$

where

$$\begin{aligned}
T_{ijklmn} = &\delta_{ij}(\delta_{kl}\delta_{mn}+\delta_{km}\delta_{ln}+\delta_{kn}\delta_{lm}) + \delta_{ik}(\delta_{jl}\delta_{mn}+\delta_{jm}\delta_{ln}+\delta_{jn}\delta_{lm}) \\
&+ \delta_{il}(\delta_{jk}\delta_{mn}+\delta_{jm}\delta_{kn}+\delta_{jn}\delta_{km}) + \delta_{im}(\delta_{jk}\delta_{ln}+\delta_{jl}\delta_{kn}+\delta_{jn}\delta_{kl}) \\
&+ \delta_{in}(\delta_{jk}\delta_{lm}+\delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl}).
\end{aligned}$$

Proof. Consider the left-hand side of eq. (F.13). The tensorial structure, and even or odd nature of the integrand on the left-hand side are taken care by the combination of delta functions given by T_{ijklmn} above. Also, because of symmetry,

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^6 = \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_2^6 = \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_3^6$$

and

$$\begin{aligned}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^4\tilde{u}_2^2 &= \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^2\tilde{u}_2^4 = \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_2^4\tilde{u}_3^2 \\ &= \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_2^2\tilde{u}_3^4 = \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_3^4\tilde{u}_1^2 = \int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_3^2\tilde{u}_1^4.\end{aligned}$$

Note that for $i = j = k = l = m = n$, $T_{ijklmn} = 15$, hence in this case

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n = \frac{1}{15}T_{ijklmn}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^6.$$

Let us transform the integration over $\tilde{\boldsymbol{u}}$ to the spherical polar coordinate system $(\tilde{u}, \theta, \phi)$. Using eq. (***), we get

$$\begin{aligned}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n &= \frac{1}{15}T_{ijklmn}\int_{\tilde{u}=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\mathrm{d}\tilde{u}\,\mathrm{d}\theta\,\mathrm{d}\phi\,\tilde{u}^2\sin\theta\, f(\tilde{u})(\tilde{u}^6\sin^6\theta\cos^6\phi) \\ &= \frac{1}{15}T_{ijklmn}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8\int_{\theta=0}^{\pi}\mathrm{d}\theta\,\sin^7\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi\,\cos^6\phi \\ &= \frac{1}{15}T_{ijklmn}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8\times\frac{32}{35}\times\frac{5\pi}{8} \\ &= \frac{4\pi}{105}T_{ijklmn}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8.\end{aligned}$$

Next consider the case: ($i = j = k = l$ & $m = n$ & $i \neq m$) or any similar case. In this case $T_{ijklmn} = 3$, hence in this case

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n = \frac{1}{3}T_{ijklmn}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^4\tilde{u}_2^2.$$

Let us transform the integration over $\tilde{\boldsymbol{u}}$ to the spherical polar coordinate system $(\tilde{u}, \theta, \phi)$. Using eq. (***), we get

$$\begin{aligned}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n &= \frac{1}{3}T_{ijklmn}\int_{\tilde{u}=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\mathrm{d}\tilde{u}\,\mathrm{d}\theta\,\mathrm{d}\phi\,\tilde{u}^2\sin\theta\, f(\tilde{u})(\tilde{u}^6\sin^6\theta\cos^4\phi\sin^2\phi) \\ &= \frac{1}{3}T_{ijklmn}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8\int_{\theta=0}^{\pi}\mathrm{d}\theta\,\sin^7\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi\,\sin^2\phi\cos^4\phi \\ &= \frac{1}{3}T_{ijklmn}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8\times\frac{32}{35}\times\frac{\pi}{8} \\ &= \frac{4\pi}{105}T_{ijklmn}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8.\end{aligned}$$

Next consider the case: $(i = j \; \& \; k = l \; \& \; m = n \; \& \; i \neq k \; \& \; i \neq m \; \& \; k \neq m)$ or any similar case. In this case $T_{ijklmn} = 1$, hence in this case

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n = T_{ijklmn}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_1^2\tilde{u}_2^2\tilde{u}_3^2.$$

Let us transform the integration over $\tilde{\boldsymbol{u}}$ to the spherical polar coordinate system $(\tilde{u}, \theta, \phi)$. Using eq. (***), we get

$$\begin{aligned}
&\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n \\
&= T_{ijklmn}\int_{\tilde{u}=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \mathrm{d}\tilde{u}\,\mathrm{d}\theta\,\mathrm{d}\phi\,\tilde{u}^2\sin\theta\, f(\tilde{u})(\tilde{u}^6\sin^4\theta\cos^2\theta\cos^2\phi\sin^2\phi) \\
&= T_{ijklmn}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8\int_{\theta=0}^{\pi}\mathrm{d}\theta\,\sin^5\theta\cos^2\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi\,\sin^2\phi\cos^2\phi \\
&= T_{ijklmn}\int_{\tilde{u}=0}^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8 \times \frac{16}{105}\times\frac{\pi}{4} = \frac{4\pi}{105}T_{ijklmn}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8.
\end{aligned}$$

Therefore, in any case

$$\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n = \frac{4\pi}{105}T_{ijklmn}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8. \qquad \blacksquare$$

Claim 8.

$$(i) \quad \overline{\frac{\partial V_i}{\partial r_j}}\,\frac{\partial V_k}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l = \frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \tag{F.14a}$$

$$(ii) \quad \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l = \frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 = \frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \tag{F.14b}$$

Proof. Using eq. (F.11) and the facts: $\overline{\frac{\partial V_i}{\partial r_i}} = 0$ and $\overline{\frac{\partial V_j}{\partial r_i}} = \overline{\frac{\partial V_i}{\partial r_j}}$,

$$\begin{aligned}
(i) \quad \overline{\frac{\partial V_i}{\partial r_j}}\,\frac{\partial V_k}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}^2\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l &= \frac{4\pi}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\overline{\frac{\partial V_i}{\partial r_j}}\,\frac{\partial V_k}{\partial r_l}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{4\pi}{15}\left(\overline{\frac{\partial V_i}{\partial r_j}}\,\frac{\partial V_i}{\partial r_j} + \overline{\frac{\partial V_i}{\partial r_j}}\,\frac{\partial V_j}{\partial r_i}\right)\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{4\pi}{15}\left(\overline{\frac{\partial V_i}{\partial r_j}}\,\frac{\partial V_i}{\partial r_j} + \overline{\frac{\partial V_i}{\partial r_j}}\,\frac{\partial V_j}{\partial r_i} - \frac{2}{3}\overline{\frac{\partial V_i}{\partial r_i}}\,\frac{\partial V_k}{\partial r_k}\right)\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}}\left\{\frac{1}{2}\left(\frac{\partial V_i}{\partial r_j}+\frac{\partial V_j}{\partial r_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial V_k}{\partial r_k}\right\}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}\int_0^{\infty}\mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6,
\end{aligned}$$

$$(ii)\quad \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}^2\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6$$
$$= \frac{4\pi}{15}\left(\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}}\right)\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6$$
$$= \frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6$$
$$= \frac{8\pi}{15}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6. \qquad \blacksquare$$

Claim 9.

$$\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\overline{\overline{\tilde{u}_i\tilde{u}_j}}\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n = \frac{32\pi}{105}\overline{\overline{\frac{\partial V_i}{\partial r_k}\,\frac{\partial V_k}{\partial r_j}}}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8, \tag{F.15}$$

Proof. Consider the left-hand side of eq. (F.15).

$$\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\overline{\overline{\tilde{u}_i\tilde{u}_j}}\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n$$
$$= \overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n - \frac{1}{3}\delta_{ij}\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}^2\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n. \tag{\#}$$

Using eq. (F.13),

$$\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n = \frac{4\pi}{105}\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}T_{ijklmn}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8,$$

where

$$\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}T_{ijklmn} = \overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\Big\{\delta_{ij}(\delta_{kl}\delta_{mn}+\delta_{km}\delta_{ln}+\delta_{kn}\delta_{lm})$$
$$+\delta_{ik}(\delta_{jl}\delta_{mn}+\delta_{jm}\delta_{ln}+\delta_{jn}\delta_{lm}) + \delta_{il}(\delta_{jk}\delta_{mn}+\delta_{jm}\delta_{kn}+\delta_{jn}\delta_{km})$$
$$+\delta_{im}(\delta_{jk}\delta_{ln}+\delta_{jl}\delta_{kn}+\delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jk}\delta_{lm}+\delta_{jl}\delta_{km}+\delta_{jm}\delta_{kl})\Big\}.$$

Using the facts: $\overline{\frac{\partial V_i}{\partial r_i}} = 0$ and $\overline{\frac{\partial V_i}{\partial r_j}} = \overline{\frac{\partial V_j}{\partial r_i}}$,

$$\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}T_{ijklmn} = \delta_{ij}\left(\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_k}{\partial r_l}} + \overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_l}{\partial r_k}}\right) + \overline{\frac{\partial V_i}{\partial r_l}}\left(\overline{\frac{\partial V_j}{\partial r_l}} + \overline{\frac{\partial V_l}{\partial r_j}}\right)$$
$$+\overline{\frac{\partial V_k}{\partial r_i}}\left(\overline{\frac{\partial V_j}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}}\right) + \overline{\frac{\partial V_i}{\partial r_n}}\left(\overline{\frac{\partial V_j}{\partial r_n}} + \overline{\frac{\partial V_n}{\partial r_j}}\right) + \overline{\frac{\partial V_m}{\partial r_i}}\left(\overline{\frac{\partial V_j}{\partial r_m}} + \overline{\frac{\partial V_m}{\partial r_j}}\right)$$
$$= 2\delta_{ij}\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_l}{\partial r_k}} + 2\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}} + 2\overline{\frac{\partial V_k}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_k}} + 2\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}} + 2\overline{\frac{\partial V_k}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_k}}$$
$$= 2\delta_{ij}\overline{\frac{\partial V_l}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_l}} + 4\left(\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_i}}\right)$$
$$= 8\left\{\frac{1}{2}\left(\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_i}}\right) - \frac{1}{3}\delta_{ij}\overline{\frac{\partial V_l}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_l}}\right\} + \frac{14}{3}\delta_{ij}\overline{\frac{\partial V_l}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_l}}$$
$$= 8\overline{\overline{\frac{\partial V_i}{\partial r_k}\,\frac{\partial V_k}{\partial r_j}}} + \frac{14}{3}\delta_{ij}\overline{\frac{\partial V_l}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_l}}.$$

Hence

$$\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n = \frac{4\pi}{105}\left\{8\overline{\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}} + \frac{14}{3}\delta_{ij}\overline{\frac{\partial V_l}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_l}}\right\}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8.$$

With above equation and eq. (F.14b), eq. (#) changes to

$$\begin{aligned}
&\overline{\frac{\partial V_k}{\partial r_l}}\,\overline{\frac{\partial V_m}{\partial r_n}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\overline{\tilde{u}_i\tilde{u}_j}\tilde{u}_k\tilde{u}_l\tilde{u}_m\tilde{u}_n \\
&= \frac{4\pi}{105}\left\{8\overline{\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}} + \frac{14}{3}\delta_{ij}\overline{\frac{\partial V_l}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_l}}\right\}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8 - \frac{1}{3}\delta_{ij}\frac{8\pi}{15}\overline{\frac{\partial V_l}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8 \\
&= \frac{32\pi}{105}\overline{\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^8. \qquad \blacksquare
\end{aligned}$$

Claim 10.

$$\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l = \frac{8\pi}{15}\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial \Theta}{\partial r_j}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \tag{F.16}$$

Proof. Using eq. (F.11),

$$\begin{aligned}
\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l &= \overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\times\frac{4\pi}{15}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{4\pi}{15}\left(\overline{\frac{\partial V_i}{\partial r_k}}\frac{\partial \Theta}{\partial r_k} + \overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial \Theta}{\partial r_j} + \overline{\frac{\partial V_j}{\partial r_j}}\frac{\partial \Theta}{\partial r_i}\right)\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{4\pi}{15}\left(\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_i}}\right)\frac{\partial \Theta}{\partial r_j}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6 \\
&= \frac{8\pi}{15}\overline{\frac{\partial V_j}{\partial r_i}}\frac{\partial \Theta}{\partial r_j}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6. \qquad \blacksquare
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial r_l}\overline{\frac{\partial V_j}{\partial r_k}}\int \mathrm{d}\tilde{\boldsymbol{u}}\, f(\tilde{u})\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l = \frac{8\pi}{15}\frac{\partial}{\partial r_j}\overline{\frac{\partial V_j}{\partial r_i}}\int_0^\infty \mathrm{d}\tilde{u}\, f(\tilde{u})\tilde{u}^6. \tag{F.17}$$

Appendix G

The Evaluation of some Integrals

1. $\displaystyle\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^n = ?,\quad \text{where}\quad n\in\mathbb{N}.$

Let $\displaystyle\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^n = I_1$ and let us transform the integral over $\hat{\boldsymbol{k}}$ in a spherical coordinate system $(\hat{k},\theta,\phi)$ whose z-axis coincides with $\tilde{\boldsymbol{u}}_{12}$ (see Appendix C). Hence

$$\hat{\boldsymbol{k}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}},$$
$$\tilde{\boldsymbol{u}}_{12} = \tilde{u}_{12}\,\hat{\boldsymbol{z}},$$
$$\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12} = \tilde{u}_{12}\cos\theta.$$

Note that $\hat{k}$ varies from 0 to 1 because $\hat{\boldsymbol{k}}$ is a unit vector. Therefore

$$\begin{aligned} I_1 &= \int_{\hat{k}=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}\hat{k}\,(\tilde{u}_{12}\cos\theta)^n \\ &= \tilde{u}_{12}^n\int_{\hat{k}=0}^{1}\mathrm{d}\hat{k}\int_{\theta=0}^{\pi/2}\sin\theta\cos^n\theta\,\mathrm{d}\theta\int_{\phi=0}^{2\pi}\mathrm{d}\phi = \tilde{u}_{12}^n\times 1\times\frac{1}{2}B\left(\frac{n+1}{2},1\right)\times 2\pi, \end{aligned}$$

where $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is beta function and $\Gamma(x)$ is gamma function (Abramowitz & Stegun 1965). Hence

$$I_1 = \pi\tilde{u}_{12}^n\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma(1)}{\Gamma\left(\frac{n+3}{2}\right)} = \pi\tilde{u}_{12}^n\frac{\Gamma\left(\frac{n+1}{2}\right)\times 1}{\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)} = \frac{2\pi}{(n+1)}\tilde{u}_{12}^n$$

or

$$\boxed{\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^n = \frac{2\pi}{(n+1)}\tilde{u}_{12}^n \quad \text{for}\quad n\in\mathbb{N}.} \tag{G.1a}$$

In particular,

$$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) = \pi\,\tilde{u}_{12}, \tag{G.1b}$$
$$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^3 = \frac{\pi}{2}\,\tilde{u}_{12}^3, \tag{G.1c}$$
$$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^5 = \frac{\pi}{3}\,\tilde{u}_{12}^5. \tag{G.1d}$$

2. $\displaystyle\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} = ?$

Let us make the transformation $(\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2) \to (\boldsymbol{g}_1, \boldsymbol{g}_2)$ such that,

$$\boldsymbol{g}_1 = \tilde{\boldsymbol{u}}_1 + \tilde{\boldsymbol{u}}_2,$$
$$\boldsymbol{g}_2 = \tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2.$$

Squaring and adding the above equations, we get $g_1^2 + g_2^2 = 2(\tilde{u}_1^2 + \tilde{u}_2^2)$, and $\tilde{u}_{12} = |\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2| = |\boldsymbol{g}_2| = g_2$. Using eq. (J.5), we have

$$\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} = \int \left(\frac{1}{8} \mathrm{d}\boldsymbol{g}_1 \mathrm{d}\boldsymbol{g}_2\right) g_2^3\, \mathrm{e}^{-\frac{1}{2}(g_1^2+g_2^2)} = \frac{1}{8} \int \mathrm{d}\boldsymbol{g}_1\, \mathrm{e}^{-\frac{1}{2}g_1^2} \times \int \mathrm{d}\boldsymbol{g}_2\, g_2^3\, \mathrm{e}^{-\frac{1}{2}g_2^2}\,.$$

Changing both the integrals to spherical polar coordinate systems (g_1, θ_1, ϕ_1), (g_2, θ_2, ϕ_2) respectively, we get

$$\begin{aligned}
&\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} \\
&= \frac{1}{8} \int_{g_1=0}^{\infty} \int_{\theta_1=0}^{\pi} \int_{\phi_1=0}^{2\pi} g_1^2\, \mathrm{e}^{-\frac{1}{2}g_1^2} \sin\theta_1\, \mathrm{d}g_1\, \mathrm{d}\theta_1\, \mathrm{d}\phi_1 \int_{g_2=0}^{\infty} \int_{\theta_2=0}^{\pi} \int_{\phi_2=0}^{2\pi} g_2^5\, \mathrm{e}^{-\frac{1}{2}g_2^2} \sin\theta_2\, \mathrm{d}g_2\, \mathrm{d}\theta_2\, \mathrm{d}\phi_2 \\
&= \frac{1}{8} \int_{g_1=0}^{\infty} g_1^2\, \mathrm{e}^{-\frac{1}{2}g_1^2}\, \mathrm{d}g_1 \int_{\theta_1=0}^{\pi} \sin\theta_1\, \mathrm{d}\theta_1 \int_{\phi_1=0}^{2\pi} \mathrm{d}\phi_1 \int_{g_2=0}^{\infty} g_2^5\, \mathrm{e}^{-\frac{1}{2}g_2^2}\, \mathrm{d}g_2 \int_{\theta_2=0}^{\pi} \sin\theta_2\, \mathrm{d}\theta_2 \int_{\phi_2=0}^{2\pi} \mathrm{d}\phi_2 \\
&= \frac{1}{8} \times \sqrt{\frac{\pi}{2}} \times 2 \times 2\pi \times 8 \times 2 \times 2\pi
\end{aligned}$$

or

$$\boxed{\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)} = 8\sqrt{2}\,\pi^{5/2}} \tag{G.2}$$

3. $\displaystyle\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-\tilde{u}_2^2} = ?$

Let $\displaystyle\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \mathrm{e}^{-\tilde{u}_2^2} = I_2$ and $\tilde{\boldsymbol{u}}_{12} = \tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2 = \boldsymbol{g}$. Hence

$$\begin{aligned}
I_2 &= \int_{\tilde{u}_{2x}=-\infty}^{\infty} \int_{\tilde{u}_{2y}=-\infty}^{\infty} \int_{\tilde{u}_{2z}=-\infty}^{\infty} \mathrm{d}\tilde{u}_{2x}\, \mathrm{d}\tilde{u}_{2y}\, \mathrm{d}\tilde{u}_{2z}\, \tilde{u}_{12}\ \mathrm{e}^{-\tilde{u}_2^2} \\
&= \int_{g_x=\infty}^{-\infty} \int_{g_y=\infty}^{-\infty} \int_{g_z=\infty}^{-\infty} (-\mathrm{d}g_x)(-\mathrm{d}g_y)(-\mathrm{d}g_z)\, g\ \mathrm{e}^{-(\tilde{u}_1^2+g^2-2\tilde{\boldsymbol{u}}_1\cdot\boldsymbol{g})} \\
&= \mathrm{e}^{-\tilde{u}_1^2} \int_{g_x=-\infty}^{\infty} \int_{g_y=-\infty}^{\infty} \int_{g_z=-\infty}^{\infty} \mathrm{d}g_x\, \mathrm{d}g_y\, \mathrm{d}g_z\, g\ \mathrm{e}^{-g^2}\, \mathrm{e}^{2\tilde{\boldsymbol{u}}_1\cdot\boldsymbol{g}}\,.
\end{aligned}$$

Let us replace $\tilde{\boldsymbol{u}}_1$ by $\boldsymbol{u}$ for simplicity. Therefore

$$I_2 = \mathrm{e}^{-u^2} \int_{g_x=-\infty}^{\infty} \int_{g_y=-\infty}^{\infty} \int_{g_z=-\infty}^{\infty} \mathrm{d}g_x\, \mathrm{d}g_y\, \mathrm{d}g_z\, g\ \mathrm{e}^{-g^2}\, \mathrm{e}^{2\boldsymbol{u}\cdot\boldsymbol{g}}\,.$$

Changing the above integral to spherical polar coordinate system (g, θ, ϕ), whose z-axis coincides with $\boldsymbol{u}$, i.e.,

$$\boldsymbol{u} = u\,\hat{\boldsymbol{z}},$$
$$\boldsymbol{g} = g \sin\theta \cos\phi\, \hat{\boldsymbol{x}} + g \sin\theta \sin\phi\, \hat{\boldsymbol{y}} + g\cos\theta\, \hat{\boldsymbol{z}}.$$

Therefore

$$\begin{aligned} I_2 &= \mathrm{e}^{-u^2} \int_{g=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(g\, \mathrm{e}^{-g^2}\, \mathrm{e}^{2gu\cos\theta} \right) \left(g^2 \sin\theta\, \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}g \right) \\ &= \mathrm{e}^{-u^2} \int_{g=0}^{\infty} g^3\, \mathrm{e}^{-g^2} \left\{ \int_{\theta=0}^{\pi} \mathrm{e}^{2gu\cos\theta} \sin\theta\, \mathrm{d}\theta \right\} \mathrm{d}g \int_{\phi=0}^{2\pi} \mathrm{d}\phi. \end{aligned}$$

Let $\cos\theta = -t \Rightarrow \sin\theta\, \mathrm{d}\theta = \mathrm{d}t$, hence

$$\begin{aligned} I_2 &= \mathrm{e}^{-u^2} \int_{g=0}^{\infty} g^3\, \mathrm{e}^{-g^2} \left\{ \int_{t=-1}^{1} \mathrm{e}^{-2gut}\, \mathrm{d}t \right\} \mathrm{d}g \times 2\pi = 2\pi\, \mathrm{e}^{-u^2} \int_{g=0}^{\infty} g^3\, \mathrm{e}^{-g^2} \left[\frac{\mathrm{e}^{-2gut}}{-2gu} \right]_{t=-1}^{1} \mathrm{d}g \\ &= \frac{\pi\, \mathrm{e}^{-u^2}}{u} \int_{g=0}^{\infty} g^2\, \mathrm{e}^{-g^2} \left(\mathrm{e}^{2gu} - \mathrm{e}^{-2gu} \right) \mathrm{d}g \\ &= \frac{\pi}{u} \left[\int_{g=0}^{\infty} g^2\, \mathrm{e}^{-(u^2+g^2-2gu)}\, \mathrm{d}g - \int_{g=0}^{\infty} g^2\, \mathrm{e}^{-(u^2+g^2+2gu)}\, \mathrm{d}g \right] \\ &= \frac{\pi}{u} \int_{g=0}^{\infty} g^2\, \mathrm{e}^{-(g-u)^2}\, \mathrm{d}g - \frac{\pi}{u} \int_{g=0}^{\infty} g^2\, \mathrm{e}^{-(g+u)^2}\, \mathrm{d}g. \end{aligned}$$

Let $g - u = x$ and $g + u = y$, hence

$$\begin{aligned} I_2 &= \frac{\pi}{u} \int_{x=-u}^{\infty} (x+u)^2\, \mathrm{e}^{-x^2}\, \mathrm{d}x - \frac{\pi}{u} \int_{y=u}^{\infty} (y-u)^2\, \mathrm{e}^{-y^2}\, \mathrm{d}y \\ &= \frac{\pi}{u} \left[\int_{x=-u}^{0} (x+u)^2\, \mathrm{e}^{-x^2}\, \mathrm{d}x + \int_{x=0}^{\infty} (x+u)^2\, \mathrm{e}^{-x^2}\, \mathrm{d}x \right] \\ &\quad - \frac{\pi}{u} \left[\int_{y=0}^{\infty} (y-u)^2\, \mathrm{e}^{-y^2}\, \mathrm{d}y - \int_{y=0}^{u} (y-u)^2\, \mathrm{e}^{-y^2}\, \mathrm{d}y \right] \\ &= \frac{\pi}{u} \left[\int_{x=-u}^{0} (x+u)^2\, \mathrm{e}^{-x^2}\, \mathrm{d}x + \int_{y=0}^{u} (y-u)^2\, \mathrm{e}^{-y^2}\, \mathrm{d}y \right] \\ &\quad + \frac{\pi}{u} \left[\int_{x=0}^{\infty} (x+u)^2\, \mathrm{e}^{-x^2}\, \mathrm{d}x - \int_{y=0}^{\infty} (y-u)^2\, \mathrm{e}^{-y^2}\, \mathrm{d}y \right]. \end{aligned}$$

Now, replacing x by $-y$ in the first term above, we see that

$$\int_{x=-u}^{0} (x+u)^2\, \mathrm{e}^{-x^2}\, \mathrm{d}x = \int_{y=u}^{0} (-y+u)^2\, \mathrm{e}^{-y^2} (-\mathrm{d}y) = \int_{y=0}^{u} (y-u)^2\, \mathrm{e}^{-y^2}\, \mathrm{d}y.$$

Therefore

$$\begin{aligned} I_2 &= \frac{2\pi}{u} \int_{y=0}^{u} (y-u)^2\, \mathrm{e}^{-y^2}\, \mathrm{d}y + \frac{\pi}{u} \int_{x=0}^{\infty} \left[(x+u)^2 - (x-u)^2 \right] \mathrm{e}^{-x^2}\, \mathrm{d}x \\ &= \frac{2\pi}{u} I_{2a} + \frac{\pi}{u} \int_{x=0}^{\infty} 4ux\, \mathrm{e}^{-x^2}\, \mathrm{d}x, \end{aligned}$$

where $I_{2a} = \int_{y=0}^{u} (y-u)^2 \, \mathrm{e}^{-y^2} \, \mathrm{d}y$. Hence

$$I_2 = \frac{2\pi}{u} I_{2a} + 2\pi \int_{x=0}^{\infty} 2x \, \mathrm{e}^{-x^2} \, \mathrm{d}x \underset{(x^2=t)}{=} \frac{2\pi}{u} I_{2a} + 2\pi \int_{t=0}^{\infty} \mathrm{e}^{-t} \, \mathrm{d}t = \frac{2\pi}{u} I_{2a} + 2\pi.$$

We shall evaluate I_{2a} separately.

$$\begin{aligned} I_{2a} &= \int_{y=0}^{u} (y-u)^2 \, \mathrm{e}^{-y^2} \, \mathrm{d}y = \int_{y=0}^{u} (y^2 - 2yu + u^2) \, \mathrm{e}^{-y^2} \, \mathrm{d}y \\ &= \int_0^u y^2 \, \mathrm{e}^{-y^2} \, \mathrm{d}y - 2u \int_0^u y \, \mathrm{e}^{-y^2} \, \mathrm{d}y + u^2 \int_0^u \mathrm{e}^{-y^2} \, \mathrm{d}y. \end{aligned}$$

Let us evaluate the integrals on the right-hand side of the above equation separately.

- $\displaystyle\int_0^u \mathrm{e}^{-y^2} \, \mathrm{d}y = \frac{\pi^{1/2}}{2} \operatorname{erf}(u)$,
- $\displaystyle\int_0^u y \, \mathrm{e}^{-y^2} \, \mathrm{d}y \underset{y^2=t}{=} \frac{1}{2} \int_0^{u^2} \mathrm{e}^{-t} \, \mathrm{d}t = \frac{1}{2} \left[-\mathrm{e}^{-t} \right]_0^{u^2} = \frac{1}{2} (1 - \mathrm{e}^{-u^2})$,
- $\displaystyle\int_0^u y^2 \, \mathrm{e}^{-y^2} \, \mathrm{d}y = -\frac{1}{2} \int_0^u y(-2y \, \mathrm{e}^{-y^2}) \mathrm{d}y = -\frac{1}{2} \left[\left\{ y \, \mathrm{e}^{-y^2} \right\}_0^u - \int_0^u \mathrm{e}^{-y^2} \, \mathrm{d}y \right]$
  $\displaystyle= -\frac{1}{2} \left[u \, \mathrm{e}^{-u^2} - \frac{\pi^{1/2}}{2} \operatorname{erf}(u) \right] = -\frac{1}{2} u \, \mathrm{e}^{-u^2} + \frac{\pi^{1/2}}{4} \operatorname{erf}(u)$.

Substituting these values of the integrals in the expression of I_{2a}, we get

$$\begin{aligned} I_{2a} &= \left(-\frac{1}{2} u \, \mathrm{e}^{-u^2} + \frac{\pi^{1/2}}{4} \operatorname{erf}(u) \right) - 2u \left\{ \frac{1}{2} (1 - \mathrm{e}^{-u^2}) \right\} + u^2 \left(\frac{\pi^{1/2}}{2} \operatorname{erf}(u) \right) \\ &= \frac{1}{2} u \, \mathrm{e}^{-u^2} + \frac{\pi^{1/2} (1 + 2u^2)}{4} \operatorname{erf}(u) - u. \end{aligned}$$

Therefore

$$\begin{aligned} I_2 &= \frac{2\pi}{u} \left[\frac{1}{2} u \, \mathrm{e}^{-u^2} + \frac{\pi^{1/2} (1 + 2u^2)}{4} \operatorname{erf}(u) - u \right] + 2\pi \\ &= \pi \left[\mathrm{e}^{-u^2} + \frac{\pi^{1/2} (1 + 2u^2)}{2u} \operatorname{erf}(u) \right] - 2\pi + 2\pi = \pi \left[\mathrm{e}^{-u^2} + \frac{\pi^{1/2} (1 + 2u^2)}{2u} \operatorname{erf}(u) \right] \end{aligned}$$

or

$$\boxed{\int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \tilde{u}_{12} \, \mathrm{e}^{-\tilde{u}_2^2} = \pi \left[\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2} (1 + 2\tilde{u}_1^2)}{2\tilde{u}_1} \operatorname{erf}(\tilde{u}_1) \right]} \tag{G.3}$$

4. $\displaystyle\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-\tilde{u}_2^2} = ?$

Let $\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}^3\, \mathrm{e}^{-\tilde{u}_2^2} = I_3$ and $\tilde{\boldsymbol{u}}_{12} = \tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2 = \boldsymbol{g}$. Hence

$$\begin{aligned} I_3 &= \int_{\tilde{u}_{2x}=-\infty}^{\infty} \int_{\tilde{u}_{2y}=-\infty}^{\infty} \int_{\tilde{u}_{2z}=-\infty}^{\infty} \mathrm{d}\tilde{u}_{2x}\, \mathrm{d}\tilde{u}_{2y}\, \mathrm{d}\tilde{u}_{2z}\, \tilde{u}_{12}^3\, \mathrm{e}^{-\tilde{u}_2^2} \\ &= \int_{g_x=\infty}^{-\infty} \int_{g_y=\infty}^{-\infty} \int_{g_z=\infty}^{-\infty} (-\mathrm{d}g_x)(-\mathrm{d}g_y)(-\mathrm{d}g_z)\, g^3\, \mathrm{e}^{-(\tilde{u}_1^2+g^2-2\tilde{\boldsymbol{u}}_1\cdot\boldsymbol{g})} \\ &= \mathrm{e}^{-\tilde{u}_1^2} \int_{g_x=-\infty}^{\infty} \int_{g_y=-\infty}^{\infty} \int_{g_z=-\infty}^{\infty} \mathrm{d}g_x\, \mathrm{d}g_y\, \mathrm{d}g_z\, g^3\, \mathrm{e}^{-g^2}\, \mathrm{e}^{2\tilde{\boldsymbol{u}}_1\cdot\boldsymbol{g}}\,. \end{aligned}$$

Let us replace $\tilde{\boldsymbol{u}}_1$ by $\boldsymbol{u}$ for simplicity.

$$I_3 = \mathrm{e}^{-u^2} \int_{g_x=-\infty}^{\infty} \int_{g_y=-\infty}^{\infty} \int_{g_z=-\infty}^{\infty} \mathrm{d}g_x\, \mathrm{d}g_y\, \mathrm{d}g_z\, g^3\, \mathrm{e}^{-g^2}\, \mathrm{e}^{2\boldsymbol{u}\cdot\boldsymbol{g}}\,.$$

Changing the above integral to spherical polar coordinate system (g, θ, ϕ), whose z-axis coincides with $\boldsymbol{u}$, i.e.,

$$\begin{aligned} \boldsymbol{u} &= u\,\hat{\boldsymbol{z}}, \\ \boldsymbol{g} &= g\sin\theta\cos\phi\,\hat{\boldsymbol{x}} + g\sin\theta\sin\phi\,\hat{\boldsymbol{y}} + g\cos\theta\,\hat{\boldsymbol{z}}. \end{aligned}$$

Therefore

$$\begin{aligned} I_3 &= \mathrm{e}^{-u^2} \int_{g=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(g^3\, \mathrm{e}^{-g^2}\, \mathrm{e}^{2gu\cos\theta}\right) \left(g^2 \sin\theta\, \mathrm{d}\phi\, \mathrm{d}\theta\, \mathrm{d}g\right) \\ &= \mathrm{e}^{-u^2} \int_{g=0}^{\infty} g^5\, \mathrm{e}^{-g^2} \left\{ \int_{\theta=0}^{\pi} \mathrm{e}^{2gu\cos\theta} \sin\theta\, \mathrm{d}\theta \right\} \mathrm{d}g \int_{\phi=0}^{2\pi} \mathrm{d}\phi. \end{aligned}$$

Let $\cos\theta = -t \Rightarrow \sin\theta\, \mathrm{d}\theta = \mathrm{d}t$, hence

$$\begin{aligned} I_3 &= \mathrm{e}^{-u^2} \int_{g=0}^{\infty} g^5\, \mathrm{e}^{-g^2} \left\{ \int_{t=-1}^{1} \mathrm{e}^{-2gut}\, \mathrm{d}t \right\} \mathrm{d}g \times 2\pi = 2\pi\, \mathrm{e}^{-u^2} \int_{g=0}^{\infty} g^5\, \mathrm{e}^{-g^2} \left[\frac{\mathrm{e}^{-2gut}}{-2gu} \right]_{t=-1}^{1} \mathrm{d}g \\ &= \frac{\pi\, \mathrm{e}^{-u^2}}{u} \int_{g=0}^{\infty} g^4\, \mathrm{e}^{-g^2} \left(\mathrm{e}^{2gu} - \mathrm{e}^{-2gu}\right) \mathrm{d}g \\ &= \frac{\pi}{u} \left[\int_{g=0}^{\infty} g^4\, \mathrm{e}^{-(u^2+g^2-2gu)}\, \mathrm{d}g - \int_{g=0}^{\infty} g^4\, \mathrm{e}^{-(u^2+g^2+2gu)}\, \mathrm{d}g \right] \\ &= \frac{\pi}{u} \int_{g=0}^{\infty} g^4\, \mathrm{e}^{-(g-u)^2}\, \mathrm{d}g - \frac{\pi}{u} \int_{g=0}^{\infty} g^4\, \mathrm{e}^{-(g+u)^2}\, \mathrm{d}g. \end{aligned}$$

Let $g - u = x$ and $g + u = y$, hence

$$I_3 = \frac{\pi}{u} \int_{x=-u}^{\infty} (x+u)^4\, \mathrm{e}^{-x^2}\, \mathrm{d}x - \frac{\pi}{u} \int_{y=u}^{\infty} (y-u)^4\, \mathrm{e}^{-y^2}\, \mathrm{d}y$$

or

$$
\begin{aligned}
I_3 &= \frac{\pi}{u}\left[\int_{x=-u}^{0}(x+u)^4\,\mathrm{e}^{-x^2}\,\mathrm{d}x + \int_{x=0}^{\infty}(x+u)^4\,\mathrm{e}^{-x^2}\,\mathrm{d}x\right] \\
&\quad - \frac{\pi}{u}\left[\int_{y=0}^{\infty}(y-u)^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y - \int_{y=0}^{u}(y-u)^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y\right] \\
&= \frac{\pi}{u}\left[\int_{x=-u}^{0}(x+u)^4\,\mathrm{e}^{-x^2}\,\mathrm{d}x + \int_{y=0}^{u}(y-u)^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y\right] \\
&\quad + \frac{\pi}{u}\left[\int_{x=0}^{\infty}(x+u)^4\,\mathrm{e}^{-x^2}\,\mathrm{d}x - \int_{y=0}^{\infty}(y-u)^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y\right].
\end{aligned}
$$

Now, replacing x by $-y$ in the first term above we see that

$$
\int_{x=-u}^{0}(x+u)^4\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \int_{y=u}^{0}(-y+u)^4\,\mathrm{e}^{-y^2}(-\mathrm{d}y) = \int_{y=0}^{u}(y-u)^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y.
$$

Therefore

$$
\begin{aligned}
I_3 &= \frac{2\pi}{u}\int_{y=0}^{u}(y-u)^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y + \frac{\pi}{u}\int_{x=0}^{\infty}\left[(x+u)^4-(x-u)^4\right]\mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= \frac{2\pi}{u}\int_{y=0}^{u}(y-u)^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y + \frac{\pi}{u}\int_{x=0}^{\infty}8ux(x^2+u^2)\,\mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= \frac{2\pi}{u}\int_{y=0}^{u}(y-u)^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y + 8\pi\int_{x=0}^{\infty}(x^3+xu^2)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{2\pi}{u}\,I_{3a} + 8\pi\,I_{3b},
\end{aligned}
$$

where $I_{3a} = \int_{y=0}^{u}(y-u)^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y$ and $I_{3b} = \int_{x=0}^{\infty}(x^3+xu^2)\,\mathrm{e}^{-x^2}\,\mathrm{d}x$. We shall evaluate I_{3a} and I_{3b} separately.

$$
\begin{aligned}
I_{3a} &= \int_{y=0}^{u}(y-u)^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y = \int_{y=0}^{u}(y^4-4y^3u+6u^2y^2-4u^3y+u^4)\,\mathrm{e}^{-y^2}\,\mathrm{d}y \\
&= \int_0^u y^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y - 4u\int_0^u y^3\,\mathrm{e}^{-y^2}\,\mathrm{d}y + 6u^2\int_0^u y^2\,\mathrm{e}^{-y^2}\,\mathrm{d}y - 4u^3\int_0^u y\,\mathrm{e}^{-y^2}\,\mathrm{d}y + u^4\int_0^u \mathrm{e}^{-y^2}\,\mathrm{d}y.
\end{aligned}
$$

Let us evaluate the integrals on the right-hand side of the above equation separately.

- $\displaystyle\int_0^u \mathrm{e}^{-y^2}\,\mathrm{d}y = \frac{\pi^{1/2}}{2}\,\mathrm{erf}(u),$

- $\displaystyle\int_0^u y\,\mathrm{e}^{-y^2}\,\mathrm{d}y \underset{y^2=t}{=} \frac{1}{2}\int_0^{u^2}\mathrm{e}^{-t}\,\mathrm{d}t = \frac{1}{2}\left[-\mathrm{e}^{-t}\right]_0^{u^2} = \frac{1}{2}(1-\mathrm{e}^{-u^2}),$

- $\displaystyle\int_0^u y^2\,\mathrm{e}^{-y^2}\,\mathrm{d}y = -\frac{1}{2}\int_0^u y(-2y\,\mathrm{e}^{-y^2})\mathrm{d}y = -\frac{1}{2}\left[\left\{y\,\mathrm{e}^{-y^2}\right\}_0^u - \int_0^u \mathrm{e}^{-y^2}\,\mathrm{d}y\right]$
  $\displaystyle= -\frac{1}{2}\left[u\,\mathrm{e}^{-u^2} - \frac{\pi^{1/2}}{2}\,\mathrm{erf}(u)\right] = -\frac{1}{2}u\,\mathrm{e}^{-u^2} + \frac{\pi^{1/2}}{4}\,\mathrm{erf}(u),$

- $\displaystyle\int_0^u y^3\,\mathrm{e}^{-y^2}\,\mathrm{d}y \underset{y^2=t}{=} \frac{1}{2}\int_0^{u^2} t\,\mathrm{e}^{-t}\,\mathrm{d}t = \frac{1}{2}\left[\left\{t(-\mathrm{e}^{-t})\right\}_0^{u^2} - \left\{\mathrm{e}^{-t}\right\}_0^{u^2}\right]$
  $\displaystyle= \frac{1}{2}\left[-u^2\,\mathrm{e}^{-u^2} - \mathrm{e}^{-u^2} + 1\right] = -\frac{1}{2}u^2\,\mathrm{e}^{-u^2} - \frac{1}{2}\,\mathrm{e}^{-u^2} + \frac{1}{2},$

- $$\int_0^u y^4\,\mathrm{e}^{-y^2}\,\mathrm{d}y = -\frac{1}{2}\int_0^u y^3(-2y\,\mathrm{e}^{-y^2})\mathrm{d}y = -\frac{1}{2}\left[\left\{y^3\,\mathrm{e}^{-y^2}\right\}_0^u - \int_0^u 3y^2\,\mathrm{e}^{-y^2}\,\mathrm{d}y\right]$$
$$= -\frac{1}{2}\left[u^3\,\mathrm{e}^{-u^2} - 3\left(-\frac{1}{2}u\,\mathrm{e}^{-u^2} + \frac{\pi^{1/2}}{4}\,\mathrm{erf}(u)\right)\right] = -\frac{1}{2}u^3\,\mathrm{e}^{-u^2} - \frac{3}{4}u\,\mathrm{e}^{-u^2} + \frac{3}{8}\pi^{1/2}\,\mathrm{erf}(u).$$

Substituting these values of the integrals in the expression of I_{3a}, we get

$$\begin{aligned}
I_{3a} &= \left(-\frac{1}{2}u^3\,\mathrm{e}^{-u^2} - \frac{3}{4}u\,\mathrm{e}^{-u^2} + \frac{3}{8}\pi^{1/2}\,\mathrm{erf}(u)\right) - 4u\left(-\frac{1}{2}u^2\,\mathrm{e}^{-u^2} - \frac{1}{2}\,\mathrm{e}^{-u^2} + \frac{1}{2}\right) \\
&\quad + 6u^2\left(-\frac{1}{2}u\,\mathrm{e}^{-u^2} + \frac{\pi^{1/2}}{4}\,\mathrm{erf}(u)\right) - 4u^3\left\{\frac{1}{2}(1-\mathrm{e}^{-u^2})\right\} + u^4\left(\frac{\pi^{1/2}}{2}\,\mathrm{erf}(u)\right) \\
&= \left(-\frac{1}{2}u^3 - \frac{3}{4}u + 2u^3 + 2u - 3u^3 + 2u^3\right)\mathrm{e}^{-u^2} - 2u - 2u^3 + \pi^{1/2}\,\mathrm{erf}(u)\left(\frac{3}{8} + \frac{3}{2}u^2 + \frac{1}{2}u^4\right) \\
&= \left(\frac{1}{2}u^3 + \frac{5}{4}u\right)\mathrm{e}^{-u^2} + \pi^{1/2}\,\mathrm{erf}(u)\left(\frac{3+12u^2+4u^4}{8}\right) - 2u(1+u^2) \\
&= \frac{1}{2}u\left(u^2 + \frac{5}{2}\right)\mathrm{e}^{-u^2} + \pi^{1/2}\,\mathrm{erf}(u)\left(\frac{3+12u^2+4u^4}{8}\right) - 2u(1+u^2)
\end{aligned}$$

and

$$\begin{aligned}
I_{3b} &= \int_{x=0}^{\infty}(x^3 + xu^2)\,\mathrm{e}^{-x^2}\,\mathrm{d}x \underset{x^2=t}{=} \frac{1}{2}\int_{t=0}^{\infty}(t+u^2)\,\mathrm{e}^{-t}\,\mathrm{d}t \\
&= \frac{1}{2}\left[\left\{(t+u^2)(-e^{-t})\right\}_{t=0}^{\infty} - \left\{e^{-t}\right\}_{t=0}^{\infty}\right] = \frac{1}{2}(1+u^2).
\end{aligned}$$

Substituting the values of I_{3a} and I_{3b} in the expression of I_3, we get

$$\begin{aligned}
I_3 &= \frac{2\pi}{u}\left[\frac{1}{2}u\left(u^2 + \frac{5}{2}\right)\mathrm{e}^{-u^2} + \pi^{1/2}\,\mathrm{erf}(u)\left(\frac{3+12u^2+4u^4}{8}\right) - 2u(1+u^2)\right] + 8\pi \times \frac{1}{2}(1+u^2) \\
&= \pi\left[\left(u^2 + \frac{5}{2}\right)\mathrm{e}^{-u^2} + \frac{\pi^{1/2}(3+12u^2+4u^4)}{4u}\,\mathrm{erf}(u)\right]
\end{aligned}$$

or

$$\boxed{\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\,\mathrm{e}^{-\tilde{u}_2^2} = \pi\left[\left(\tilde{u}_1^2 + \frac{5}{2}\right)\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(3+12\tilde{u}_1^2+4\tilde{u}_1^4)}{4\tilde{u}_1}\,\mathrm{erf}(\tilde{u}_1)\right]} \tag{G.4}$$

5. $\displaystyle\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^5\,\mathrm{e}^{-\tilde{u}_2^2} = ?$

This integral can be evaluated by following a similar procedure as in evaluating the integrals in eqs. (G.3) and (G.4). The result is†:

$$\boxed{\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^5\,\mathrm{e}^{-\tilde{u}_2^2} = \pi\left[\frac{1}{8}(33+28\tilde{u}_1^2+4\tilde{u}_1^4)\,\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(15+90\tilde{u}_1^2+60\tilde{u}_1^4+8\tilde{u}_1^6)}{16\tilde{u}_1}\,\mathrm{erf}(\tilde{u}_1)\right]} \tag{G.5}$$

†This result is obtained by using the generating function given in Appendix E of Bar-Lev (2005) and Mathematica.

6. $$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) = ?$$

Let $\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right) = I_4$. The integrations over $\hat{\boldsymbol{k}}$ are given in eqs. (G.1b) and (G.1c). Hence using these equations

$$I_4 = \int \mathrm{d}\tilde{\boldsymbol{u}}_2\,(\pi\,\tilde{u}_{12})\,\mathrm{e}^{-\tilde{u}_2^2} - \frac{1}{2}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\left(\frac{\pi}{2}\,\tilde{u}_{12}^3\right)\mathrm{e}^{-\tilde{u}_2^2} = \pi\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-\tilde{u}_2^2} - \frac{\pi}{4}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\,\mathrm{e}^{-\tilde{u}_2^2}\,.$$

Now, with the help of eqs. (G.3) and (G.4),

$$\begin{aligned}
I_4 &= \pi\times\pi\left\{\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(1+2\tilde{u}_1^2)}{2\tilde{u}_1}\,\mathrm{erf}(\tilde{u}_1)\right\}\\
&\quad - \frac{\pi}{4}\times\pi\left\{\left(\tilde{u}_1^2+\frac{5}{2}\right)\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(3+12\tilde{u}_1^2+4\tilde{u}_1^4)}{4\tilde{u}_1}\,\mathrm{erf}(\tilde{u}_1)\right\}\\
&= \pi^2\left[\left(1-\frac{\tilde{u}_1^2}{4}-\frac{5}{8}\right)\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}}{2\tilde{u}_1}\left\{1+2\tilde{u}_1^2-\frac{(3+12\tilde{u}_1^2+4\tilde{u}_1^4)}{8}\right\}\mathrm{erf}(\tilde{u}_1)\right]\\
&= \pi^2\left[\left(\frac{3-2\tilde{u}_1^2}{8}\right)\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}}{2\tilde{u}_1}\left\{\frac{5+4\tilde{u}_1^2-4\tilde{u}_1^4}{8}\right\}\mathrm{erf}(\tilde{u}_1)\right]
\end{aligned}$$

or

$$\boxed{\begin{aligned}
&\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1-\frac{1}{2}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2\right)\\
&= \pi^{5/2}\left[\left(\frac{3-2\tilde{u}_1^2}{8\,\pi^{1/2}}\right)\mathrm{e}^{-\tilde{u}_1^2} + \frac{(5+4\tilde{u}_1^2-4\tilde{u}_1^4)\,\mathrm{erf}(\tilde{u}_1)}{16\tilde{u}_1}\right]
\end{aligned}} \tag{G.6}$$

7. $$\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1-(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2+\frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right)=?$$

Let $\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1-(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2+\frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right)=I_5$. The integrations over $\hat{\boldsymbol{k}}$ are given in eqs. (G.1b)-(G.1d). Hence using these equations

$$\begin{aligned}
I_5 &= \int \mathrm{d}\tilde{\boldsymbol{u}}_2\,(\pi\,\tilde{u}_{12})\,\mathrm{e}^{-\tilde{u}_2^2} - \int \mathrm{d}\tilde{\boldsymbol{u}}_2\left(\frac{\pi}{2}\,\tilde{u}_{12}^3\right)\mathrm{e}^{-\tilde{u}_2^2} + \frac{1}{8}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\left(\frac{\pi}{3}\,\tilde{u}_{12}^5\right)\mathrm{e}^{-\tilde{u}_2^2} \\
&= \pi\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\mathrm{e}^{-\tilde{u}_2^2} - \frac{\pi}{2}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^3\,\mathrm{e}^{-\tilde{u}_2^2} + \frac{\pi}{24}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}^5\,\mathrm{e}^{-\tilde{u}_2^2}\,.
\end{aligned}$$

Now, with the help of eqs. (G.3)-(G.5),

$$\begin{aligned}
I_5 &= \pi\times\pi\left\{\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(1+2\tilde{u}_1^2)}{2\tilde{u}_1}\,\mathrm{erf}(\tilde{u}_1)\right\} \\
&\quad - \frac{\pi}{2}\times\pi\left\{\left(\tilde{u}_1^2+\frac{5}{2}\right)\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(3+12\tilde{u}_1^2+4\tilde{u}_1^4)}{4\tilde{u}_1}\,\mathrm{erf}(\tilde{u}_1)\right\} \\
&\quad + \frac{\pi}{24}\times\pi\left\{\frac{1}{8}(33+28\tilde{u}_1^2+4\tilde{u}_1^4)\,\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(15+90\tilde{u}_1^2+60\tilde{u}_1^4+8\tilde{u}_1^6)}{16\tilde{u}_1}\,\mathrm{erf}(\tilde{u}_1)\right\} \\
&= \pi^2\left[\left(1-\frac{\tilde{u}_1^2}{2}-\frac{5}{4}+\frac{33}{192}+\frac{7\tilde{u}_1^2}{48}+\frac{\tilde{u}_1^4}{48}\right)\mathrm{e}^{-\tilde{u}_1^2}\right. \\
&\quad \left. + \frac{\pi^{1/2}}{2\tilde{u}_1}\left\{1+2\tilde{u}_1^2-\frac{(3+12\tilde{u}_1^2+4\tilde{u}_1^4)}{4}+\frac{(15+90\tilde{u}_1^2+60\tilde{u}_1^4+8\tilde{u}_1^6)}{192}\right\}\mathrm{erf}(\tilde{u}_1)\right] \\
&= \pi^2\left[\frac{(-15-68\tilde{u}_1^2+4\tilde{u}_1^4)}{192}\,\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}}{2\tilde{u}_1}\left\{\frac{(63-102\tilde{u}_1^2-132\tilde{u}_1^4+8\tilde{u}_1^6)}{192}\right\}\mathrm{erf}(\tilde{u}_1)\right]
\end{aligned}$$

or

$$\boxed{\begin{aligned}
&\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\left(1-(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^2+\frac{1}{8}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})^4\right) \\
&= \pi^{5/2}\left[\left(\frac{-15-68\tilde{u}_1^2+4\tilde{u}_1^4}{192\,\pi^{1/2}}\right)\mathrm{e}^{-\tilde{u}_1^2} + \frac{(63-102\tilde{u}_1^2-132\tilde{u}_1^4+8\tilde{u}_1^6)\,\mathrm{erf}(\tilde{u}_1)}{384\tilde{u}_1}\right]
\end{aligned}} \tag{G.7}$$

Appendix H

Integrals over an Azimuthal angle in doubly rotated Spherical Coordinate System

In this appendix, few expressions used in the evaluation of integrals in Chapter 4 are derived. The following identities will be used latter in this Appendix:

$$\int_0^{2\pi} \sin x \, \mathrm{d}x = \int_0^{2\pi} \cos x \, \mathrm{d}x = \int_0^{2\pi} \sin x \cos x \, \mathrm{d}x = 0 \tag{H.1a}$$

$$\int_0^{2\pi} \sin^2 x \, \mathrm{d}x = \int_0^{2\pi} \cos^2 x \, \mathrm{d}x = \pi \tag{H.1b}$$

Let the spherical coordinates of $\tilde{\boldsymbol{u}}$ in the original coordinate system be $(\tilde{u}, \theta_{\tilde{u}}, \phi_{\tilde{u}})$ and the integration over $\tilde{\boldsymbol{s}}$ is performed in a (rotated) spherical coordinate system $(\tilde{s}, \theta', \phi')$, which results from two rotations: (i) rotation of xy-plane in positive direction (x towards y) around z-axis by an angle $\phi_{\tilde{u}}$ and (ii) rotation of new zx-plane in positive direction (z towards x) around new y-axis by an angle $\theta_{\tilde{u}}$, so that $\tilde{\boldsymbol{u}}$ coincides with the new z-axis (figure H.1) and $\tilde{\boldsymbol{s}} \cdot \tilde{\boldsymbol{u}} = \tilde{s}\tilde{u} \cos\theta'$.

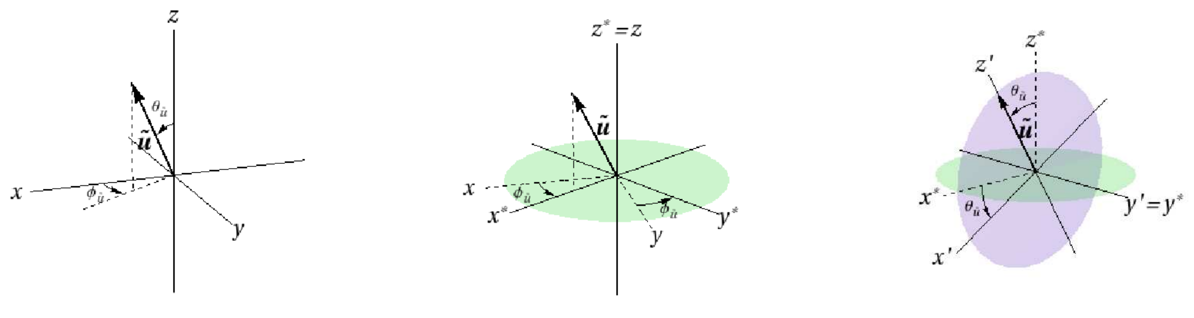


Figure H.1: (a) The original coordinate system xyz, (b) The new coordinate system $x^*y^*z^*$, obtained by rotation of xy-plane in positive direction (x towards y) around z-axis by an angle $\phi_{\tilde{u}}$, and (c) The new coordinate system $x'y'z'$, obtained by rotation of z^*x^*-plane in positive direction (z^* towards x^*) around y^*-axis by an angle $\theta_{\tilde{u}}$.

Note that, θ' and ϕ' are the spherical angles of $\tilde{\boldsymbol{s}}$ in the rotated frame of reference. Hence

$$\left.\begin{aligned} \tilde{u}_x &= \tilde{u} \sin\theta_{\tilde{u}} \cos\phi_{\tilde{u}} \\ \tilde{u}_y &= \tilde{u} \sin\theta_{\tilde{u}} \sin\phi_{\tilde{u}} \\ \tilde{u}_z &= \tilde{u} \cos\theta_{\tilde{u}} \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} \tilde{s}'_x &= \tilde{s} \sin\theta' \cos\phi' \\ \tilde{s}'_y &= \tilde{s} \sin\theta' \sin\phi' \\ \tilde{s}'_z &= \tilde{s} \cos\theta'. \end{aligned}\right. \tag{H.2}$$

The rotation matrix for "the rotation of xy-plane in positive direction (x towards y) around z-axis by an angle $\phi_{\tilde{u}}$" is:

$$R_z = \begin{bmatrix} \cos\phi_{\tilde{u}} & \sin\phi_{\tilde{u}} & 0 \\ -\sin\phi_{\tilde{u}} & \cos\phi_{\tilde{u}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the rotation matrix for "the rotation of new zx-plane in positive direction (z towards x) around new y-axis by an angle $\theta_{\tilde{u}}$" is:

$$R_y = \begin{bmatrix} \cos\theta_{\tilde{u}} & 0 & -\sin\theta_{\tilde{u}} \\ 0 & 1 & 0 \\ \sin\theta_{\tilde{u}} & 0 & \cos\theta_{\tilde{u}} \end{bmatrix}$$

Hence the cartesian-components of any vector $\tilde{\boldsymbol{s}}$ in this new coordinate system are given by

$$\begin{bmatrix} \tilde{s}'_x \\ \tilde{s}'_y \\ \tilde{s}'_z \end{bmatrix} = A \begin{bmatrix} \tilde{s}_x \\ \tilde{s}_y \\ \tilde{s}_z \end{bmatrix}, \qquad \text{where} \qquad A = R_y R_z. \tag{H.3}$$

In the above equation $(\tilde{s}_x, \tilde{s}_y, \tilde{s}_z)$ and $(\tilde{s}'_x, \tilde{s}'_y, \tilde{s}'_z)$ denote the cartesian-components of $\tilde{\boldsymbol{s}}$ in the original and rotated coordinate system respectively. Now

$$A = R_y R_z = \begin{bmatrix} \cos\theta_{\tilde{u}}\cos\phi_{\tilde{u}} & \cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}} & -\sin\theta_{\tilde{u}} \\ -\sin\phi_{\tilde{u}} & \cos\phi_{\tilde{u}} & 0 \\ \sin\theta_{\tilde{u}}\cos\phi_{\tilde{u}} & \sin\theta_{\tilde{u}}\sin\phi_{\tilde{u}} & \cos\theta_{\tilde{u}} \end{bmatrix}$$

and since the rotation matrices are orthogonal,

$$A^{-1} = A^{\intercal} = \begin{bmatrix} \cos\theta_{\tilde{u}}\cos\phi_{\tilde{u}} & -\sin\phi_{\tilde{u}} & \sin\theta_{\tilde{u}}\cos\phi_{\tilde{u}} \\ \cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}} & \cos\phi_{\tilde{u}} & \sin\theta_{\tilde{u}}\sin\phi_{\tilde{u}} \\ -\sin\theta_{\tilde{u}} & 0 & \cos\theta_{\tilde{u}} \end{bmatrix}$$

Hence $(\tilde{s}_x, \tilde{s}_y, \tilde{s}_z)$ in terms of $(\tilde{s}'_x, \tilde{s}'_y, \tilde{s}'_z)$ are given by

$$\begin{bmatrix} \tilde{s}_x \\ \tilde{s}_y \\ \tilde{s}_z \end{bmatrix} = A^{-1} \begin{bmatrix} \tilde{s}'_x \\ \tilde{s}'_y \\ \tilde{s}'_z \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \tilde{s}_x \\ \tilde{s}_y \\ \tilde{s}_z \end{bmatrix} = \begin{bmatrix} \cos\theta_{\tilde{u}}\cos\phi_{\tilde{u}} & -\sin\phi_{\tilde{u}} & \sin\theta_{\tilde{u}}\cos\phi_{\tilde{u}} \\ \cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}} & \cos\phi_{\tilde{u}} & \sin\theta_{\tilde{u}}\sin\phi_{\tilde{u}} \\ -\sin\theta_{\tilde{u}} & 0 & \cos\theta_{\tilde{u}} \end{bmatrix} \begin{bmatrix} \tilde{s}'_x \\ \tilde{s}'_y \\ \tilde{s}'_z \end{bmatrix}$$

or

$$\left.\begin{aligned} \tilde{s}_x &= \tilde{s}'_x \cos\theta_{\tilde{u}}\cos\phi_{\tilde{u}} - \tilde{s}'_y \sin\phi_{\tilde{u}} + \tilde{s}'_z \sin\theta_{\tilde{u}}\cos\phi_{\tilde{u}} \\ \tilde{s}_y &= \tilde{s}'_x \cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}} + \tilde{s}'_y \cos\phi_{\tilde{u}} + \tilde{s}'_z \sin\theta_{\tilde{u}}\sin\phi_{\tilde{u}} \\ \tilde{s}_z &= -\tilde{s}'_x \sin\theta_{\tilde{u}} + \tilde{s}'_z \cos\theta_{\tilde{u}}. \end{aligned}\right\} \tag{H.4}$$

Using eq. (H.2), eq. (H.4) can be written as

$$\left.\begin{aligned}\tilde{s}_x &= \tilde{s}\sin\theta'\cos\phi'\cos\theta_{\tilde{u}}\cos\phi_{\tilde{u}} - \tilde{s}\sin\theta'\sin\phi'\sin\phi_{\tilde{u}} + \tilde{s}\cos\theta'\sin\theta_{\tilde{u}}\cos\phi_{\tilde{u}}\\ \tilde{s}_y &= \tilde{s}\sin\theta'\cos\phi'\cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}} + \tilde{s}\sin\theta'\sin\phi'\cos\phi_{\tilde{u}} + \tilde{s}\cos\theta'\sin\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\\ \tilde{s}_z &= -\tilde{s}\sin\theta'\cos\phi'\sin\theta_{\tilde{u}} + \tilde{s}\cos\theta'\cos\theta_{\tilde{u}}.\end{aligned}\right\} \tag{H.5}$$

Using the identities given in eq. (H.1a), and eq. (H.2),

$$\begin{aligned}\int_{\phi'=0}^{2\pi}\tilde{s}_x\,\mathrm{d}\phi' &= 2\pi\tilde{s}\cos\theta'\sin\theta_{\tilde{u}}\cos\phi_{\tilde{u}} = 2\pi\tilde{s}\frac{\tilde{u}_x}{\tilde{u}}\cos\theta',\\ \int_{\phi'=0}^{2\pi}\tilde{s}_y\,\mathrm{d}\phi' &= 2\pi\tilde{s}\cos\theta'\sin\theta_{\tilde{u}}\sin\phi_{\tilde{u}} = 2\pi\tilde{s}\frac{\tilde{u}_y}{\tilde{u}}\cos\theta',\\ \int_{\phi'=0}^{2\pi}\tilde{s}_z\,\mathrm{d}\phi' &= 2\pi\tilde{s}\cos\theta'\cos\theta_{\tilde{u}} = 2\pi\tilde{s}\frac{\tilde{u}_z}{\tilde{u}}\cos\theta',\end{aligned}$$

that means one can conclude that

$$\int_{\phi'=0}^{2\pi}\tilde{s}_j\,\mathrm{d}\phi' = 2\pi\frac{\tilde{s}}{\tilde{u}}\tilde{u}_j\cos\theta'. \tag{H.6}$$

From eq. (H.5),

$$\begin{aligned}\tilde{s}_x^2 = \tilde{s}^2\big(&\sin^2\theta'\cos^2\phi'\cos^2\theta_{\tilde{u}}\cos^2\phi_{\tilde{u}} + \sin^2\theta'\sin^2\phi'\sin^2\phi_{\tilde{u}} + \cos^2\theta'\sin^2\theta_{\tilde{u}}\cos^2\phi_{\tilde{u}}\\ &-2\sin^2\theta'\sin\phi'\cos\phi'\cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\cos\phi_{\tilde{u}} - 2\sin\theta'\cos\theta'\sin\phi'\sin\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\cos\phi_{\tilde{u}}\\ &+\sin\theta'\cos\theta'\cos\phi'\sin\theta_{\tilde{u}}\cos\theta_{\tilde{u}}\cos^2\phi_{\tilde{u}}\big),\\ \tilde{s}_y^2 = \tilde{s}^2\big(&\sin^2\theta'\cos^2\phi'\cos^2\theta_{\tilde{u}}\sin^2\phi_{\tilde{u}} + \sin^2\theta'\sin^2\phi'\cos^2\phi_{\tilde{u}} + \cos^2\theta'\sin^2\theta_{\tilde{u}}\sin^2\phi_{\tilde{u}}\\ &+2\sin^2\theta'\sin\phi'\cos\phi'\cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\cos\phi_{\tilde{u}} + 2\sin\theta'\cos\theta'\sin\phi'\sin\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\cos\phi_{\tilde{u}}\\ &+\sin\theta'\cos\theta'\cos\phi'\sin\theta_{\tilde{u}}\cos\theta_{\tilde{u}}\sin^2\phi_{\tilde{u}}\big),\\ \tilde{s}_z^2 = \tilde{s}^2\big(&\sin^2\theta'\cos^2\phi'\sin^2\theta_{\tilde{u}} + \cos^2\theta'\cos^2\theta_{\tilde{u}} - 2\sin\theta'\cos\theta'\cos\phi'\sin\theta_{\tilde{u}}\cos\theta_{\tilde{u}}\big),\end{aligned}$$

and

$$\begin{aligned}\tilde{s}_x\tilde{s}_y = \tilde{s}^2\big(&\sin^2\theta'\cos^2\phi'\cos^2\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\cos\phi_{\tilde{u}} + \sin^2\theta'\sin\phi'\cos\phi'\cos\theta_{\tilde{u}}\cos^2\phi_{\tilde{u}}\\ &+\sin\theta'\cos\theta'\cos\phi'\sin\theta_{\tilde{u}}\cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\cos\phi_{\tilde{u}} - \sin^2\theta'\sin\phi'\cos\phi'\cos\theta_{\tilde{u}}\sin^2\phi_{\tilde{u}}\\ &-\sin^2\theta'\sin^2\phi'\sin\phi_{\tilde{u}}\cos\phi_{\tilde{u}} - \sin\theta'\cos\theta'\sin\phi'\sin\theta_{\tilde{u}}\sin^2\phi_{\tilde{u}}\\ &+\sin\theta'\cos\theta'\cos\phi'\sin\theta_{\tilde{u}}\cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\cos\phi_{\tilde{u}} + \sin\theta'\cos\theta'\sin\phi'\sin\theta_{\tilde{u}}\cos^2\phi_{\tilde{u}}\\ &+\cos^2\theta'\sin^2\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\cos\phi_{\tilde{u}}\big),\\ \tilde{s}_y\tilde{s}_z = \tilde{s}^2\big(&-\sin^2\theta'\cos^2\phi'\sin\theta_{\tilde{u}}\cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}} + \sin\theta'\cos\theta'\cos\phi'\cos^2\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\\ &+\sin^2\theta'\sin\phi'\cos\phi'\sin\theta_{\tilde{u}}\sin\phi_{\tilde{u}} - \sin\theta'\cos\theta'\sin\phi'\cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\\ &-\sin\theta'\cos\theta'\cos\phi'\sin^2\theta_{\tilde{u}}\sin\phi_{\tilde{u}} + \cos^2\theta'\sin\theta_{\tilde{u}}\cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\big),\\ \tilde{s}_z\tilde{s}_x = \tilde{s}^2\big(&-\sin^2\theta'\cos^2\phi'\sin\theta_{\tilde{u}}\cos\theta_{\tilde{u}}\cos\phi_{\tilde{u}} + \sin^2\theta'\sin\phi'\cos\phi'\sin\theta_{\tilde{u}}\sin\phi_{\tilde{u}}\\ &-\sin\theta'\cos\theta'\cos\phi'\sin^2\theta_{\tilde{u}}\cos\phi_{\tilde{u}} + \sin\theta'\cos\theta'\cos\phi'\cos^2\theta_{\tilde{u}}\cos\phi_{\tilde{u}}\\ &-\sin\theta'\cos\theta'\sin\phi'\cos\theta_{\tilde{u}}\sin\phi_{\tilde{u}} + \cos^2\theta'\sin\theta_{\tilde{u}}\cos\theta_{\tilde{u}}\cos\phi_{\tilde{u}}\big).\end{aligned}$$

Using the identities given in eqs. (H.1a), (H.1b), and eq. (H.2),

$$\begin{aligned}
\int_{\phi'=0}^{2\pi} \tilde{s}_x^2 \, \mathrm{d}\phi' &= \tilde{s}^2 \left(\pi \sin^2\theta' \cos^2\theta_{\tilde{u}} \cos^2\phi_{\tilde{u}} + \pi \sin^2\theta' \sin^2\phi_{\tilde{u}} + 2\pi \cos^2\theta' \sin^2\theta_{\tilde{u}} \cos^2\phi_{\tilde{u}}\right) \\
&= \pi\tilde{s}^2 \left[\sin^2\theta' \left\{(1-\sin^2\theta_{\tilde{u}})\cos^2\phi_{\tilde{u}} + (1-\cos^2\phi_{\tilde{u}})\right\} + 2\cos^2\theta' \sin^2\theta_{\tilde{u}} \cos^2\phi_{\tilde{u}}\right] \\
&= \pi\tilde{s}^2 \left[(1-\cos^2\theta')\left\{-\sin^2\theta_{\tilde{u}}\cos^2\phi_{\tilde{u}} + 1\right\} + 2\cos^2\theta' \sin^2\theta_{\tilde{u}} \cos^2\phi_{\tilde{u}}\right] \\
&= \pi\tilde{s}^2 \left[1-\cos^2\theta' - \sin^2\theta_{\tilde{u}}\cos^2\phi_{\tilde{u}} + 3\cos^2\theta' \sin^2\theta_{\tilde{u}} \cos^2\phi_{\tilde{u}}\right] \\
&= \pi\tilde{s}^2 \left[1-\cos^2\theta' + (3\cos^2\theta' - 1)\frac{\tilde{u}_x^2}{\tilde{u}^2}\right], \\
\int_{\phi'=0}^{2\pi} \tilde{s}_y^2 \, \mathrm{d}\phi' &= \tilde{s}^2 \left(\pi \sin^2\theta' \cos^2\theta_{\tilde{u}} \sin^2\phi_{\tilde{u}} + \pi \sin^2\theta' \cos^2\phi_{\tilde{u}} + 2\pi \cos^2\theta' \sin^2\theta_{\tilde{u}} \sin^2\phi_{\tilde{u}}\right) \\
&= \pi\tilde{s}^2 \left[\sin^2\theta' \left\{(1-\sin^2\theta_{\tilde{u}})\sin^2\phi_{\tilde{u}} + (1-\sin^2\phi_{\tilde{u}})\right\} + 2\cos^2\theta' \sin^2\theta_{\tilde{u}} \sin^2\phi_{\tilde{u}}\right] \\
&= \pi\tilde{s}^2 \left[(1-\cos^2\theta')\left\{-\sin^2\theta_{\tilde{u}}\sin^2\phi_{\tilde{u}} + 1\right\} + 2\cos^2\theta' \sin^2\theta_{\tilde{u}} \sin^2\phi_{\tilde{u}}\right] \\
&= \pi\tilde{s}^2 \left[1-\cos^2\theta' - \sin^2\theta_{\tilde{u}}\sin^2\phi_{\tilde{u}} + 3\cos^2\theta' \sin^2\theta_{\tilde{u}} \sin^2\phi_{\tilde{u}}\right] \\
&= \pi\tilde{s}^2 \left[1-\cos^2\theta' + (3\cos^2\theta' - 1)\frac{\tilde{u}_y^2}{\tilde{u}^2}\right], \\
\int_{\phi'=0}^{2\pi} \tilde{s}_z^2 \, \mathrm{d}\phi' &= \tilde{s}^2 \left(\pi \sin^2\theta' \sin^2\theta_{\tilde{u}} + 2\pi \cos^2\theta' \cos^2\theta_{\tilde{u}}\right) \\
&= \pi\tilde{s}^2 \left[(1-\cos^2\theta')(1-\cos^2\theta_{\tilde{u}}) + 2\cos^2\theta' \cos^2\theta_{\tilde{u}}\right] \\
&= \pi\tilde{s}^2 \left[1-\cos^2\theta' + (3\cos^2\theta' - 1)\cos^2\theta_{\tilde{u}}\right] \\
&= \pi\tilde{s}^2 \left[1-\cos^2\theta' + (3\cos^2\theta' - 1)\frac{\tilde{u}_z^2}{\tilde{u}^2}\right],
\end{aligned}$$

that means one can conclude that

$$\int_{\phi'=0}^{2\pi} \tilde{s}_j^2 \, \mathrm{d}\phi' = \pi\tilde{s}^2 \left[1-\cos^2\theta' + (3\cos^2\theta' - 1)\frac{\tilde{u}_j^2}{\tilde{u}^2}\right], \tag{H.7}$$

and

$$\begin{aligned}
\int_{\phi'=0}^{2\pi} \tilde{s}_x \tilde{s}_y \, \mathrm{d}\phi' &= \tilde{s}^2 \left(\pi \sin^2\theta' \cos^2\theta_{\tilde{u}} \sin\phi_{\tilde{u}} \cos\phi_{\tilde{u}} - \pi \sin^2\theta' \sin\phi_{\tilde{u}} \cos\phi_{\tilde{u}}\right. \\
&\qquad \left. + 2\pi \cos^2\theta' \sin^2\theta_{\tilde{u}} \sin\phi_{\tilde{u}} \cos\phi_{\tilde{u}}\right) \\
&= \pi\tilde{s}^2 \sin\phi_{\tilde{u}} \cos\phi_{\tilde{u}} \left\{\sin^2\theta'(\cos^2\theta_{\tilde{u}} - 1) + 2\cos^2\theta' \sin^2\theta_{\tilde{u}}\right\} \\
&= \pi\tilde{s}^2 (\sin\theta_{\tilde{u}} \cos\phi_{\tilde{u}})(\sin\theta_{\tilde{u}} \sin\phi_{\tilde{u}}) \left\{-(1-\cos^2\theta') + 2\cos^2\theta'\right\} \\
&= \pi\frac{\tilde{s}^2}{\tilde{u}^2}\tilde{u}_x\tilde{u}_y(3\cos^2\theta' - 1), \\
\int_{\phi'=0}^{2\pi} \tilde{s}_y \tilde{s}_z \, \mathrm{d}\phi' &= \tilde{s}^2 \left(-\pi \sin^2\theta' \sin\theta_{\tilde{u}} \cos\theta_{\tilde{u}} \sin\phi_{\tilde{u}} + 2\pi \cos^2\theta' \sin\theta_{\tilde{u}} \cos\theta_{\tilde{u}} \sin\phi_{\tilde{u}}\right) \\
&= \pi\tilde{s}^2 (\sin\theta_{\tilde{u}} \sin\phi_{\tilde{u}}) \cos\theta_{\tilde{u}} \left\{-(1-\cos^2\theta') + 2\cos^2\theta'\right\} \\
&= \pi\frac{\tilde{s}^2}{\tilde{u}^2}\tilde{u}_y\tilde{u}_z(3\cos^2\theta' - 1),
\end{aligned}$$

$$\begin{aligned}\int_{\phi'=0}^{2\pi} \tilde{s}_z\tilde{s}_x \, \mathrm{d}\phi' &= \tilde{s}^2 \left(-\pi \sin^2\theta' \sin\theta_{\tilde{u}} \cos\theta_{\tilde{u}} \cos\phi_{\tilde{u}} + 2\pi\cos^2\theta' \sin\theta_{\tilde{u}}\cos\theta_{\tilde{u}}\cos\phi_{\tilde{u}}\right) \\ &= \pi\tilde{s}^2 \cos\theta_{\tilde{u}} (\sin\theta_{\tilde{u}}\cos\phi_{\tilde{u}}) \left\{-(1-\cos^2\theta') + 2\cos^2\theta'\right\} \\ &= \pi\frac{\tilde{s}^2}{\tilde{u}^2}\tilde{u}_z\tilde{u}_x(3\cos^2\theta' - 1),\end{aligned}$$

that means (for $i \neq j$) one can conclude that

$$\int_{\phi'=0}^{2\pi} \tilde{s}_i\tilde{s}_j \, \mathrm{d}\phi' = \pi\frac{\tilde{s}^2}{\tilde{u}^2}\tilde{u}_i\tilde{u}_j(3\cos^2\theta' - 1). \tag{H.8}$$

The integrals involving 3 and 4 components of $\tilde{\boldsymbol{s}}$ can be evaluated by following similar procedure as above and using expressions (H.5) and (H.2). The results are:

$$\int_{\phi'=0}^{2\pi} \tilde{s}_i\tilde{s}_j\tilde{s}_k \, \mathrm{d}\phi' = \pi\tilde{s}^3\frac{\tilde{u}_i\tilde{u}_j\tilde{u}_k}{\tilde{u}^3}(5\cos^3\theta' - 3\cos\theta'), \quad \text{for} \quad i \neq j \neq k. \tag{H.9}$$

$$\int_{\phi'=0}^{2\pi} \tilde{s}_i\tilde{s}_j^2 \, \mathrm{d}\phi' = \pi\tilde{s}^3\frac{\tilde{u}_i\tilde{u}_j^2}{\tilde{u}^3}(5\cos^3\theta' - 3\cos\theta') + \pi\tilde{s}^3\frac{\tilde{u}_i}{\tilde{u}}\cos\theta'(1-\cos^2\theta'), \quad \text{for} \quad i \neq j. \tag{H.10}$$

$$\int_{\phi'=0}^{2\pi} \tilde{s}_i^3 \, \mathrm{d}\phi' = \pi\tilde{s}^3\frac{\tilde{u}_i^3}{\tilde{u}^3}(5\cos^3\theta' - 3\cos\theta') + 3\pi\tilde{s}^3\frac{\tilde{u}_i}{\tilde{u}}\cos\theta'(1-\cos^2\theta'). \tag{H.11}$$

$$\begin{aligned}\int_{\phi'=0}^{2\pi} \tilde{s}_i\tilde{s}_j\tilde{s}_k^2 \, \mathrm{d}\phi' &= \frac{\pi}{4}\tilde{s}^4\frac{\tilde{u}_i\tilde{u}_j\tilde{u}_k^2}{\tilde{u}^4}(35\cos^4\theta' - 30\cos^2\theta' + 3) \\ &+ \frac{\pi}{4}\tilde{s}^4\frac{\tilde{u}_i\tilde{u}_j}{\tilde{u}^2}(5\cos^2\theta' - 1)(1-\cos^2\theta'), \quad \text{for} \quad i \neq j \neq k.\end{aligned} \tag{H.12}$$

$$\begin{aligned}\int_{\phi'=0}^{2\pi} \tilde{s}_i^2\tilde{s}_j^2 \, \mathrm{d}\phi' &= \frac{\pi}{4}\tilde{s}^4\frac{\tilde{u}_i^2\tilde{u}_j^2}{\tilde{u}^4}(35\cos^4\theta' - 30\cos^2\theta' + 3) + \frac{\pi}{4}\tilde{s}^4(1-\cos^2\theta')^2 \\ &+ \frac{\pi}{4}\tilde{s}^4\frac{(\tilde{u}_i^2 + \tilde{u}_j^2)}{\tilde{u}^2}(5\cos^2\theta' - 1)(1-\cos^2\theta'), \quad \text{for} \quad i \neq j.\end{aligned} \tag{H.13}$$

$$\begin{aligned}\int_{\phi'=0}^{2\pi} \tilde{s}_i\tilde{s}_j^3 \, \mathrm{d}\phi' &= \frac{\pi}{4}\tilde{s}^4\frac{\tilde{u}_i\tilde{u}_j^3}{\tilde{u}^4}(35\cos^4\theta' - 30\cos^2\theta' + 3) \\ &+ \frac{3\pi}{4}\tilde{s}^4\frac{\tilde{u}_i\tilde{u}_j}{\tilde{u}^2}(5\cos^2\theta' - 1)(1-\cos^2\theta'), \quad \text{for} \quad i \neq j.\end{aligned} \tag{H.14}$$

$$\begin{aligned}\int_{\phi'=0}^{2\pi} \tilde{s}_i^4 \, \mathrm{d}\phi' &= \frac{\pi}{4}\tilde{s}^4\frac{\tilde{u}_i^4}{\tilde{u}^4}(35\cos^4\theta' - 30\cos^2\theta' + 3) + \frac{3\pi}{4}\tilde{s}^4(1-\cos^2\theta')^2 \\ &+ \frac{3\pi}{2}\tilde{s}^4\frac{\tilde{u}_i^2}{\tilde{u}^2}(5\cos^2\theta' - 1)(1-\cos^2\theta').\end{aligned} \tag{H.15}$$

Next, applying eqs. (H.6)-(H.15) and using the facts (for a second order tensor A): $\overline{A_{ii}} = 0$ and $\overline{A_{ji}} = \overline{A_{ij}}$, we shall evaluate few integrals which have appeared while deriving the constitutive relations of $O(K\epsilon)$ and $O(KK)$.

- Let A_{ij} be the components of a second order tensor. Then $A_{ij}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_j = ?$

  Since
$$A_{ij}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_j = \sum_{i=1}^{3}A_{ii}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_i + \sum_{\substack{i,j=1\\i\neq j}}^{3}A_{ij}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_j,$$
  using eqs. (H.7) and (H.8),
$$\begin{aligned}A_{ij}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_j &= \sum_{i=1}^{3}A_{ii}\pi\tilde{s}^2\left[1-\cos^2\theta' + (3\cos^2\theta'-1)\frac{\tilde{u}_i^2}{\tilde{u}^2}\right]\\ &\quad+\sum_{\substack{i,j=1\\i\neq j}}^{3}A_{ij}\pi\frac{\tilde{s}^2}{\tilde{u}^2}\tilde{u}_i\tilde{u}_j(3\cos^2\theta'-1)\\ &= \pi\tilde{s}^2(1-\cos^2\theta')\sum_{i=1}^{3}A_{ii} + \pi\frac{\tilde{s}^2}{\tilde{u}^2}\sum_{i,j=1}^{3}A_{ij}\tilde{u}_i\tilde{u}_j(3\cos^2\theta'-1)\\ &= \pi\tilde{s}^2A_{ii}(1-\cos^2\theta') + \pi\frac{\tilde{s}^2}{\tilde{u}^2}A_{ij}\tilde{u}_i\tilde{u}_j(3\cos^2\theta'-1).\end{aligned} \tag{H.16}$$
  Therefore, using eqs. (H.6) and (H.16),
$$\begin{aligned}&A_{ij}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,(\tilde{u}_i-\tilde{s}_i)\tilde{s}_j\\ &= 2\pi\frac{\tilde{s}}{\tilde{u}}A_{ij}\tilde{u}_i\tilde{u}_j\cos\theta' - \pi\tilde{s}^2A_{ii}(1-\cos^2\theta') - \pi\frac{\tilde{s}^2}{\tilde{u}^2}A_{ij}\tilde{u}_i\tilde{u}_j(3\cos^2\theta'-1)\\ &= -\pi\tilde{s}^2A_{ii}(1-\cos^2\theta') + 2\pi\frac{\tilde{s}}{\tilde{u}^2}A_{ij}\tilde{u}_i\tilde{u}_j\left\{\tilde{u}\cos\theta' - \frac{1}{2}\tilde{s}(3\cos^2\theta'-1)\right\}.\end{aligned} \tag{H.17}$$

- $\overline{\frac{\partial V_i}{\partial r_j}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_j = ?$

  Replacing A_{ij} by $\overline{\frac{\partial V_i}{\partial r_j}}$ in eq. (H.16),
$$\begin{aligned}\overline{\frac{\partial V_i}{\partial r_j}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_j &= \pi\tilde{s}^2(1-\cos^2\theta')\overline{\frac{\partial V_i}{\partial r_i}} + \pi\frac{\tilde{s}^2}{\tilde{u}^2}\overline{\frac{\partial V_i}{\partial r_j}}\tilde{u}_i\tilde{u}_j(3\cos^2\theta'-1)\\ &= \pi\frac{\tilde{s}^2}{\tilde{u}^2}\overline{\frac{\partial V_i}{\partial r_j}}\tilde{u}_i\tilde{u}_j(3\cos^2\theta'-1).\end{aligned} \tag{H.18}$$
  Therefore using eqs. (H.6) and (H.18),
$$\begin{aligned}&\overline{\frac{\partial V_i}{\partial r_j}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j) = \overline{\frac{\partial V_i}{\partial r_j}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,(\tilde{u}_i\tilde{u}_j - \tilde{s}_i\tilde{u}_j - \tilde{s}_i\tilde{u}_j + \tilde{s}_i\tilde{s}_j)\\ &= \overline{\frac{\partial V_i}{\partial r_j}}\left\{2\pi\tilde{u}_i\tilde{u}_j - 2\pi\frac{\tilde{s}}{\tilde{u}}\tilde{u}_i\tilde{u}_j\cos\theta' - 2\pi\frac{\tilde{s}}{\tilde{u}}\tilde{u}_i\tilde{u}_j\cos\theta' + \pi\frac{\tilde{s}^2}{\tilde{u}^2}\overline{\frac{\partial V_i}{\partial r_j}}\tilde{u}_i\tilde{u}_j(3\cos^2\theta'-1)\right\}\\ &= 2\pi\frac{1}{\tilde{u}^2}\overline{\frac{\partial V_i}{\partial r_j}}\tilde{u}_i\tilde{u}_j\left\{\tilde{u}^2 - 2\tilde{u}\tilde{s}\cos\theta' + \frac{1}{2}\tilde{s}^2(3\cos^2\theta'-1)\right\}.\end{aligned} \tag{H.19}$$

- $\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{u}_j\tilde{s}_k\tilde{s}_l = ?$

Since

$$\begin{aligned}
&\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{u}_j\tilde{s}_k\tilde{s}_l \\
&= \sum_{\substack{i,k,l=1\\ i\neq k\neq l\neq i}}^{3}\left(\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}+\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_l}{\partial r_k}}+\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_l}}+\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_l}{\partial r_i}}+\overline{\frac{\partial V_l}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}}+\overline{\frac{\partial V_l}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}}\right)\tilde{u}_j \\
&\quad\times\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_k\tilde{s}_l+\sum_{\substack{i,k=1\\ i\neq k}}^{3}\left(\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}}+\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}}+\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_k}}\right)\tilde{u}_j\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_k^2 \\
&\quad+\sum_{i=1}^{3}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_i}}\tilde{u}_j\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i^3,
\end{aligned}$$

using eqs. (H.9)-(H.11),

$$\begin{aligned}
&\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{u}_j\tilde{s}_k\tilde{s}_l \\
&= \sum_{\substack{i,k,l=1\\ i\neq k\neq l\neq i}}^{3}\left(\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}+\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_l}{\partial r_k}}+\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_l}}+\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_l}{\partial r_i}}+\overline{\frac{\partial V_l}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}}+\overline{\frac{\partial V_l}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}}\right)\tilde{u}_j \\
&\quad\times\pi\tilde{s}^3\frac{\tilde{u}_i\tilde{u}_k\tilde{u}_l}{\tilde{u}^3}(5\cos^3\theta'-3\cos\theta') \\
&\quad+\sum_{\substack{i,k=1\\ i\neq k}}^{3}\left(\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}}+\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}}+\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_k}}\right)\tilde{u}_j \\
&\quad\times\pi\tilde{s}^3\left[\frac{\tilde{u}_i\tilde{u}_k^2}{\tilde{u}^3}(5\cos^3\theta'-3\cos\theta')+\frac{\tilde{u}_i}{\tilde{u}}\cos\theta'(1-\cos^2\theta')\right] \\
&\quad+\sum_{i=1}^{3}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_i}}\tilde{u}_j\pi\tilde{s}^3\left[\frac{\tilde{u}_i^3}{\tilde{u}^3}(5\cos^3\theta'-3\cos\theta')+3\frac{\tilde{u}_i}{\tilde{u}}\cos\theta'(1-\cos^2\theta')\right] \\
&=\pi\frac{\tilde{s}^3}{\tilde{u}^3}\sum_{i,k,l=1}^{3}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(5\cos^3\theta'-3\cos\theta') \\
&\quad+\pi\frac{\tilde{s}^3}{\tilde{u}}\sum_{i,k=1}^{3}\left(\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}}+\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}}+\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_k}}\right)\tilde{u}_i\tilde{u}_j\cos\theta'(1-\cos^2\theta') \\
&=\pi\frac{\tilde{s}^3}{\tilde{u}^3}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(5\cos^3\theta'-3\cos\theta') \\
&\quad+\pi\frac{\tilde{s}^3}{\tilde{u}}\left(\overline{\frac{\partial V_k}{\partial r_i}}+\overline{\frac{\partial V_i}{\partial r_k}}\right)\overline{\frac{\partial V_k}{\partial r_j}}\tilde{u}_i\tilde{u}_j\cos\theta'(1-\cos^2\theta') \\
&=\pi\frac{\tilde{s}^3}{\tilde{u}^3}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(5\cos^3\theta'-3\cos\theta')+2\pi\frac{\tilde{s}^3}{\tilde{u}}\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}\tilde{u}_i\tilde{u}_j\cos\theta'(1-\cos^2\theta')
\end{aligned}\tag{H.20}$$

and hence

$$\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{u}_i\tilde{s}_j\tilde{s}_k\tilde{s}_l=\overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{u}_j\tilde{s}_i\tilde{s}_k\tilde{s}_l$$
$$=\pi\frac{\tilde{s}^3}{\tilde{u}^3}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(5\cos^3\theta'-3\cos\theta')+2\pi\frac{\tilde{s}^3}{\tilde{u}}\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}\tilde{u}_i\tilde{u}_j\cos\theta'(1-\cos^2\theta'). \tag{H.21}$$

- $\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_j\tilde{s}_k\tilde{s}_l=?$

Since

$$\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_j\tilde{s}_k\tilde{s}_l$$
$$=\sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3}\left(\overline{\frac{\partial V_k}{\partial r_k}}\,\overline{\frac{\partial V_j}{\partial r_i}}+\overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_i}}+\overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_k}{\partial r_k}}+\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}}+\overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_i}{\partial r_k}}+\overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}}\right)$$
$$\times\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_j\tilde{s}_k^2$$
$$+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\left(\overline{\frac{\partial V_j}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}}+\overline{\frac{\partial V_j}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}+\overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_j}}+\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_j}}\right)\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i\tilde{s}_j^3$$
$$+\sum_{\substack{i,j=1\\ i\neq j}}^{3}\left(\overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_j}}+\overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_j}}+\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}}\right)\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i^2\tilde{s}_j^2+\sum_{i=1}^{3}\overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_i}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_i^4.$$

Using eqs. (H.12)-(H.15),

$$
\begin{aligned}
&\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}} \int_{\phi'=0}^{2\pi} \mathrm{d}\phi'\, \tilde{s}_i \tilde{s}_j \tilde{s}_k \tilde{s}_l \\
&= \sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3} \left(\overline{\frac{\partial V_k}{\partial r_k}}\,\overline{\frac{\partial V_j}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_k}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_i}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}} \right) \\
&\quad \times \left[\frac{\pi}{4} \tilde{s}^4 \frac{\tilde{u}_i \tilde{u}_j \tilde{u}_k^2}{\tilde{u}^4} (35\cos^4\theta' - 30\cos^2\theta' + 3) + \frac{\pi}{4} \tilde{s}^4 \frac{\tilde{u}_i \tilde{u}_j}{\tilde{u}^2} (5\cos^2\theta' - 1)(1 - \cos^2\theta') \right] \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \left(\overline{\frac{\partial V_j}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_j}} + \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_j}} \right) \\
&\quad \times \left[\frac{\pi}{4} \tilde{s}^4 \frac{\tilde{u}_i \tilde{u}_j^3}{\tilde{u}^4} (35\cos^4\theta' - 30\cos^2\theta' + 3) + \frac{3\pi}{4} \tilde{s}^4 \frac{\tilde{u}_i \tilde{u}_j}{\tilde{u}^2} (5\cos^2\theta' - 1)(1 - \cos^2\theta') \right] \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \left(\overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} \right) \left[\frac{\pi}{4} \tilde{s}^4 \frac{\tilde{u}_i^2 \tilde{u}_j^2}{\tilde{u}^4} (35\cos^4\theta' - 30\cos^2\theta' + 3) \right. \\
&\quad \left. + \frac{\pi}{4} \tilde{s}^4 (1 - \cos^2\theta')^2 + \frac{\pi}{4} \tilde{s}^4 \frac{(\tilde{u}_i^2 + \tilde{u}_j^2)}{\tilde{u}^2} (5\cos^2\theta' - 1)(1 - \cos^2\theta') \right] \\
&\quad + \sum_{i=1}^{3} \overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_i}} \left[\frac{\pi}{4} \tilde{s}^4 \frac{\tilde{u}_i^4}{\tilde{u}^4} (35\cos^4\theta' - 30\cos^2\theta' + 3) + \frac{3\pi}{4} \tilde{s}^4 (1 - \cos^2\theta')^2 \right. \\
&\quad \left. + \frac{3\pi}{2} \tilde{s}^4 \frac{\tilde{u}_i^2}{\tilde{u}^2} (5\cos^2\theta' - 1)(1 - \cos^2\theta') \right] \\
&= \frac{\pi}{4} \frac{\tilde{s}^4}{\tilde{u}^4} \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}} \tilde{u}_i \tilde{u}_j \tilde{u}_k \tilde{u}_l (35\cos^4\theta' - 30\cos^2\theta' + 3) \\
&\quad \sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3} \left(\overline{\frac{\partial V_k}{\partial r_k}}\,\overline{\frac{\partial V_j}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_k}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_i}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}} \right) \\
&\quad \times \frac{\pi}{4} \tilde{s}^4 \frac{\tilde{u}_i \tilde{u}_j}{\tilde{u}^2} (5\cos^2\theta' - 1)(1 - \cos^2\theta') \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \left(\overline{\frac{\partial V_j}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_j}} + \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_j}} \right) \\
&\quad \times \left[\frac{3\pi}{4} \tilde{s}^4 \frac{\tilde{u}_i \tilde{u}_j}{\tilde{u}^2} (5\cos^2\theta' - 1)(1 - \cos^2\theta') \right] \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \left(\overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} \right) \\
&\quad \times \left[\frac{\pi}{4} \tilde{s}^4 (1 - \cos^2\theta')^2 + \frac{\pi}{4} \tilde{s}^4 \frac{(\tilde{u}_i^2 + \tilde{u}_j^2)}{\tilde{u}^2} (5\cos^2\theta' - 1)(1 - \cos^2\theta') \right] \\
&\quad + \sum_{i=1}^{3} \overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_i}} \left[\frac{3\pi}{4} \tilde{s}^4 (1 - \cos^2\theta')^2 + \frac{3\pi}{2} \tilde{s}^4 \frac{\tilde{u}_i^2}{\tilde{u}^2} (5\cos^2\theta' - 1)(1 - \cos^2\theta') \right]
\end{aligned}
$$

or

$$
\begin{aligned}
&\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}} \int_{\phi'=0}^{2\pi} \mathrm{d}\phi'\, \tilde{s}_i \tilde{s}_j \tilde{s}_k \tilde{s}_l \\
&= \frac{\pi}{4}\frac{\tilde{s}^4}{\tilde{u}^4}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(35\cos^4\theta' - 30\cos^2\theta' + 3) + \frac{\pi}{4}\tilde{s}^4\frac{1}{\tilde{u}^2}(5\cos^2\theta' - 1)(1-\cos^2\theta') \\
&\quad \times \Bigg[\sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3} \left(\overline{\frac{\partial V_k}{\partial r_k}}\,\overline{\frac{\partial V_j}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_k}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_i}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}} \right) \tilde{u}_i\tilde{u}_j \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \left(\overline{\frac{\partial V_j}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_j}} + \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_j}} \right) 3\tilde{u}_i\tilde{u}_j \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{3} \left(\overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} \right)(\tilde{u}_i^2 + \tilde{u}_j^2) + \sum_{i=1}^{3} \overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_i}} 6\tilde{u}_i^2 \Bigg] \\
&\quad + \frac{\pi}{4}\tilde{s}^4(1-\cos^2\theta')^2 \left[\sum_{\substack{i,j=1\\ i\neq j}}^{3} \left(\overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} \right) + 3\sum_{i=1}^{3} \overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_i}} \right] \\
&= \frac{\pi}{4}\frac{\tilde{s}^4}{\tilde{u}^4}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(35\cos^4\theta' - 30\cos^2\theta' + 3) + \frac{\pi}{4}\tilde{s}^4\frac{1}{\tilde{u}^2}(5\cos^2\theta' - 1)(1-\cos^2\theta') \\
&\quad \times \sum_{i,j,k=1}^{3} \left(\overline{\frac{\partial V_k}{\partial r_k}}\,\overline{\frac{\partial V_j}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_k}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_i}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}} \right) \\
&\quad \times \tilde{u}_i\tilde{u}_j + \frac{\pi}{4}\tilde{s}^4(1-\cos^2\theta')^2 \sum_{i,j=1}^{3} \left(\overline{\frac{\partial V_i}{\partial r_i}}\,\overline{\frac{\partial V_j}{\partial r_j}} + \overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} \right) \\
&= \frac{\pi}{4}\frac{\tilde{s}^4}{\tilde{u}^4}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(35\cos^4\theta' - 30\cos^2\theta' + 3) + \frac{\pi}{4}\tilde{s}^4\frac{1}{\tilde{u}^2}(5\cos^2\theta' - 1)(1-\cos^2\theta') \\
&\quad \times \left(\overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_i}} + \overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_i}} + \overline{\frac{\partial V_j}{\partial r_k}}\,\overline{\frac{\partial V_i}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_k}} \right) \tilde{u}_i\tilde{u}_j \\
&\quad + \frac{\pi}{4}\tilde{s}^4(1-\cos^2\theta')^2 \left(\overline{\frac{\partial V_j}{\partial r_i}}\,\overline{\frac{\partial V_i}{\partial r_j}} + \overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_i}{\partial r_j}} \right)
\end{aligned}
$$

or

$$
\begin{aligned}
&\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}} \int_{\phi'=0}^{2\pi} \mathrm{d}\phi'\, \tilde{s}_i \tilde{s}_j \tilde{s}_k \tilde{s}_l \\
&= \frac{\pi}{4}\frac{\tilde{s}^4}{\tilde{u}^4}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(35\cos^4\theta' - 30\cos^2\theta' + 3) \\
&\quad + \pi\tilde{s}^4\frac{1}{\tilde{u}^2}(5\cos^2\theta' - 1)(1-\cos^2\theta')\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}\tilde{u}_i\tilde{u}_j + \frac{\pi}{2}\tilde{s}^4(1-\cos^2\theta')^2\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}}. \qquad \text{(H.22)}
\end{aligned}
$$

Therefore using eqs. (H.18) and (H.20)-(H.22),

$$
\begin{aligned}
&\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\tilde{s}_k\tilde{s}_l \\
&=\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,(\tilde{u}_i\tilde{u}_j\tilde{s}_k\tilde{s}_l-\tilde{s}_i\tilde{u}_j\tilde{s}_k\tilde{s}_l-\tilde{u}_i\tilde{s}_j\tilde{s}_k\tilde{s}_l+\tilde{s}_i\tilde{s}_j\tilde{s}_k\tilde{s}_l) \\
&=\pi\frac{\tilde{s}^2}{\tilde{u}^2}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(3\cos^2\theta'-1) \\
&\quad-\pi\frac{\tilde{s}^3}{\tilde{u}^3}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(5\cos^3\theta'-3\cos\theta')-2\pi\frac{\tilde{s}^3}{\tilde{u}}\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}\tilde{u}_i\tilde{u}_j\cos\theta'(1-\cos^2\theta') \\
&\quad-\pi\frac{\tilde{s}^3}{\tilde{u}^3}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(5\cos^3\theta'-3\cos\theta')-2\pi\frac{\tilde{s}^3}{\tilde{u}}\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}\tilde{u}_i\tilde{u}_j\cos\theta'(1-\cos^2\theta') \\
&\quad+\frac{\pi}{4}\frac{\tilde{s}^4}{\tilde{u}^4}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l(35\cos^4\theta'-30\cos^2\theta'+3) \\
&\quad+\pi\tilde{s}^4\frac{1}{\tilde{u}^2}(5\cos^2\theta'-1)(1-\cos^2\theta')\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}\tilde{u}_i\tilde{u}_j+\frac{\pi}{2}\tilde{s}^4(1-\cos^2\theta')^2\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}}
\end{aligned}
$$

or

$$
\begin{aligned}
&\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,(\tilde{u}_i-\tilde{s}_i)(\tilde{u}_j-\tilde{s}_j)\tilde{s}_k\tilde{s}_l \\
&=2\pi\frac{\tilde{s}^2}{\tilde{u}^4}\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_k}{\partial r_l}}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\left\{\tilde{u}^2\frac{1}{2}(3\cos^2\theta'-1)-2\tilde{u}\tilde{s}\frac{1}{2}(5\cos^3\theta'-3\cos\theta')\right. \\
&\quad\left.+\tilde{s}^2\frac{1}{8}(35\cos^4\theta'-30\cos^2\theta'+3)\right\} \\
&\quad+\pi\frac{\tilde{s}^3}{\tilde{u}^2}\overline{\frac{\partial V_i}{\partial r_k}}\,\overline{\frac{\partial V_k}{\partial r_j}}\tilde{u}_i\tilde{u}_j(1-\cos^2\theta')\{\tilde{s}(5\cos^2\theta'-1)-4\tilde{u}\cos\theta'\} \\
&\quad+\frac{\pi}{2}\tilde{s}^4(1-\cos^2\theta')^2\overline{\frac{\partial V_i}{\partial r_j}}\,\overline{\frac{\partial V_j}{\partial r_i}}.
\end{aligned}
\tag{H.23}
$$

- $\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_j\tilde{s}_k\tilde{s}_l=?$

Since

$$
\begin{aligned}
&\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_j\tilde{s}_k\tilde{s}_l \\
&=\sum_{\substack{j,k,l=1\\ j\neq k\neq l\neq j}}^{3}\left(\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}+\overline{\frac{\partial V_k}{\partial r_j}}\frac{\partial \Theta}{\partial r_l}+\overline{\frac{\partial V_l}{\partial r_k}}\frac{\partial \Theta}{\partial r_j}+\overline{\frac{\partial V_k}{\partial r_l}}\frac{\partial \Theta}{\partial r_j}+\overline{\frac{\partial V_j}{\partial r_l}}\frac{\partial \Theta}{\partial r_k}+\overline{\frac{\partial V_l}{\partial r_j}}\frac{\partial \Theta}{\partial r_k}\right) \\
&\quad\times\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_j\tilde{s}_k\tilde{s}_l+\sum_{\substack{j,k=1\\ j\neq k}}^{3}\left(\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_k}+\overline{\frac{\partial V_k}{\partial r_j}}\frac{\partial \Theta}{\partial r_k}+\overline{\frac{\partial V_k}{\partial r_k}}\frac{\partial \Theta}{\partial r_j}\right)\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_j\tilde{s}_k^2 \\
&\quad+\sum_{j=1}^{3}\overline{\frac{\partial V_j}{\partial r_j}}\frac{\partial \Theta}{\partial r_j}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{s}_j^3,
\end{aligned}
$$

using eqs. (H.9)-(H.11),

$$
\begin{aligned}
&\overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \int_{\phi'=0}^{2\pi} \mathrm{d}\phi' \, \tilde{s}_j \tilde{s}_k \tilde{s}_l \\
&= \sum_{\substack{j,k,l=1 \\ j \neq k \neq l \neq j}}^{3} \left(\overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} + \overline{\frac{\partial V_k}{\partial r_j}} \frac{\partial \Theta}{\partial r_l} + \overline{\frac{\partial V_l}{\partial r_k}} \frac{\partial \Theta}{\partial r_j} + \overline{\frac{\partial V_k}{\partial r_l}} \frac{\partial \Theta}{\partial r_j} + \overline{\frac{\partial V_j}{\partial r_l}} \frac{\partial \Theta}{\partial r_k} + \overline{\frac{\partial V_l}{\partial r_j}} \frac{\partial \Theta}{\partial r_k} \right) \\
&\quad \times \pi \tilde{s}^3 \frac{\tilde{u}_j \tilde{u}_k \tilde{u}_l}{\tilde{u}^3} (5 \cos^3 \theta' - 3 \cos \theta') \\
&\quad + \sum_{\substack{j,k=1 \\ j \neq k}}^{3} \left(\overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_k} + \overline{\frac{\partial V_k}{\partial r_j}} \frac{\partial \Theta}{\partial r_k} + \overline{\frac{\partial V_k}{\partial r_k}} \frac{\partial \Theta}{\partial r_j} \right) \\
&\quad \times \pi \tilde{s}^3 \left[\frac{\tilde{u}_j \tilde{u}_k^2}{\tilde{u}^3} (5 \cos^3 \theta' - 3 \cos \theta') + \frac{\tilde{u}_j}{\tilde{u}} \cos \theta' (1 - \cos^2 \theta') \right] \\
&\quad + \sum_{j=1}^{3} \overline{\frac{\partial V_j}{\partial r_j}} \frac{\partial \Theta}{\partial r_j} \pi \tilde{s}^3 \left[\frac{\tilde{u}_j^3}{\tilde{u}^3} (5 \cos^3 \theta' - 3 \cos \theta') + 3 \frac{\tilde{u}_j}{\tilde{u}} \cos \theta' (1 - \cos^2 \theta') \right] \\
&= \pi \frac{\tilde{s}^3}{\tilde{u}^3} \sum_{j,k,l=1}^{3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \tilde{u}_j \tilde{u}_k \tilde{u}_l (5 \cos^3 \theta' - 3 \cos \theta') \\
&\quad + \pi \frac{\tilde{s}^3}{\tilde{u}} \sum_{j,k=1}^{3} \left(\overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_k} + \overline{\frac{\partial V_k}{\partial r_j}} \frac{\partial \Theta}{\partial r_k} + \overline{\frac{\partial V_k}{\partial r_k}} \frac{\partial \Theta}{\partial r_j} \right) \tilde{u}_j \cos \theta' (1 - \cos^2 \theta') \\
&= \pi \frac{\tilde{s}^3}{\tilde{u}^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \tilde{u}_j \tilde{u}_k \tilde{u}_l (5 \cos^3 \theta' - 3 \cos \theta') + \pi \frac{\tilde{s}^3}{\tilde{u}} \left(\overline{\frac{\partial V_j}{\partial r_k}} + \overline{\frac{\partial V_k}{\partial r_j}} \right) \frac{\partial \Theta}{\partial r_k} \tilde{u}_j \cos \theta' (1 - \cos^2 \theta')
\end{aligned}
$$

or

$$
\begin{aligned}
&\overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \int_{\phi'=0}^{2\pi} \mathrm{d}\phi' \, \tilde{s}_j \tilde{s}_k \tilde{s}_l \\
&= \pi \frac{\tilde{s}^3}{\tilde{u}^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \tilde{u}_j \tilde{u}_k \tilde{u}_l (5 \cos^3 \theta' - 3 \cos \theta') + 2\pi \frac{\tilde{s}^3}{\tilde{u}} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_k} \tilde{u}_j \cos \theta' (1 - \cos^2 \theta').
\end{aligned} \tag{H.24}
$$

Hence using eqs. (H.18) and (H.24),

$$
\begin{aligned}
&\overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \int_{\phi'=0}^{2\pi} \mathrm{d}\phi' \, \tilde{u}_i \tilde{s}_j \tilde{s}_k (\tilde{u}_l - \tilde{s}_l) \\
&= \tilde{u}_i \tilde{u}_l \frac{\partial \Theta}{\partial r_l} \times \pi \frac{\tilde{s}^2}{\tilde{u}^2} \overline{\frac{\partial V_j}{\partial r_k}} \tilde{u}_j \tilde{u}_k (3 \cos^2 \theta' - 1) \\
&\quad - \tilde{u}_i \left\{ \pi \frac{\tilde{s}^3}{\tilde{u}^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \tilde{u}_j \tilde{u}_k \tilde{u}_l (5 \cos^3 \theta' - 3 \cos \theta') + 2\pi \frac{\tilde{s}^3}{\tilde{u}} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_k} \tilde{u}_j \cos \theta' (1 - \cos^2 \theta') \right\} \\
&= 2\pi \frac{\tilde{s}^2}{\tilde{u}^3} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_l} \tilde{u}_i \tilde{u}_j \tilde{u}_k \tilde{u}_l \left\{ \tilde{u} \frac{1}{2} (3 \cos^2 \theta' - 1) - \tilde{s} \frac{1}{2} (5 \cos^3 \theta' - 3 \cos \theta') \right\} \\
&\quad - 2\pi \frac{\tilde{s}^3}{\tilde{u}} \overline{\frac{\partial V_j}{\partial r_k}} \frac{\partial \Theta}{\partial r_k} \tilde{u}_i \tilde{u}_j \cos \theta' (1 - \cos^2 \theta').
\end{aligned} \tag{H.25}
$$

Also, using eqs. (H.6), (H.16) and (H.24),

$$
\begin{aligned}
&\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\int_{\phi'=0}^{2\pi}\mathrm{d}\phi'\,\tilde{u}_i(\tilde{u}_j-\tilde{s}_j)(\tilde{u}_k-\tilde{s}_k)\tilde{s}_l \\
&=\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\tilde{u}_i\tilde{u}_j\tilde{u}_k\times 2\pi\frac{\tilde{s}}{\tilde{u}}\tilde{u}_l\cos\theta' \\
&\quad-\tilde{u}_i\tilde{u}_j\left\{\pi\tilde{s}^2\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_k}(1-\cos^2\theta')+\pi\frac{\tilde{s}^2}{\tilde{u}^2}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\tilde{u}_k\tilde{u}_l(3\cos^2\theta'-1)\right\} \\
&\quad-\tilde{u}_i\tilde{u}_k\left\{\pi\tilde{s}^2\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_j}(1-\cos^2\theta')+\pi\frac{\tilde{s}^2}{\tilde{u}^2}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\tilde{u}_j\tilde{u}_l(3\cos^2\theta'-1)\right\} \\
&\quad+\tilde{u}_i\left\{\pi\frac{\tilde{s}^3}{\tilde{u}^3}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\tilde{u}_j\tilde{u}_k\tilde{u}_l(5\cos^3\theta'-3\cos\theta')+2\pi\frac{\tilde{s}^3}{\tilde{u}}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_k}\tilde{u}_j\cos\theta'(1-\cos^2\theta')\right\} \\
&=2\pi\frac{\tilde{s}}{\tilde{u}^3}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_l}\tilde{u}_i\tilde{u}_j\tilde{u}_k\tilde{u}_l\left\{\tilde{u}^2\cos\theta'-2\tilde{u}\tilde{s}\frac{1}{2}(3\cos^2\theta'-1)+\tilde{s}^2\frac{1}{2}(5\cos^3\theta'-3\cos\theta')\right\} \\
&\quad-2\pi\frac{\tilde{s}^2}{\tilde{u}}\overline{\frac{\partial V_j}{\partial r_k}}\frac{\partial \Theta}{\partial r_k}\tilde{u}_i\tilde{u}_j(\tilde{u}-\tilde{s}\cos\theta')(1-\cos^2\theta'). \qquad \text{(H.26)}
\end{aligned}
$$

Appendix I

Fredholm-form of operator $\tilde{\mathscr{L}}$

In this Appendix we shall transform the operator $\tilde{\mathscr{L}}$ into a Fredholm-type integral operator (in few equations), following a similar idea as in Sela *et al.* (1996) and then we shall show that the unknown functions are even. In fact, when we say that the unknown functions $\hat{\Phi}_v$, $\hat{\Phi}_c$, $\hat{\Phi}_e$ and $\bar{\eta}$ are functions of speed $\tilde{u}$ alone, that itself means that the functions are even because magnitude of velocity can not be negative. Nevertheless, we shall use the Fredholm form of operator $\tilde{\mathscr{L}}$ to show mathematically that these functions are even.

I.1 Fredholm-form of operator $\tilde{\mathscr{L}}$

Consider eq. (3.20). This equation can be written as

$$\tilde{\mathscr{L}}\left[\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\right] = \overline{\tilde{u}_{1i}\tilde{u}_{1j}}. \tag{I.1}$$

Using the definition of operator $\tilde{\mathscr{L}}$ (eq. (2.37)), the left-hand side of eq. (I.1) can be written as

$$\tilde{\mathscr{L}}\left[\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\right] = L_1'[\Phi(\tilde{\boldsymbol{u}}_1)] + L_2'[\Phi(\tilde{\boldsymbol{u}}_1)] - L_1[\Phi(\tilde{\boldsymbol{u}}_1)] - L_2[\Phi(\tilde{\boldsymbol{u}}_1)], \tag{I.2}$$

where

$$L_1'[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_1')\overline{\tilde{u}_{1i}'\tilde{u}_{1j}'}, \tag{I.3a}$$

$$L_2'[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2')\overline{\tilde{u}_{2i}'\tilde{u}_{2j}'}, \tag{I.3b}$$

$$L_1[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}, \tag{I.3c}$$

$$L_2[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}. \tag{I.3d}$$

First consider $L_1'[\Phi(\tilde{\boldsymbol{u}}_1)]$. Multiplying eq. (I.3a) by $\tilde{f}_0(\tilde{u}_1)\delta(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1)$ (where $\tilde{f}_0(\tilde{u})$ is given in eq. (2.17)), and integrating over $\tilde{\boldsymbol{u}}_1$, one obtains

$$\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1')\overline{\tilde{u}_{1i}'\tilde{u}_{1j}'}\,\delta(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1).$$

Since the operator $\tilde{\mathscr{L}}$ is defined for *elastic* collisions, one may replace $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}$ by $-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'$, $\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2$ by $\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'$ and $\tilde{u}_1^2 + \tilde{u}_2^2$ by $\tilde{u}_1'^2 + \tilde{u}_2'^2$, and using eq. (2.1) with $e = 1$, we get

$$\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'<0} \mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\,\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\,\hat{\Phi}_v(\tilde{u}_1')\,\overline{\tilde{u}_{1i}'\tilde{u}_{1j}'}\,\delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1'+(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\hat{\boldsymbol{k}}\Big).$$

In the above integral, if we replace $\hat{\boldsymbol{k}}$ by $-\hat{\boldsymbol{k}}$ and adjust the limits of integrations over the components of $\hat{\boldsymbol{k}}$ accordingly, we get

$$\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0} \mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\,\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\,\hat{\Phi}_v(\tilde{u}_1')\,\overline{\tilde{u}_{1i}'\tilde{u}_{1j}'}\,\delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1'+(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\hat{\boldsymbol{k}}\Big).$$

Since $\tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2'$ are integration variables, we can omit the prime signs to get

$$\begin{aligned}\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] &= \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1+(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\hat{\boldsymbol{k}}\Big)\\ &= \frac{1}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}I_\delta^{(0)},\end{aligned}$$

where $I_\delta^{(0)}$ is given in eq. (4.48). Let $\tilde{\boldsymbol{s}} = \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1$ and the spherical coordinates of $\tilde{\boldsymbol{s}}$ in the original coordinate system be $(\tilde{s}, \theta_{\tilde{s}}, \phi_{\tilde{s}})$ and the integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2'', \phi_2'')$, which results from two rotations: (i) rotation of xy-plane in positive direction (x towards y) around z-axis by an angle $\phi_{\tilde{s}}$ and (ii) rotation of new zx-plane in positive direction (z towards x) around new y-axis by an angle $\theta_{\tilde{s}}$, so that $\tilde{\boldsymbol{s}}$ coincides with the new z-axis (see figure H.1) and $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_2 = \tilde{s}\tilde{u}_2\cos\theta_2''$. Hence

$$\begin{aligned}\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] &= \frac{1}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2''=0}^{\pi}\int_{\phi_2''=0}^{2\pi}\mathrm{d}\phi_2''\mathrm{d}\theta_2''\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2''\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}I_\delta^{(0)}\\ &= \frac{2}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\mathrm{e}^{-\tilde{u}_1^2}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^2\,\mathrm{e}^{-\tilde{u}_2^2}\left(\int_{\theta_2''=0}^{\pi}\sin\theta_2'' I_\delta^{(0)}\mathrm{d}\theta_2''\right)\mathrm{d}\tilde{u}_2.\end{aligned}$$

Using eq. (C.12),

$$\int_{\theta_2''=0}^{\pi}\sin\theta_2'' I_\delta^{(0)}\mathrm{d}\theta_2'' = \frac{1}{\tilde{s}\tilde{u}_2}H\left(\tilde{u}_2 - \left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|\right).$$

Hence

$$\begin{aligned}\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] &= \frac{1}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\mathrm{e}^{-\tilde{u}_1^2}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\,\frac{1}{\tilde{s}}\int_{\tilde{u}_2=0}^{\infty}(2\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,H\left(\tilde{u}_2-\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|\right)\mathrm{d}\tilde{u}_2\\ &= \frac{1}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\mathrm{e}^{-\tilde{u}_1^2}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\,\frac{1}{\tilde{s}}\int_{\tilde{u}_2=\left|\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right|}^{\infty}(2\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.\end{aligned}$$

Let $\tilde{u}_2^2 = t \Rightarrow 2\tilde{u}_2\,\mathrm{d}\tilde{u}_2 = \mathrm{d}t$. This implies that

$$\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\mathrm{e}^{-\tilde{u}_1^2}\,\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\,\frac{1}{\tilde{s}}\,\mathrm{e}^{-\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}.$$

Therefore

$$L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \, \mathrm{e}^{-\tilde{u}_1^2} \, \hat{\Phi}_v(\tilde{u}_1) \overline{\tilde{u}_{1i}\tilde{u}_{1j}} \, \frac{1}{\tilde{s}} \, \mathrm{e}^{\tilde{u}^2 - \left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}$$
$$= \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \, \mathrm{e}^{-\tilde{u}_1^2} \, \hat{\Phi}_v(\tilde{u}_1) \overline{\tilde{u}_{1i}\tilde{u}_{1j}} \, \frac{1}{|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1|} \, \mathrm{e}^{\tilde{u}^2 - \left(\frac{(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot\tilde{\boldsymbol{u}}}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1|}\right)^2} .$$

Since $\tilde{\boldsymbol{u}}_1$ is the integration variable, we can change it to $\tilde{\boldsymbol{u}}_2$, i.e.,

$$L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \mathrm{e}^{-\tilde{u}_2^2} \, \hat{\Phi}_v(\tilde{u}_2) \overline{\tilde{u}_{2i}\tilde{u}_{2j}} \, \frac{1}{|\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_2|} \, \mathrm{e}^{\tilde{u}^2 - \left(\frac{(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\right)^2} ,$$

$$\therefore \quad L_1'[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \mathrm{e}^{-\tilde{u}_2^2} \, \hat{\Phi}_v(\tilde{u}_2) \overline{\tilde{u}_{2i}\tilde{u}_{2j}} \, \frac{1}{|\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2|} \, \mathrm{e}^{\tilde{u}_1^2 - \left(\frac{(\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}_1}{|\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2|}\right)^2} .$$

Let $|\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2| = R$. Here

$$\tilde{u}_1^2 - \left(\frac{(\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}_1}{|\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2|}\right)^2 = \frac{1}{R^2}\left[\tilde{u}_1^2(\tilde{u}_1^2 - 2\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2 + \tilde{u}_2^2) - (\tilde{u}_1^2 - \tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2)^2\right]$$
$$= \frac{1}{R^2}\left[\tilde{u}_1^2\tilde{u}_2^2 - (\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2)^2\right] = \frac{1}{R^2}\left[\tilde{u}_1^2\tilde{u}_2^2 - \tilde{u}_1^2\tilde{u}_2^2\cos^2\theta_2'\right]$$
$$= \frac{\tilde{u}_1^2\tilde{u}_2^2\sin^2\theta_2'}{R^2} = \frac{|\tilde{\boldsymbol{u}}_1 \times \tilde{\boldsymbol{u}}_2|^2}{R^2} = w^2,$$

where θ_2' is the angle between $\tilde{\boldsymbol{u}}_1$ and $\tilde{\boldsymbol{u}}_2$ and $w \equiv \frac{\tilde{\boldsymbol{u}}_1\times\tilde{\boldsymbol{u}}_2}{R}$. Hence

$$L_1'[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \hat{\Phi}_v(\tilde{u}_2) \overline{\tilde{u}_{2i}\tilde{u}_{2j}} \, \mathrm{e}^{-\tilde{u}_2^2} \, \frac{1}{R} \, \mathrm{e}^{w^2} . \tag{I.4}$$

Next consider $L_2'[\Phi(\tilde{\boldsymbol{u}}_1)]$. Multiplying eq. (I.3b) by $\tilde{f}_0(\tilde{u}_1)\delta(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1)$ and integrating over $\tilde{\boldsymbol{u}}_1$ one obtains

$$\tilde{f}_0(\tilde{u}) L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4} \int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}) \, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)} \, \hat{\Phi}_v(\tilde{u}_2') \overline{\tilde{u}_{2i}'\tilde{u}_{2j}'} \, \delta(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1).$$

By using similar procedure as above, we get

$$\tilde{f}_0(\tilde{u}) L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)} \, \hat{\Phi}_v(\tilde{u}_2) \overline{\tilde{u}_{2i}\tilde{u}_{2j}} I_\delta^{(0)} .$$

Let $\tilde{\boldsymbol{t}} = \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_2$ and the spherical coordinates of $\tilde{\boldsymbol{t}}$ in the original coordinate system be $(\tilde{t}, \theta_{\tilde{t}}, \phi_{\tilde{t}})$ and the integration over $\tilde{\boldsymbol{u}}_1$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_1, \theta_1'', \phi_1'')$, which results from two rotations: (i) rotation of xy-plane in positive direction (x towards y) around z-axis by an angle $\phi_{\tilde{t}}$ and (ii) rotation of new zx-plane in positive direction (z towards x) around new y-axis by an angle $\theta_{\tilde{t}}$, so that $\tilde{\boldsymbol{t}}$ coincides with the new z-axis (see figure H.1) and $\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}_1 = \tilde{t}\tilde{u}_1\cos\theta_1''$. Hence

$$\tilde{f}_0(\tilde{u}) L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \int_{\tilde{u}_1=0}^{\infty} \int_{\theta_1''=0}^{\pi} \int_{\phi_1''=0}^{2\pi} \mathrm{d}\phi_1''\mathrm{d}\theta_1''\mathrm{d}\tilde{u}_1 \, \tilde{u}_1^2 \sin\theta_1'' \, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)} \, \hat{\Phi}_v(\tilde{u}_2) \overline{\tilde{u}_{2i}\tilde{u}_{2j}} I_\delta^{(0)}$$

or

$$\tilde{f}_0(\tilde{u})L_2'[\varPhi(\tilde{\boldsymbol{u}})] = \frac{2}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-\tilde{u}_2^2}\,\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\int_{\tilde{u}_1=0}^{\infty}\tilde{u}_1^2\,\mathrm{e}^{-\tilde{u}_1^2}\left(\int_{\theta_1''=0}^{\pi}\sin\theta_1'' I_\delta^{(0)}\mathrm{d}\theta_1''\right)\mathrm{d}\tilde{u}_1.$$

Using eq. (C.18),

$$\int_{\theta_1''=0}^{\pi}\sin\theta_1'' I_\delta^{(0)}\mathrm{d}\theta_1'' = \frac{1}{\tilde{t}\tilde{u}_1}H\left(\tilde{u}_1 - \left|\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}}\right|\right).$$

Hence

$$\begin{aligned}\tilde{f}_0(\tilde{u})L_2'[\varPhi(\tilde{\boldsymbol{u}})] &= \frac{1}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\frac{1}{\tilde{t}}\int_{\tilde{u}_1=0}^{\infty}(2\tilde{u}_1)\,\mathrm{e}^{-\tilde{u}_1^2}\,H\left(\tilde{u}_1-\left|\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}}\right|\right)\mathrm{d}\tilde{u}_1\\ &= \frac{1}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\frac{1}{\tilde{t}}\int_{\tilde{u}_1=\left|\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}}\right|}^{\infty}(2\tilde{u}_1)\,\mathrm{e}^{-\tilde{u}_1^2}\,\mathrm{d}\tilde{u}_1.\end{aligned}$$

Let $\tilde{u}_1^2 = z \Rightarrow 2\tilde{u}_1\,\mathrm{d}\tilde{u}_1 = \mathrm{d}z$. This implies that

$$\tilde{f}_0(\tilde{u})L_2'[\varPhi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\frac{1}{\tilde{t}}\,\mathrm{e}^{-\left(\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}}\right)^2}.$$

Therefore

$$\begin{aligned}L_2'[\varPhi(\tilde{\boldsymbol{u}})] &= \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\frac{1}{\tilde{t}}\,\mathrm{e}^{\tilde{u}^2-\left(\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}}\right)^2}\\ &= \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\frac{1}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\,\mathrm{e}^{\tilde{u}^2-\left(\frac{(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\right)^2}.\end{aligned}$$

$$\therefore \quad L_2'[\varPhi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\frac{1}{|\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2|}\,\mathrm{e}^{\tilde{u}_1^2-\left(\frac{(\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}_1}{|\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2|}\right)^2},$$

i.e.,

$$L_2'[\varPhi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\mathrm{e}^{-\tilde{u}_2^2}\,\frac{1}{R}\,\mathrm{e}^{w^2}. \tag{I.5}$$

Next consider $L_1[\varPhi(\tilde{\boldsymbol{u}}_1)]$. In eq. (I.3c), the integration over $\hat{\boldsymbol{k}}$ is trivial. Using eq. (G.1b),

$$L_1[\varPhi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\,\mathrm{e}^{-\tilde{u}_2^2}.$$

The integration over $\tilde{\boldsymbol{u}}_2$ is given in eq. (G.3); using that, we get

$$\begin{aligned}L_1[\varPhi(\tilde{\boldsymbol{u}}_1)] &= \frac{1}{\sqrt{\pi}}\left[\mathrm{e}^{-\tilde{u}_1^2}+\frac{\pi^{1/2}(1+2\tilde{u}_1^2)}{2\tilde{u}_1}\,\mathrm{erf}(\tilde{u}_1)\right]\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\\ &= \frac{1}{\sqrt{\pi}}Q(\tilde{u}_1)\hat{\varPhi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}},\end{aligned} \tag{I.6}$$

where $Q(\tilde{u}) \equiv \mathrm{e}^{-\tilde{u}^2}+\frac{\sqrt{\pi}}{2}\left(2\tilde{u}+\frac{1}{\tilde{u}}\right)\mathrm{erf}(\tilde{u})$. Next consider $L_2[\varPhi(\tilde{\boldsymbol{u}}_1)]$. In eq. (I.3d) also, the integration over $\hat{\boldsymbol{k}}$ is trivial. Using eq. (G.1b) and the relation: $\tilde{u}_{12} = |\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2| = R$,

$$L_2[\varPhi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\tilde{u}_{12}\,\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\mathrm{e}^{-\tilde{u}_2^2} = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\varPhi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\mathrm{e}^{-\tilde{u}_2^2}\,R. \tag{I.7}$$

Substituting the values of $L_1'[\Phi(\tilde{\boldsymbol{u}}_1)]$, $L_2'[\Phi(\tilde{\boldsymbol{u}}_1)]$, $L_1[\Phi(\tilde{\boldsymbol{u}}_1)]$ and $L_2[\Phi(\tilde{\boldsymbol{u}}_1)]$ from eqs. (I.4)-(I.7) respectively, into eq. (I.2), we get

$$\tilde{\mathscr{L}}\left[\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}\right] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\mathrm{e}^{-\tilde{u}_2^2}\left(\frac{2}{R}\,\mathrm{e}^{w^2} - R\right) - \frac{1}{\sqrt{\pi}}Q(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}}.$$

Hence eq. (I.1) can be written as

$$-\frac{1}{\sqrt{\pi}}\left[Q(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}} + \frac{1}{\pi}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\mathrm{e}^{-\tilde{u}_2^2}\left(R - \frac{2}{R}\,\mathrm{e}^{w^2}\right)\right] = \overline{\tilde{u}_{1i}\tilde{u}_{1j}}. \tag{I.8}$$

We shall multiply both sides of eq. (I.8) by $\frac{\partial V_i}{\partial r_j}$ for further simplification; recall that eq. (I.1) resulted from comparing the coefficients $\frac{\partial V_i}{\partial r_j}$ in eqs. (3.5) and (3.18).

$$-\frac{1}{\sqrt{\pi}}\left[Q(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_1)\overline{\tilde{u}_{1i}\tilde{u}_{1j}} + \frac{1}{\pi}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\mathrm{e}^{-\tilde{u}_2^2}\left(R - \frac{2}{R}\,\mathrm{e}^{w^2}\right)\right]\frac{\partial V_i}{\partial r_j} = \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\frac{\partial V_i}{\partial r_j}. \tag{I.9}$$

Next, we shall simplify the integral in the left-hand side of eq. (I.9) as following. Let

$$I_v = \frac{\partial V_i}{\partial r_j}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\Phi}_v(\tilde{u}_2)\overline{\tilde{u}_{2i}\tilde{u}_{2j}}\,\mathrm{e}^{-\tilde{u}_2^2}\left(R - \frac{2}{R}\,\mathrm{e}^{w^2}\right).$$

Using eq. (F.3), the above equation can be written as

$$I_v = \overline{\frac{\partial V_i}{\partial r_j}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2i}\tilde{u}_{2j}\,\mathrm{e}^{-\tilde{u}_2^2}\left(R - \frac{2}{R}\,\mathrm{e}^{w^2}\right).$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1 \cdot \tilde{\boldsymbol{u}}_2 = \tilde{u}_1\tilde{u}_2\cos\theta_2'$, i.e.,

$$I_v = \overline{\frac{\partial V_i}{\partial r_j}}\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\hat{\Phi}_v(\tilde{u}_2)\tilde{u}_{2i}\tilde{u}_{2j}\,\mathrm{e}^{-\tilde{u}_2^2}\left(R - \frac{2}{R}\,\mathrm{e}^{w^2}\right).$$

Note that only components of $\tilde{\boldsymbol{u}}_2$ are the functions of ϕ_2'. Hence the integration over ϕ_2' results into (cf. eq. (H.18))

$$I_v = \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\hat{\Phi}_v(\tilde{u}_2)\left\{\pi\frac{\tilde{u}_2^2}{\tilde{u}_1^2}\overline{\frac{\partial V_i}{\partial r_j}}\tilde{u}_{1i}\tilde{u}_{1j}(3\cos^2\theta_2' - 1)\right\}\mathrm{e}^{-\tilde{u}_2^2}\left(R - \frac{2}{R}\,\mathrm{e}^{w^2}\right).$$

Again, using eq. (F.3), I_v can be written as

$$I_v = \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\frac{\partial V_i}{\partial r_j}\frac{2\pi}{\tilde{u}_1^2}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^4\,\hat{\Phi}_v(\tilde{u}_2)\left\{\int_{\theta_2'=0}^{\pi}\sin\theta_2'\left(R - \frac{2}{R}\,\mathrm{e}^{w^2}\right)P_2(\cos\theta_2')\,\mathrm{d}\theta_2'\right\}\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2,$$

where $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the second order Legendre polynomial. Let

$$A_n(\tilde{u}_1, \tilde{u}_2) = \int_{\theta_2'=0}^{\pi}\sin\theta_2'\left(R - \frac{2}{R}\,\mathrm{e}^{w^2}\right)P_n(\cos\theta_2')\,\mathrm{d}\theta_2'. \tag{I.10}$$

Therefore

$$I_v = \overline{\tilde{u}_{1i}\tilde{u}_{1j}}\frac{\partial V_i}{\partial r_j}\frac{2\pi}{\tilde{u}_1^2}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^4\,\hat{\Phi}_v(\tilde{u}_2)A_2(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.$$

Using the above equation, eq. (I.9) reduces to

$$-\frac{1}{\sqrt{\pi}}\left[Q(\tilde{u}_1)\hat{\Phi}_v(\tilde{u}_1)+\frac{2}{\tilde{u}_1^2}\int_0^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^4\,\hat{\Phi}_v(\tilde{u}_2)A_2(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\right]=1. \tag{I.11}$$

Eq. (I.11) is the Fredholm-form of eq. (I.1), where, in particular, $A_2(\tilde{u}_1,\tilde{u}_2)$ (for $\tilde{u}_1>\tilde{u}_2$) is given by (Pekeris 1955):

$$\begin{aligned}A_2(\tilde{u}_1,\tilde{u}_2)=\frac{1}{\tilde{u}_1^3\tilde{u}_2^3}&\left[\frac{2}{35}\tilde{u}_2^7-\frac{2}{15}\tilde{u}_2^5\tilde{u}_1^2+3\tilde{u}_1^2\tilde{u}_2-3\tilde{u}_2^3+18\tilde{u}_2\right.\\&\left.+\frac{\sqrt{\pi}}{2}(-6\tilde{u}_2^4+2\tilde{u}_1^2\tilde{u}_2^2-3\tilde{u}_1^2+15\tilde{u}_2^2-18)\,\mathrm{e}^{\tilde{u}_2^2}\,\mathrm{erf}(\tilde{u}_2)\right].\end{aligned} \tag{I.12}$$

The value of $A_2(\tilde{u}_1,\tilde{u}_2)$ for $\tilde{u}_1<\tilde{u}_2$ is obtained from eq. (I.12) by interchanging $\tilde{u}_1$ and $\tilde{u}_2$.

Next, consider eq. (3.21). This equation can be written as

$$\tilde{\mathscr{L}}\left[\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\right]=\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}. \tag{I.13}$$

Using the definition of operator $\tilde{\mathscr{L}}$ (eq. (2.37)), the left-hand side of eq. (I.13) can be written as

$$\tilde{\mathscr{L}}\left[\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\right]=L_1'[\Phi(\tilde{\boldsymbol{u}}_1)]+L_2'[\Phi(\tilde{\boldsymbol{u}}_1)]-L_1[\Phi(\tilde{\boldsymbol{u}}_1)]-L_2[\Phi(\tilde{\boldsymbol{u}}_1)], \tag{I.14}$$

where

$$L_1'[\Phi(\tilde{\boldsymbol{u}}_1)]=\frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_1')\left(\tilde{u}_1'^2-\frac{5}{2}\right)\tilde{u}_{1i}', \tag{I.15a}$$

$$L_2'[\Phi(\tilde{\boldsymbol{u}}_1)]=\frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2')\left(\tilde{u}_2'^2-\frac{5}{2}\right)\tilde{u}_{2i}', \tag{I.15b}$$

$$L_1[\Phi(\tilde{\boldsymbol{u}}_1)]=\frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}, \tag{I.15c}$$

$$L_2[\Phi(\tilde{\boldsymbol{u}}_1)]=\frac{1}{\pi^{5/2}}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}. \tag{I.15d}$$

First consider $L_1'[\Phi(\tilde{\boldsymbol{u}}_1)]$. Multiplying eq. (I.15a) by $\tilde{f}_0(\tilde{u}_1)\delta(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)$ and integrating over $\tilde{\boldsymbol{u}}_1$ one obtains

$$\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})]=\frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1')\left(\tilde{u}_1'^2-\frac{5}{2}\right)\tilde{u}_{1i}'\,\delta(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1).$$

Since the operator $\tilde{\mathscr{L}}$ is defined for *elastic* collisions, one may replace $\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}$ by $-\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'$, $\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2$ by $\mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'$ and $\tilde{u}_1^2+\tilde{u}_2^2$ by $\tilde{u}_1'^2+\tilde{u}_2'^2$; following a similar procedure as above, we get

$$\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}'>0} \mathrm{d}\tilde{\boldsymbol{u}}_1'\mathrm{d}\tilde{\boldsymbol{u}}_2'\mathrm{d}\hat{\boldsymbol{k}}\,(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\,\mathrm{e}^{-(\tilde{u}_1'^2+\tilde{u}_2'^2)}\,\hat{\Phi}_c(\tilde{u}_1')\left(\tilde{u}_1'^2-\frac{5}{2}\right)\tilde{u}_{1i}'\,\delta\Big(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1'+(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}')\hat{\boldsymbol{k}}\Big).$$

Since $\tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2'$ are integration parameters, we can omit the prime signs to get

$$\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}I_\delta^{(0)}.$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a rotated spherical coordinate system $(\tilde{u}_2,\theta_2'',\phi_2'')$ (defined as above), whose z-axis coincides with $\tilde{\boldsymbol{s}}$ so that $\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}_2=\tilde{s}\tilde{u}_2\cos\theta_2''$. Hence

$$\begin{aligned}\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] &= \frac{1}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2''=0}^{\pi}\int_{\phi_2''=0}^{2\pi}\mathrm{d}\phi_2''\mathrm{d}\theta_2''\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2''\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}I_\delta^{(0)}\\ &= \frac{2}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\mathrm{e}^{-\tilde{u}_1^2}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^2\,\mathrm{e}^{-\tilde{u}_2^2}\left(\int_{\theta_2''=0}^{\pi}\sin\theta_2''I_\delta^{(0)}\mathrm{d}\theta_2''\right)\mathrm{d}\tilde{u}_2.\end{aligned}$$

We integrate over θ_2'' and $\tilde{u}_2$ by following exactly similar procedure as above and get

$$\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\mathrm{e}^{-\tilde{u}_1^2}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\frac{1}{\tilde{s}}\,\mathrm{e}^{-\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}$$

or

$$L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\,\mathrm{e}^{-\tilde{u}_1^2}\,\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\frac{1}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1|}\,\mathrm{e}^{\tilde{u}^2-\left(\frac{(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot\tilde{\boldsymbol{u}}}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1|}\right)^2}.$$

Since $\tilde{\boldsymbol{u}}_1$ is the integration variable, we can change it to $\tilde{\boldsymbol{u}}_2$, i.e.,

$$L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\frac{1}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\,\mathrm{e}^{\tilde{u}^2-\left(\frac{(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\right)^2},$$

$$\begin{aligned}\therefore\quad L_1'[\Phi(\tilde{\boldsymbol{u}}_1)] &= \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-\tilde{u}_2^2}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\frac{1}{|\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2|}\,\mathrm{e}^{\tilde{u}_1^2-\left(\frac{(\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}_1}{|\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2|}\right)^2}\\ &= \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\mathrm{e}^{-\tilde{u}_2^2}\,\frac{1}{R}\,\mathrm{e}^{w^2}.\end{aligned}\tag{I.16}$$

Next consider $L_2'[\Phi(\tilde{\boldsymbol{u}}_1)]$. Multiplying eq. (I.15b) by $\tilde{f}_0(\tilde{u}_1)\delta(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)$ and integrating over $\tilde{\boldsymbol{u}}_1$ one obtains

$$\tilde{f}_0(\tilde{u})L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0}\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_2')\left(\tilde{u}_2'^2-\frac{5}{2}\right)\tilde{u}_{2i}'\,\delta(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1).$$

By using similar procedure as above, we get

$$\tilde{f}_0(\tilde{u})L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\,\mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}I_\delta^{(0)}.$$

The integration over $\tilde{\boldsymbol{u}}_1$ is performed in a rotated spherical coordinate system $(\tilde{u}_1, \theta_1'', \phi_1'')$ (defined as above), whose z-axis coincides with $\tilde{\boldsymbol{t}}$ so that $\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}_1 = \tilde{t}\tilde{u}_1\cos\theta_1''$. Hence

$$\begin{aligned}
&\tilde{f}_0(\tilde{u})L_2'[\Phi(\tilde{\boldsymbol{u}})]\\
&= \frac{1}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_2 \int_{\tilde{u}_1=0}^{\infty}\int_{\theta_1''=0}^{\pi}\int_{\phi_1''=0}^{2\pi} \mathrm{d}\phi_1''\mathrm{d}\theta_1''\mathrm{d}\tilde{u}_1\, \tilde{u}_1^2 \sin\theta_1''\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i} I_\delta^{(0)}\\
&= \frac{2}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\int_{\tilde{u}_1=0}^{\infty}\tilde{u}_1^2\,\mathrm{e}^{-\tilde{u}_1^2}\left(\int_{\theta_1''=0}^{\pi}\sin\theta_1'' I_\delta^{(0)}\mathrm{d}\theta_1''\right)\mathrm{d}\tilde{u}_1.
\end{aligned}$$

We integrate over θ_1'' and $\tilde{u}_1$ by following exactly similar procedure as above and get

$$\tilde{f}_0(\tilde{u})L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\frac{1}{\tilde{t}}\,\mathrm{e}^{-\left(\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}}\right)^2}$$

or

$$L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\frac{1}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\,\mathrm{e}^{\tilde{u}^2-\left(\frac{(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\right)^2}.$$

$$\begin{aligned}
\therefore\quad L_2'[\Phi(\tilde{\boldsymbol{u}}_1)] &= \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\frac{1}{|\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2|}\,\mathrm{e}^{\tilde{u}_1^2-\left(\frac{(\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}_1}{|\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2|}\right)^2}\\
&= \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\mathrm{e}^{-\tilde{u}_2^2}\,\frac{1}{R}\,\mathrm{e}^{w^2}. \qquad \text{(I.17)}
\end{aligned}$$

Next consider $L_1[\Phi(\tilde{\boldsymbol{u}}_1)]$. In eq. (I.15c), the integration over $\hat{\boldsymbol{k}}$ is trivial. Using eq. (G.1b),

$$L_1[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\,\mathrm{e}^{-\tilde{u}_2^2}.$$

The integration over $\tilde{\boldsymbol{u}}_2$ is given in eq. (G.3); using that we get

$$\begin{aligned}
L_1[\Phi(\tilde{\boldsymbol{u}}_1)] &= \frac{1}{\sqrt{\pi}}\left[\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(1+2\tilde{u}_1^2)}{2\tilde{u}_1}\,\mathrm{erf}(\tilde{u}_1)\right]\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\\
&= \frac{1}{\sqrt{\pi}}Q(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}. \qquad \text{(I.18)}
\end{aligned}$$

Next consider $L_2[\Phi(\tilde{\boldsymbol{u}}_1)]$. In eq. (I.15d) also, the integration over $\hat{\boldsymbol{k}}$ is trivial. Using eq. (G.1b),

$$\begin{aligned}
L_2[\Phi(\tilde{\boldsymbol{u}}_1)] &= \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \tilde{u}_{12}\, \hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\mathrm{e}^{-\tilde{u}_2^2}\\
&= \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\mathrm{e}^{-\tilde{u}_2^2}\,R. \qquad \text{(I.19)}
\end{aligned}$$

Substituting the values of $L_1'[\Phi(\tilde{\boldsymbol{u}}_1)]$, $L_2'[\Phi(\tilde{\boldsymbol{u}}_1)]$, $L_1[\Phi(\tilde{\boldsymbol{u}}_1)]$ and $L_2[\Phi(\tilde{\boldsymbol{u}}_1)]$ from eqs. (I.16)-(I.19) respectively, into eq. (I.14), we get

$$\tilde{\mathscr{L}}\left[\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}\right]=\frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\mathrm{e}^{-\tilde{u}_2^2}\left(\frac{2}{R}\,\mathrm{e}^{w^2}-R\right)$$
$$-\frac{1}{\sqrt{\pi}}Q(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}.$$

Hence eq. (I.13) can be written as

$$-\frac{1}{\sqrt{\pi}}\left[Q(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}+\frac{1}{\pi}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\mathrm{e}^{-\tilde{u}_2^2}\left(R-\frac{2}{R}\,\mathrm{e}^{w^2}\right)\right]$$
$$=\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}. \tag{I.20}$$

Next, we shall simplify the integral in the left-hand side of eq. (I.20). Let

$$I_c=\int \mathrm{d}\tilde{\boldsymbol{u}}_2\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\mathrm{e}^{-\tilde{u}_2^2}\left(R-\frac{2}{R}\,\mathrm{e}^{w^2}\right). \tag{I.21}$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2,\theta_2',\phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1\cdot\tilde{\boldsymbol{u}}_2=\tilde{u}_1\tilde{u}_2\cos\theta_2'$, i.e.,

$$I_c=\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\int_{\phi_2'=0}^{2\pi}\mathrm{d}\phi_2'\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\tilde{u}_{2i}\,\mathrm{e}^{-\tilde{u}_2^2}\left(R-\frac{2}{R}\,\mathrm{e}^{w^2}\right).$$

Note that only components of $\tilde{\boldsymbol{u}}_2$ are the functions of ϕ_2' (see Appendix H). The integration over ϕ_2' results into (cf. eq. (H.6))

$$I_c=\int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2'=0}^{\pi}\mathrm{d}\theta_2'\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\sin\theta_2'\,\hat{\Phi}_c(\tilde{u}_2)\left(\tilde{u}_2^2-\frac{5}{2}\right)\left(2\pi\frac{\tilde{u}_2}{\tilde{u}_1}\tilde{u}_{1i}\cos\theta_2'\right)\mathrm{e}^{-\tilde{u}_2^2}\left(R-\frac{2}{R}\,\mathrm{e}^{w^2}\right)$$
$$=\frac{2\pi}{\tilde{u}_1}\tilde{u}_{1i}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^3\left(\tilde{u}_2^2-\frac{5}{2}\right)\hat{\Phi}_c(\tilde{u}_2)\left\{\int_{\theta_2'=0}^{\pi}\sin\theta_2'\left(R-\frac{2}{R}\,\mathrm{e}^{w^2}\right)P_1(\cos\theta_2')\,\mathrm{d}\theta_2'\right\}\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2,$$

where $P_1(x)=x$ is the first order Legendre polynomial. Hence

$$I_c=\frac{2\pi}{\tilde{u}_1}\tilde{u}_{1i}\int_{\tilde{u}_2=0}^{\infty}\tilde{u}_2^3\left(\tilde{u}_2^2-\frac{5}{2}\right)\hat{\Phi}_c(\tilde{u}_2)A_1(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\,\mathrm{d}\tilde{u}_2.$$

Hence eq. (I.20) changes to

$$-\frac{1}{\sqrt{\pi}}\left[Q(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}+\frac{2}{\tilde{u}_1}\tilde{u}_{1i}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^3\left(\tilde{u}_2^2-\frac{5}{2}\right)\hat{\Phi}_c(\tilde{u}_2)A_1(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\right]$$
$$=\left(\tilde{u}_1^2-\frac{5}{2}\right)\tilde{u}_{1i}$$

or

$$-\frac{1}{\sqrt{\pi}}\left[Q(\tilde{u}_1)\hat{\Phi}_c(\tilde{u}_1)+\frac{2}{\tilde{u}_1\left(\tilde{u}_1^2-\frac{5}{2}\right)}\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^3\left(\tilde{u}_2^2-\frac{5}{2}\right)\hat{\Phi}_c(\tilde{u}_2)A_1(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\right]=1. \tag{I.22}$$

Eq. (I.22) is the Fredholm-form of eq. (I.13), where, in particular, $A_1(\tilde{u}_1, \tilde{u}_2)$ (for $\tilde{u}_1 > \tilde{u}_2$) is given by (Pekeris 1955):

$$A_1(\tilde{u}_1, \tilde{u}_2) = \frac{1}{\tilde{u}_1^2 \tilde{u}_2^2} \left[\frac{2}{15} \tilde{u}_2^5 - \frac{2}{3} \tilde{u}_2^3 \tilde{u}_1^2 - 4 \left\{ \tilde{u}_2 + \frac{\sqrt{\pi}}{2} (\tilde{u}_2^2 - 1)\, \mathrm{e}^{\tilde{u}_2^2} \operatorname{erf}(\tilde{u}_2) \right\} \right]. \tag{I.23}$$

The value of $A_1(\tilde{u}_1, \tilde{u}_2)$ for $\tilde{u}_1 < \tilde{u}_2$ is obtained from eq. (I.23) by interchanging $\tilde{u}_1$ and $\tilde{u}_2$.

Next, we shall change the operator $\tilde{\mathscr{L}}$ in eq. (3.29) into a Fredholm-type integral operator. Using eq. (3.35), eq. (3.29) can be written as

$$\tilde{\mathscr{L}} \left[\hat{\Phi}_e(\tilde{u}_1) \right] = h(\tilde{u}_1), \tag{I.24}$$

where $h(\tilde{u})$ is given in eq. (3.30). Using the definition of operator $\tilde{\mathscr{L}}$ (eq. (2.37)), the left-hand side of eq. (I.24) can be written as

$$\tilde{\mathscr{L}} \left[\hat{\Phi}_e(\tilde{u}_1) \right] = L_1'[\Phi(\tilde{\boldsymbol{u}}_1)] + L_2'[\Phi(\tilde{\boldsymbol{u}}_1)] - L_1[\Phi(\tilde{\boldsymbol{u}}_1)] - L_2[\Phi(\tilde{\boldsymbol{u}}_1)], \tag{I.25}$$

where

$$L_1'[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_e(\tilde{u}_1'), \tag{I.26a}$$

$$L_2'[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_e(\tilde{u}_2'), \tag{I.26b}$$

$$L_1[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_e(\tilde{u}_1), \tag{I.26c}$$

$$L_2[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{5/2}} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_e(\tilde{u}_2). \tag{I.26d}$$

First consider $L_1'[\Phi(\tilde{\boldsymbol{u}}_1)]$. Multiplying eq. (I.26a) by $\tilde{f}_0(\tilde{u}_1)\delta(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1)$ and integrating over $\tilde{\boldsymbol{u}}_1$ one obtains

$$\tilde{f}_0(\tilde{u}) L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12} > 0} \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2 \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_1')\, \delta(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1).$$

Since the operator $\tilde{\mathscr{L}}$ is defined for *elastic* collisions, one may replace $\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}$ by $-\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}'$, $\mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2$ by $\mathrm{d}\tilde{\boldsymbol{u}}_1' \mathrm{d}\tilde{\boldsymbol{u}}_2'$ and $\tilde{u}_1^2 + \tilde{u}_2^2$ by $\tilde{u}_1'^2 + \tilde{u}_2'^2$; following a similar procedure as above, we get

$$\tilde{f}_0(\tilde{u}) L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4} \int_{\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}' > 0} \mathrm{d}\tilde{\boldsymbol{u}}_1' \mathrm{d}\tilde{\boldsymbol{u}}_2' \mathrm{d}\hat{\boldsymbol{k}} (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}')\, \mathrm{e}^{-(\tilde{u}_1'^2 + \tilde{u}_2'^2)}\, \hat{\Phi}_e(\tilde{u}_1')\, \delta\Big(\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_1' + (\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_{12}') \hat{\boldsymbol{k}}\Big).$$

Since $\tilde{\boldsymbol{u}}_1'$ and $\tilde{\boldsymbol{u}}_2'$ are integration parameters, we can omit the prime signs to get

$$\tilde{f}_0(\tilde{u}) L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4} \int \mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-(\tilde{u}_1^2 + \tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_1) I_\delta^{(0)}.$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a rotated spherical coordinate system $(\tilde{u}_2, \theta_2'', \phi_2'')$ (defined as above), whose z-axis coincides with $\tilde{\boldsymbol{s}}$ so that $\tilde{\boldsymbol{s}} \cdot \tilde{\boldsymbol{u}}_2 = \tilde{s}\tilde{u}_2 \cos\theta_2''$. Hence

$$\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1 \int_{\tilde{u}_2=0}^{\infty}\int_{\theta_2''=0}^{\pi}\int_{\phi_2''=0}^{2\pi} \mathrm{d}\phi_2''\mathrm{d}\theta_2''\mathrm{d}\tilde{u}_2\, \tilde{u}_2^2 \sin\theta_2''\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_1) I_\delta^{(0)}$$
$$= \frac{2}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\, \mathrm{e}^{-\tilde{u}_1^2}\, \hat{\Phi}_e(\tilde{u}_1) \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2^2\, \mathrm{e}^{-\tilde{u}_2^2} \left(\int_{\theta_2''=0}^{\pi} \sin\theta_2'' I_\delta^{(0)} \mathrm{d}\theta_2''\right) \mathrm{d}\tilde{u}_2.$$

We integrate over θ_2'' and $\tilde{u}_2$ by following exactly similar procedure as above and get

$$\tilde{f}_0(\tilde{u})L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\, \mathrm{e}^{-\tilde{u}_1^2}\, \hat{\Phi}_e(\tilde{u}_1)\, \frac{1}{\tilde{s}}\, \mathrm{e}^{-\left(\frac{\tilde{\boldsymbol{s}}\cdot\tilde{\boldsymbol{u}}}{\tilde{s}}\right)^2}$$

or

$$L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\, \mathrm{e}^{-\tilde{u}_1^2}\, \hat{\Phi}_e(\tilde{u}_1)\, \frac{1}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1|}\, \mathrm{e}^{\tilde{u}^2-\left(\frac{(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)\cdot\tilde{\boldsymbol{u}}}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1|}\right)^2}.$$

Since $\tilde{\boldsymbol{u}}_1$ is the integration variable, we can change it to $\tilde{\boldsymbol{u}}_2$, i.e.,

$$L_1'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_e(\tilde{u}_2)\, \frac{1}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\, \mathrm{e}^{\tilde{u}^2-\left(\frac{(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\right)^2},$$

$$\therefore \quad L_1'[\Phi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_e(\tilde{u}_2)\, \frac{1}{|\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2|}\, \mathrm{e}^{\tilde{u}_1^2-\left(\frac{(\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}_1}{|\tilde{\boldsymbol{u}}_1-\tilde{\boldsymbol{u}}_2|}\right)^2}$$
$$= \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \hat{\Phi}_e(\tilde{u}_2)\, \mathrm{e}^{-\tilde{u}_2^2}\, \frac{1}{R}\, \mathrm{e}^{w^2}. \tag{I.27}$$

Next consider $L_2'[\Phi(\tilde{\boldsymbol{u}}_1)]$. Multiplying eq. (I.26b) by $\tilde{f}_0(\tilde{u}_1)\delta(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1)$ and integrating over $\tilde{\boldsymbol{u}}_1$ one obtains

$$\tilde{f}_0(\tilde{u})L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int_{\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12}>0} \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\mathrm{d}\hat{\boldsymbol{k}}(\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{u}}_{12})\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_2')\, \delta(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_1).$$

By using similar procedure as above, we get

$$\tilde{f}_0(\tilde{u})L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_2) I_\delta^{(0)}.$$

The integration over $\tilde{\boldsymbol{u}}_1$ is performed in a rotated spherical coordinate system $(\tilde{u}_1, \theta_1'', \phi_1'')$ (defined as above), whose z-axis coincides with $\tilde{\boldsymbol{t}}$ so that $\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}_1 = \tilde{t}\tilde{u}_1\cos\theta_1''$. Hence

$$\tilde{f}_0(\tilde{u})L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^4}\int \mathrm{d}\tilde{\boldsymbol{u}}_2 \int_{\tilde{u}_1=0}^{\infty}\int_{\theta_1''=0}^{\pi}\int_{\phi_1''=0}^{2\pi} \mathrm{d}\phi_1''\mathrm{d}\theta_1''\mathrm{d}\tilde{u}_1\, \tilde{u}_1^2 \sin\theta_1''\, \mathrm{e}^{-(\tilde{u}_1^2+\tilde{u}_2^2)}\, \hat{\Phi}_e(\tilde{u}_2) I_\delta^{(0)}$$
$$= \frac{2}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_e(\tilde{u}_2) \int_{\tilde{u}_1=0}^{\infty} \tilde{u}_1^2\, \mathrm{e}^{-\tilde{u}_1^2} \left(\int_{\theta_1''=0}^{\pi} \sin\theta_1'' I_\delta^{(0)} \mathrm{d}\theta_1''\right) \mathrm{d}\tilde{u}_1.$$

We integrate over θ_1'' and $\tilde{u}_1$ by following exactly similar procedure as above and get

$$\tilde{f}_0(\tilde{u})L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^3}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_e(\tilde{u}_2)\, \frac{1}{\tilde{t}}\, \mathrm{e}^{-\left(\frac{\tilde{\boldsymbol{t}}\cdot\tilde{\boldsymbol{u}}}{\tilde{t}}\right)^2}$$

or

$$L_2'[\Phi(\tilde{\boldsymbol{u}})] = \frac{1}{\pi^{3/2}}\int \mathrm{d}\tilde{\boldsymbol{u}}_2\, \mathrm{e}^{-\tilde{u}_2^2}\, \hat{\Phi}_e(\tilde{u}_2)\, \frac{1}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\, \mathrm{e}^{\tilde{u}^2-\left(\frac{(\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}}{|\tilde{\boldsymbol{u}}-\tilde{\boldsymbol{u}}_2|}\right)^2}.$$

$$\therefore \quad L_2'[\varPhi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \mathrm{e}^{-\tilde{u}_2^2} \, \hat{\varPhi}_e(\tilde{u}_2) \, \frac{1}{|\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2|} \, \mathrm{e}^{\tilde{u}_1^2 - \left(\frac{(\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2)\cdot\tilde{\boldsymbol{u}}_1}{|\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2|}\right)^2}$$
$$= \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \hat{\varPhi}_e(\tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} \, \frac{1}{R} \, \mathrm{e}^{w^2} \, . \tag{I.28}$$

Next consider $L_1[\varPhi(\tilde{\boldsymbol{u}}_1)]$. In eq. (I.26c), the integration over $\hat{\boldsymbol{k}}$ is trivial. Using eq. (G.1b),

$$L_1[\varPhi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \tilde{u}_{12} \, \hat{\varPhi}_e(\tilde{u}_1) \, \mathrm{e}^{-\tilde{u}_2^2} \, .$$

The integration over $\tilde{\boldsymbol{u}}_2$ is given in eq. (G.3); using that we get

$$L_1[\varPhi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\sqrt{\pi}} \left[\mathrm{e}^{-\tilde{u}_1^2} + \frac{\pi^{1/2}(1 + 2\tilde{u}_1^2)}{2\tilde{u}_1} \, \mathrm{erf}(\tilde{u}_1) \right] \hat{\varPhi}_e(\tilde{u}_1) = \frac{1}{\sqrt{\pi}} Q(\tilde{u}_1) \hat{\varPhi}_e(\tilde{u}_1), \tag{I.29}$$

where $Q(\tilde{u}) \equiv \mathrm{e}^{-\tilde{u}^2} + \frac{\sqrt{\pi}}{2} \left(2\tilde{u} + \frac{1}{\tilde{u}}\right) \mathrm{erf}(\tilde{u})$. Next consider $L_2[\varPhi(\tilde{\boldsymbol{u}}_1)]$. In eq. (I.26d) also, the integration over $\hat{\boldsymbol{k}}$ is trivial. Using eq. (G.1b),

$$L_2[\varPhi(\tilde{\boldsymbol{u}}_1)] = \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \tilde{u}_{12} \, \hat{\varPhi}_e(\tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} = \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \hat{\varPhi}_e(\tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} \, R. \tag{I.30}$$

Substituting the values of $L_1'[\varPhi(\tilde{\boldsymbol{u}}_1)]$, $L_2'[\varPhi(\tilde{\boldsymbol{u}}_1)]$, $L_1[\varPhi(\tilde{\boldsymbol{u}}_1)]$ and $L_2[\varPhi(\tilde{\boldsymbol{u}}_1)]$ from eqs. (I.27)-(I.30) respectively, into eq. (I.25), we get

$$\tilde{\mathscr{L}}\left[\hat{\varPhi}_e(\tilde{u}_1)\right] = \frac{1}{\pi^{3/2}} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \hat{\varPhi}_e(\tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} \left(\frac{2}{R} \, \mathrm{e}^{w^2} - R \right) - \frac{1}{\sqrt{\pi}} Q(\tilde{u}_1) \hat{\varPhi}_e(\tilde{u}_1).$$

Hence eq. (I.24) can be written as

$$-\frac{1}{\sqrt{\pi}} \left[Q(\tilde{u}_1) \hat{\varPhi}_e(\tilde{u}_1) + \frac{1}{\pi} \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \hat{\varPhi}_e(\tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} \left(R - \frac{2}{R} \, \mathrm{e}^{w^2} \right) \right] = h(\tilde{u}_1). \tag{I.31}$$

Next, we shall simplify the integral in the left-hand side of eq. (I.31). Let

$$I_e = \int \mathrm{d}\tilde{\boldsymbol{u}}_2 \, \hat{\varPhi}_e(\tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} \left(R - \frac{2}{R} \, \mathrm{e}^{w^2} \right). \tag{I.32}$$

The integration over $\tilde{\boldsymbol{u}}_2$ is performed in a (rotated) spherical coordinate system $(\tilde{u}_2, \theta_2', \phi_2')$ described in part (III) of $Q_{i_3}^{K\epsilon}$ such that $\tilde{\boldsymbol{u}}_1 \cdot \tilde{\boldsymbol{u}}_2 = \tilde{u}_1 \tilde{u}_2 \cos\theta_2'$, i.e.,

$$I_e = \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \int_{\phi_2'=0}^{2\pi} \mathrm{d}\phi_2' \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^2 \sin\theta_2' \hat{\varPhi}_e(\tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} \left(R - \frac{2}{R} \, \mathrm{e}^{w^2} \right)$$
$$= 2\pi \int_{\tilde{u}_2=0}^{\infty} \int_{\theta_2'=0}^{\pi} \mathrm{d}\theta_2' \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^2 \sin\theta_2' \hat{\varPhi}_e(\tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} \left(R - \frac{2}{R} \, \mathrm{e}^{w^2} \right)$$
$$= 2\pi \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2^2 \, \hat{\varPhi}_e(\tilde{u}_2) \left\{ \int_{\theta_2'=0}^{\pi} \sin\theta_2' \left(R - \frac{2}{R} \, \mathrm{e}^{w^2} \right) P_0(\cos\theta_2') \, \mathrm{d}\theta_2' \right\} \mathrm{e}^{-\tilde{u}_2^2} \, \mathrm{d}\tilde{u}_2,$$

where $P_0(x) = 1$ is the zeroth order Legendre polynomial. Hence

$$I_e = 2\pi \int_{\tilde{u}_2=0}^{\infty} \tilde{u}_2^2 \, \hat{\Phi}_e(\tilde{u}_2) A_0(\tilde{u}_1, \tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} \, \mathrm{d}\tilde{u}_2.$$

Hence eq. (I.31) changes to

$$-\frac{1}{\sqrt{\pi}} \left[Q(\tilde{u}_1) \hat{\Phi}_e(\tilde{u}_1) + 2 \int_{\tilde{u}_2=0}^{\infty} \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^2 \, \hat{\Phi}_e(\tilde{u}_2) A_0(\tilde{u}_1, \tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} \right] = h(\tilde{u}_1). \tag{I.33}$$

Eq. (I.33) is the Fredholm-form of eq. (I.24), where, in particular $A_0(\tilde{u}_1, \tilde{u}_2)$ (for $\tilde{u}_1 > \tilde{u}_2$) is given by (Pekeris 1955):

$$A_0(\tilde{u}_1, \tilde{u}_2) = \frac{1}{\tilde{u}_1 \tilde{u}_2} \left[\frac{2}{3} \tilde{u}_2^3 + 2\tilde{u}_1^2 \tilde{u}_2 - 2\sqrt{\pi} \, \mathrm{e}^{\tilde{u}_2^2} \, \mathrm{erf}(\tilde{u}_2) \right]. \tag{I.34}$$

The value of $A_0(\tilde{u}_1, \tilde{u}_2)$ for $\tilde{u}_1 < \tilde{u}_2$ is obtained from eq. (I.23) by interchanging $\tilde{u}_1$ and $\tilde{u}_2$.

Since the form of $\bar{\eta}(\tilde{u})$ is similar to $\hat{\Phi}_e(\tilde{u})$, the Fredholm-form of equation $\tilde{\mathscr{L}}(\bar{\eta}) = \bar{\chi}$ can be obtained by following a similar procedure.

I.2 To show that the unknown functions are even

Consider eq. (I.11). Replacing $\tilde{u}_1$ by $-\tilde{u}_1$ in eq. (I.11), it changes to

$$-\frac{1}{\sqrt{\pi}} \left[Q(-\tilde{u}_1) \hat{\Phi}_v(-\tilde{u}_1) + \frac{2}{\tilde{u}_1^2} \int_0^{\infty} \mathrm{d}\tilde{u}_2 \, \tilde{u}_2^4 \, \hat{\Phi}_v(\tilde{u}_2) A_2(-\tilde{u}_1, \tilde{u}_2) \, \mathrm{e}^{-\tilde{u}_2^2} \right] = 1. \tag{I.35}$$

Now, from the definition of $Q(\tilde{u})$ and the fact that error function is an odd funtion we see that

$$Q(-\tilde{u}) = \mathrm{e}^{-\tilde{u}^2} + \frac{\sqrt{\pi}}{2} \left(-2\tilde{u} + \frac{1}{(-\tilde{u})} \right) \mathrm{erf}(-\tilde{u}) = \mathrm{e}^{-\tilde{u}^2} + \frac{\sqrt{\pi}}{2} \left(2\tilde{u} + \frac{1}{\tilde{u}} \right) \mathrm{erf}(\tilde{u}) = Q(\tilde{u}), \tag{I.36}$$

i.e., $Q(\tilde{u})$ is an even function. Next, from the definition of A_n given in eq. (I.10),

$$\begin{aligned} A_2(\tilde{u}_1, \tilde{u}_2) &= \int_{\theta_2'=0}^{\pi} \sin\theta_2' \left(R - \frac{2}{R} \mathrm{e}^{w^2} \right) P_2(\cos\theta_2') \, \mathrm{d}\theta_2' \\ &= \int_{\theta_2'=0}^{\pi} \sin\theta_2' \Bigg\{ \sqrt{(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)} - \frac{2}{\sqrt{(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2)}} \\ &\quad \times \exp\left(\frac{\tilde{u}_1^2\tilde{u}_2^2\sin^2\theta_2'}{\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2\cos\theta_2' + \tilde{u}_2^2} \right) \Bigg\} P_2(\cos\theta_2') \, \mathrm{d}\theta_2'. \end{aligned}$$

Let $\cos\theta_2' = y$. Hence

$$\begin{aligned} A_2(\tilde{u}_1, \tilde{u}_2) = \int_{y=-1}^{1} \Bigg\{ & \sqrt{(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 y + \tilde{u}_2^2)} - \frac{2}{\sqrt{(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 y + \tilde{u}_2^2)}} \\ & \times \exp\left(\frac{\tilde{u}_1^2\tilde{u}_2^2(1-y^2)}{\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 y + \tilde{u}_2^2} \right) \Bigg\} P_2(y) \, \mathrm{d}y. \end{aligned}$$

Therefore

$$A_2(-\tilde{u}_1, \tilde{u}_2) = \int_{y=-1}^{1} \left\{ \sqrt{(\tilde{u}_1^2 + 2\tilde{u}_1\tilde{u}_2 y + \tilde{u}_2^2)} - \frac{2}{\sqrt{(\tilde{u}_1^2 + 2\tilde{u}_1\tilde{u}_2 y + \tilde{u}_2^2)}} \times \exp\left(\frac{\tilde{u}_1^2\tilde{u}_2^2(1-y^2)}{\tilde{u}_1^2 + 2\tilde{u}_1\tilde{u}_2 y + \tilde{u}_2^2}\right)\right\} P_2(y)\,\mathrm{d}y.$$

Substituting $y = -x$ in the above equation and using the fact that $P_2(-x) = P_2(x)$,

$$\begin{aligned} A_2(-\tilde{u}_1, \tilde{u}_2) &= \int_{x=1}^{-1} \left\{ \sqrt{(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 x + \tilde{u}_2^2)} - \frac{2}{\sqrt{(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 x + \tilde{u}_2^2)}} \times \exp\left(\frac{\tilde{u}_1^2\tilde{u}_2^2(1-x^2)}{\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 x + \tilde{u}_2^2}\right)\right\} P_2(x)\,(-\mathrm{d}x) \\ &= \int_{x=-1}^{1} \left\{ \sqrt{(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 x + \tilde{u}_2^2)} - \frac{2}{\sqrt{(\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 x + \tilde{u}_2^2)}} \times \exp\left(\frac{\tilde{u}_1^2\tilde{u}_2^2(1-x^2)}{\tilde{u}_1^2 - 2\tilde{u}_1\tilde{u}_2 x + \tilde{u}_2^2}\right)\right\} P_2(x)\,\mathrm{d}x = A_2(\tilde{u}_1, \tilde{u}_2), \end{aligned}$$

i.e., $A_2(\tilde{u}_1, \tilde{u}_2)$ is an even function in $\tilde{u}_1$. Using these, eq. (I.35) changes to

$$-\frac{1}{\sqrt{\pi}}\left[Q(\tilde{u}_1)\hat{\Phi}_v(-\tilde{u}_1) + \frac{2}{\tilde{u}_1^2}\int_0^\infty \mathrm{d}\tilde{u}_2\,\tilde{u}_2^4\,\hat{\Phi}_v(\tilde{u}_2)A_2(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\right] = 1. \tag{I.37}$$

Comparing eqs. (I.11) and (I.37), we see that $\hat{\Phi}_v(-\tilde{u}_1) = \hat{\Phi}_v(\tilde{u}_1)$, i.e., $\hat{\Phi}_v(\tilde{u}_1)$ is an even function in $\tilde{u}_1$.

Next, consider eq. (I.22). Replacing $\tilde{u}_1$ by $-\tilde{u}_1$ in eq. (I.22), it changes to

$$-\frac{1}{\sqrt{\pi}}\left[Q(-\tilde{u}_1)\hat{\Phi}_c(-\tilde{u}_1) + \frac{2}{-\tilde{u}_1\left(\tilde{u}_1^2 - \frac{5}{2}\right)}\int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\,\tilde{u}_2^3\left(\tilde{u}_2^2 - \frac{5}{2}\right)\hat{\Phi}_c(\tilde{u}_2)A_1(-\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\right] = 1. \tag{I.38}$$

As above $Q(-\tilde{u}) = Q(\tilde{u})$ and from the definition of A_n given in eq. (I.10),

$$A_1(\tilde{u}_1, \tilde{u}_2) = \int_{\theta_2'=0}^{\pi} \sin\theta_2' \left(R - \frac{2}{R}\,\mathrm{e}^{w^2}\right) P_1(\cos\theta_2')\,\mathrm{d}\theta_2'.$$

Following exactly similar procedure as above and noting that $P_1(-x) = -P_1(x)$, we get

$$A_1(-\tilde{u}_1, \tilde{u}_2) = -A_1(\tilde{u}_1, \tilde{u}_2),$$

i.e., $A_1(\tilde{u}_1, \tilde{u}_2)$ is an odd function in $\tilde{u}_1$. Using these arguments, eq. (I.38) changes to

$$-\frac{1}{\sqrt{\pi}}\left[Q(\tilde{u}_1)\hat{\Phi}_c(-\tilde{u}_1) + \frac{2}{\tilde{u}_1\left(\tilde{u}_1^2 - \frac{5}{2}\right)}\int_{\tilde{u}_2=0}^\infty \mathrm{d}\tilde{u}_2\,\tilde{u}_2^3\left(\tilde{u}_2^2 - \frac{5}{2}\right)\hat{\Phi}_c(\tilde{u}_2)A_1(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\right] = 1. \tag{I.39}$$

Comparing eqs. (I.22) and (I.39), we see that $\hat{\Phi}_c(-\tilde{u}_1) = \hat{\Phi}_c(\tilde{u}_1)$, i.e., $\hat{\Phi}_c(\tilde{u}_1)$ is an even function in $\tilde{u}_1$.

Next, consider eq. (I.33). Replacing $\tilde{u}_1$ by $-\tilde{u}_1$ in eq. (I.33), it changes to

$$-\frac{1}{\sqrt{\pi}}\left[Q(-\tilde{u}_1)\hat{\Phi}_e(-\tilde{u}_1)+2\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\,\hat{\Phi}_e(\tilde{u}_2)A_0(-\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\right]=h(-\tilde{u}_1). \tag{I.40}$$

As above $Q(-\tilde{u})=Q(\tilde{u})$ and from the definition of A_n given in eq. (I.10),

$$A_0(\tilde{u}_1,\tilde{u}_2)=\int_{\theta_2'=0}^{\pi}\sin\theta_2'\left(R-\frac{2}{R}\,\mathrm{e}^{w^2}\right)P_0(\cos\theta_2')\,\mathrm{d}\theta_2'.$$

Following exactly similar procedure as above and noting that $P_0(-x)=P_0(x)$, we get

$$A_0(-\tilde{u}_1,\tilde{u}_2)=A_0(\tilde{u}_1,\tilde{u}_2).$$

i.e., $A_0(\tilde{u}_1,\tilde{u}_2)$ is an even function in $\tilde{u}_1$. Also, from eq. (3.30),

$$h(-\tilde{u})=-\left[\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{2}{3}\tilde{u}^2-1\right)+\frac{(3-2\tilde{u}^2)}{8\pi^{1/2}}\,\mathrm{e}^{-\tilde{u}^2}+\frac{(5+4\tilde{u}^2-4\tilde{u}^4)\,\mathrm{erf}(-\tilde{u})}{16(-\tilde{u})}\right]$$

and since $\mathrm{erf}(x)$ is an odd function, the above equation changes to

$$h(-\tilde{u})=-\left[\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{2}{3}\tilde{u}^2-1\right)+\frac{(3-2\tilde{u}^2)}{8\pi^{1/2}}\,\mathrm{e}^{-\tilde{u}^2}+\frac{(5+4\tilde{u}^2-4\tilde{u}^4)\,\mathrm{erf}(\tilde{u})}{16\tilde{u}}\right]=h(\tilde{u}),$$

i.e., $h(\tilde{u})$ is also an even function. Using these, eq. (I.40) changes to

$$-\frac{1}{\sqrt{\pi}}\left[Q(\tilde{u}_1)\hat{\Phi}_e(-\tilde{u}_1)+2\int_{\tilde{u}_2=0}^{\infty}\mathrm{d}\tilde{u}_2\,\tilde{u}_2^2\,\hat{\Phi}_e(\tilde{u}_2)A_0(\tilde{u}_1,\tilde{u}_2)\,\mathrm{e}^{-\tilde{u}_2^2}\right]=h(\tilde{u}_1). \tag{I.41}$$

Comparing eqs. (I.33) and (I.41), we see that $\hat{\Phi}_e(-\tilde{u}_1)=\hat{\Phi}_e(\tilde{u}_1)$, i.e., $\hat{\Phi}_e(\tilde{u}_1)$ is an even function in $\tilde{u}_1$.

Since the form of $\bar{\eta}(\tilde{u})$ is similar to $\hat{\Phi}_e(\tilde{u})$, one can show by following a similar procedure that $\bar{\eta}(\tilde{u})$ is also an even function.

Appendix J

Jacobian of Transformations

In the component form eqns. (2.1a) and (2.1b) can be written as follows:

$$
\begin{aligned}
v_{1x} &= v'_{1x} - k_x\eta\left\{(v'_{1x}-v'_{2x})k_x + (v'_{1y}-v'_{2y})k_y + (v'_{1z}-v'_{2z})k_z\right\}\\
v_{1y} &= v'_{1y} - k_y\eta\left\{(v'_{1x}-v'_{2x})k_x + (v'_{1y}-v'_{2y})k_y + (v'_{1z}-v'_{2z})k_z\right\}\\
v_{1z} &= v'_{1z} - k_z\eta\left\{(v'_{1x}-v'_{2x})k_x + (v'_{1y}-v'_{2y})k_y + (v'_{1z}-v'_{2z})k_z\right\}\\
v_{2x} &= v'_{2x} + k_x\eta\left\{(v'_{1x}-v'_{2x})k_x + (v'_{1y}-v'_{2y})k_y + (v'_{1z}-v'_{2z})k_z\right\}\\
v_{2y} &= v'_{2y} + k_y\eta\left\{(v'_{1x}-v'_{2x})k_x + (v'_{1y}-v'_{2y})k_y + (v'_{1z}-v'_{2z})k_z\right\}\\
v_{2z} &= v'_{2z} + k_z\eta\left\{(v'_{1x}-v'_{2x})k_x + (v'_{1y}-v'_{2y})k_y + (v'_{1z}-v'_{2z})k_z\right\}
\end{aligned}
$$

where $\eta = \frac{1+e}{2}$. The relation between $\mathrm{d}\boldsymbol{v}_1\mathrm{d}\boldsymbol{v}_2$ and $\mathrm{d}\boldsymbol{v}'_1\mathrm{d}\boldsymbol{v}'_2$ is given by

$$\mathrm{d}\boldsymbol{v}_1\mathrm{d}\boldsymbol{v}_2 = |J|\,\mathrm{d}\boldsymbol{v}'_1\mathrm{d}\boldsymbol{v}'_2, \tag{J.1}$$

where

$$J = \frac{\partial(\boldsymbol{v}_1,\boldsymbol{v}_2)}{\partial(\boldsymbol{v}'_1,\boldsymbol{v}'_2)} = \begin{vmatrix}
\frac{\partial v_{1x}}{\partial v'_{1x}} & \frac{\partial v_{1x}}{\partial v'_{1y}} & \frac{\partial v_{1x}}{\partial v'_{1z}} & \frac{\partial v_{1x}}{\partial v'_{2x}} & \frac{\partial v_{1x}}{\partial v'_{2y}} & \frac{\partial v_{1x}}{\partial v'_{2z}}\\
\frac{\partial v_{1y}}{\partial v'_{1x}} & \frac{\partial v_{1y}}{\partial v'_{1y}} & \frac{\partial v_{1y}}{\partial v'_{1z}} & \frac{\partial v_{1y}}{\partial v'_{2x}} & \frac{\partial v_{1y}}{\partial v'_{2y}} & \frac{\partial v_{1y}}{\partial v'_{2z}}\\
\frac{\partial v_{1z}}{\partial v'_{1x}} & \frac{\partial v_{1z}}{\partial v'_{1y}} & \frac{\partial v_{1z}}{\partial v'_{1z}} & \frac{\partial v_{1z}}{\partial v'_{2x}} & \frac{\partial v_{1z}}{\partial v'_{2y}} & \frac{\partial v_{1z}}{\partial v'_{2z}}\\
\frac{\partial v_{2x}}{\partial v'_{1x}} & \frac{\partial v_{2x}}{\partial v'_{1y}} & \frac{\partial v_{2x}}{\partial v'_{1z}} & \frac{\partial v_{2x}}{\partial v'_{2x}} & \frac{\partial v_{2x}}{\partial v'_{2y}} & \frac{\partial v_{2x}}{\partial v'_{2z}}\\
\frac{\partial v_{2y}}{\partial v'_{1x}} & \frac{\partial v_{2y}}{\partial v'_{1y}} & \frac{\partial v_{2y}}{\partial v'_{1z}} & \frac{\partial v_{2y}}{\partial v'_{2x}} & \frac{\partial v_{2y}}{\partial v'_{2y}} & \frac{\partial v_{2y}}{\partial v'_{2z}}\\
\frac{\partial v_{2z}}{\partial v'_{1x}} & \frac{\partial v_{2z}}{\partial v'_{1y}} & \frac{\partial v_{2z}}{\partial v'_{1z}} & \frac{\partial v_{2z}}{\partial v'_{2x}} & \frac{\partial v_{2z}}{\partial v'_{2y}} & \frac{\partial v_{2z}}{\partial v'_{2z}}
\end{vmatrix}$$

or

$$J = \begin{vmatrix}
1-k_xk_x\eta & -k_xk_y\eta & -k_xk_z\eta & k_xk_x\eta & k_xk_y\eta & k_xk_z\eta\\
-k_yk_x\eta & 1-k_yk_y\eta & -k_yk_z\eta & k_yk_x\eta & k_yk_y\eta & k_yk_z\eta\\
-k_zk_x\eta & -k_zk_y\eta & 1-k_zk_z\eta & k_zk_x\eta & k_zk_y\eta & k_zk_z\eta\\
k_xk_x\eta & k_xk_y\eta & k_xk_z\eta & 1-k_xk_x\eta & -k_xk_y\eta & -k_xk_z\eta\\
k_yk_x\eta & k_yk_y\eta & k_yk_z\eta & -k_yk_x\eta & 1-k_yk_y\eta & -k_yk_z\eta\\
k_zk_x\eta & k_zk_y\eta & k_zk_z\eta & -k_zk_x\eta & -k_zk_y\eta & 1-k_zk_z\eta
\end{vmatrix}$$

$R_1 \to R_1 + R_4$, $R_2 \to R_2 + R_5$, $R_3 \to R_3 + R_6$ imply that

$$J = \begin{vmatrix}
1 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 1\\
k_xk_x\eta & k_xk_y\eta & k_xk_z\eta & 1-k_xk_x\eta & -k_xk_y\eta & -k_xk_z\eta\\
k_yk_x\eta & k_yk_y\eta & k_yk_z\eta & -k_yk_x\eta & 1-k_yk_y\eta & -k_yk_z\eta\\
k_zk_x\eta & k_zk_y\eta & k_zk_z\eta & -k_zk_x\eta & -k_zk_y\eta & 1-k_zk_z\eta
\end{vmatrix}$$

$C_4 \to C_4 - C_1$, $C_5 \to C_5 - C_2$, $C_6 \to C_6 - C_3$ imply that

$$J = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ k_xk_x\eta & k_xk_y\eta & k_xk_z\eta & 1-2k_xk_x\eta & -2k_xk_y\eta & -2k_xk_z\eta \\ k_yk_x\eta & k_yk_y\eta & k_yk_z\eta & -2k_yk_x\eta & 1-2k_yk_y\eta & -2k_yk_z\eta \\ k_zk_x\eta & k_zk_y\eta & k_zk_z\eta & -2k_zk_x\eta & -2k_zk_y\eta & 1-2k_zk_z\eta \end{vmatrix}$$

or

$$J = \begin{vmatrix} 1-2k_xk_x\eta & -2k_xk_y\eta & -2k_xk_z\eta \\ -2k_yk_x\eta & 1-2k_yk_y\eta & -2k_yk_z\eta \\ -2k_zk_x\eta & -2k_zk_y\eta & 1-2k_zk_z\eta \end{vmatrix}$$

or

$$\begin{aligned} J &= (1-2k_xk_x\eta)(1-2k_yk_y\eta-2k_zk_z\eta)+2k_xk_y\eta(-2k_yk_x\eta)-2k_xk_z\eta(2k_zk_x\eta) \\ &= 1-2k_xk_x\eta-2k_yk_y\eta-2k_zk_z\eta = 1-2\eta(k_x^2+k_y^2+k_z^2) \\ &= 1-2\times\left(\frac{1+e}{2}\right)\times 1 = -e. \end{aligned}$$

Hence from eqn. (J.1),

$$\mathrm{d}\boldsymbol{v}_1\mathrm{d}\boldsymbol{v}_2 = e\,\mathrm{d}\boldsymbol{v}_1'\mathrm{d}\boldsymbol{v}_2' \tag{J.2}$$

and hence for elastic case (i.e., $e = 1$)

$$\mathrm{d}\boldsymbol{v}_1\mathrm{d}\boldsymbol{v}_2 = \mathrm{d}\boldsymbol{v}_1'\mathrm{d}\boldsymbol{v}_2'. \tag{J.3}$$

Let us find out the Jacobian of another velocity transformation, through which velocities $(\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2)$ transform to $(\boldsymbol{g}_1, \boldsymbol{g}_2)$ such that,

$$\begin{aligned} \boldsymbol{g}_1 &= \tilde{\boldsymbol{u}}_1 + \tilde{\boldsymbol{u}}_2 \\ \boldsymbol{g}_2 &= \tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2. \end{aligned}$$

The relation between $\mathrm{d}\boldsymbol{g}_1\mathrm{d}\boldsymbol{g}_2$ and $\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2$ is given by

$$\mathrm{d}\boldsymbol{g}_1\mathrm{d}\boldsymbol{g}_2 = |J|\,\mathrm{d}\tilde{\boldsymbol{u}}_1\mathrm{d}\tilde{\boldsymbol{u}}_2, \tag{J.4}$$

where

$$J = \frac{\partial(\boldsymbol{g}_1, \boldsymbol{g}_2)}{\partial(\tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2)} = \begin{vmatrix} \frac{\partial g_{1x}}{\partial \tilde{u}_{1x}} & \frac{\partial g_{1x}}{\partial \tilde{u}_{1y}} & \frac{\partial g_{1x}}{\partial \tilde{u}_{1z}} & \frac{\partial g_{1x}}{\partial \tilde{u}_{2x}} & \frac{\partial g_{1x}}{\partial \tilde{u}_{2y}} & \frac{\partial g_{1x}}{\partial \tilde{u}_{2z}} \\ \frac{\partial g_{1y}}{\partial \tilde{u}_{1x}} & \frac{\partial g_{1y}}{\partial \tilde{u}_{1y}} & \frac{\partial g_{1y}}{\partial \tilde{u}_{1z}} & \frac{\partial g_{1y}}{\partial \tilde{u}_{2x}} & \frac{\partial g_{1y}}{\partial \tilde{u}_{2y}} & \frac{\partial g_{1y}}{\partial \tilde{u}_{2z}} \\ \frac{\partial g_{1z}}{\partial \tilde{u}_{1x}} & \frac{\partial g_{1z}}{\partial \tilde{u}_{1y}} & \frac{\partial g_{1z}}{\partial \tilde{u}_{1z}} & \frac{\partial g_{1z}}{\partial \tilde{u}_{2x}} & \frac{\partial g_{1z}}{\partial \tilde{u}_{2y}} & \frac{\partial g_{1z}}{\partial \tilde{u}_{2z}} \\ \frac{\partial g_{2x}}{\partial \tilde{u}_{1x}} & \frac{\partial g_{2x}}{\partial \tilde{u}_{1y}} & \frac{\partial g_{2x}}{\partial \tilde{u}_{1z}} & \frac{\partial g_{2x}}{\partial \tilde{u}_{2x}} & \frac{\partial g_{2x}}{\partial \tilde{u}_{2y}} & \frac{\partial g_{2x}}{\partial \tilde{u}_{2z}} \\ \frac{\partial g_{2y}}{\partial \tilde{u}_{1x}} & \frac{\partial g_{2y}}{\partial \tilde{u}_{1y}} & \frac{\partial g_{2y}}{\partial \tilde{u}_{1z}} & \frac{\partial g_{2y}}{\partial \tilde{u}_{2x}} & \frac{\partial g_{2y}}{\partial \tilde{u}_{2y}} & \frac{\partial g_{2y}}{\partial \tilde{u}_{2z}} \\ \frac{\partial g_{2z}}{\partial \tilde{u}_{1x}} & \frac{\partial g_{2z}}{\partial \tilde{u}_{1y}} & \frac{\partial g_{2z}}{\partial \tilde{u}_{1z}} & \frac{\partial g_{2z}}{\partial \tilde{u}_{2x}} & \frac{\partial g_{2z}}{\partial \tilde{u}_{2y}} & \frac{\partial g_{2z}}{\partial \tilde{u}_{2z}} \end{vmatrix}$$

or

$$J = \begin{vmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{vmatrix}$$

$C_4 \to C_4 - C_1$, $C_5 \to C_5 - C_2$, $C_6 \to C_6 - C_3$ imply that

$$J = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{vmatrix}$$

or

$$J = (-2)^3 = -8.$$

Hence from eqn. (J.4),

$$\mathrm{d}\boldsymbol{g}_1 \mathrm{d}\boldsymbol{g}_2 = 8\,\mathrm{d}\tilde{\boldsymbol{u}}_1 \mathrm{d}\tilde{\boldsymbol{u}}_2. \tag{J.5}$$

References

Abramowitz, M. & Stegun, I. A. 1965 *Handbook of Mathematical Functions*. New York: Dover.

Alam, M., Chikkadi, V. & Gupta, V. K. 2009 Density waves and the effect of wall roughness in granular poiseuille flow: Simulation and linear stability. *Eur. Phys. J. ST* **179**, 69–90.

Alam, M. & Chikkadi, V. K. 2010 Velocity distribution function and correlations in a granular poiseuille flow. *J. Fluid Mech.* **653**, 175–219.

Alam, M., Trujillo, L. & Herrmann, H. J. 2006 Hydrodynamic theory for reverse brazil nut segregation and the non-monotonic ascension dynamics. *J. Stat. Phys.* **124**, 587–623.

Ansumali, S., Karlin, I. V. & Öttinger, H. C. 2003 Minimal entropic kinetic models for hydrodynamics. *Europhys. Lett.* **63** (6), 798–804.

Bar-Lev, O. 2005 Kinetic and hydrodynamic theory of granular gases. PhD thesis, Tel-Aviv University.

Bhatnagar, P. L., Gross, E. P. & Krook, M. 1954 A model for collision processes in gases. i. small amplitude processes in charged and neutral one-component systems. *Phys. Rev. E* **94** (3), 511–525.

Brilliantov, N. V. & Pöschel, T. 2004 *Kinetic theory of granular gases*. Oxford University Press.

Campbell, C. S. 1990 Rapid granular flows. *Ann. Rev. Fluid Mech.* **22**, 57–92.

Cercignani, C. 1969 *Mathematical Methods in Kinetic Theory*. New York: Plenum.

Chapman, S. & Cowling, T. G. 1970 *The Mathematical Theory of Non-Uniform Gases*. Cambridge University Press.

Corwin, E. I., Jaeger, H. M. & Nagel, S. R. 2005 Structural signature of jamming in granular media. *Nature* **435**, 1075–1078.

Eshuis, P., van der Meer, D., Alam, M., van Gerner, H. J., van der Weele, K. & Lohse, D. 2010 Onset of convection in strongly shaken granular matter. *Phys. Rev. Lett.* **104** (3), 038001.

Ferziger, J. H. & Kaper, H. G. 1972 *Mathematical Theory of transport processes in gases*. Amsterdam, London: North-Holland Pub. Co.

Goldhirsch, I. 2003 Rapid granular flows. *Ann. Rev. Fluid Mech.* **35**, 267–293.

Goldhirsch, I. & Zanetti, G. 1993 Clustering instability in dissipative gases. *Phys. Rev. Lett.* **70** (11), 1619–1622.

Goldshtein, A. & Shapiro, M. 1995 Mechanics of collisional motion of granular materials. part 1. general hydrodynamic equations. *J. Fluid Mech.* **282**, 75–114.

Grad, H. 1949 On the kinetic theory of rarefied gases. *Comm. Pure and Appl. Math.* **2**, 331–407.

Haff, P. K. 1983 Grain flow as a fluid-mechanical phenomenon. *J. Fluid Mech.* **134**, 401–430.

Harris, S. 2004 *An Introduction to the theory of the Boltzmann equation*. New York: Dover.

Herminghaus, S. 2005 Dynamics of wet granular matter. *Advances in Physics* **54** (3), 221–261.

Jaeger, H. M., Nagel, S. R. & Behringer, R. P. 1996 Granular solids, liquids, and gases. *Rev. Mod. Phys.* **68** (4), 1259–1273.

Jenkins, J. T. & Richman, M. W. 1985*a* Grad's 13-moment system for a dense gas of inelastic spheres. *Arch. Rat. Mech. Anal.* **87**, 355–377.

Jenkins, J. T. & Richman, M. W. 1985*b* Kinetic theory for plane flows of a dense gas of identical, rough, inelastic, circular disks. *Phys. Fluids* **28**, 3485–3494.

Kogan, M. N. 1969 *Rarefied Gas Dynamics*. New York: Plenum.

Kudrolli, A., Wolpert, M. & Gollub, J. P. 1997 Cluster formation due to collisions in granular material. *Phys. Rev. Lett.* **78**, 1383–1386.

Lun, C. K. K., Savage, S. B., Jeffrey, D. J. & Chepurniy, N. 1984 Kinetic theories for granular flow: inelastic particles in couette flow and slightly inelastic particles in a general flowfield. *J. Fluid Mech.* **140**, 223–256.

Ottino, J. M. & Khakhar, D. V. 2000 Mixing and segregation of granular materials. *Ann. Rev. Fluid Mech.* **32**, 55–91.

Pekeris, C. L. 1955 Solution of the boltzmann-hilbert integral equation. *Proc. Natl. Acad. Sci.* **41**, 661–664.

Ramírez, R., Risso, D., Soto, R. & Cordero, P. 2000 Hydrodynamic theory for granular gases. *Phys. Rev. E* **62** (2), 2521–2530.

Rao, K. K. & Nott, P. R. 2008 *An Introduction to Granular Flow*. Cambridge University Press.

Richard, P., Nicodemi, M., Delannay, R., Ribière, P. & Bideau, D. 2005 Slow relaxation and compaction of granular systems. *Nature Materials* **4**, 121–128.

Santos, A., Montanero, J. M., Dufty, J. W. & Brey, J. J. 1998 Kinetic model for the hard-sphere fluid and solid. *Phys. Rev. E* **57** (2), 1644–1660.

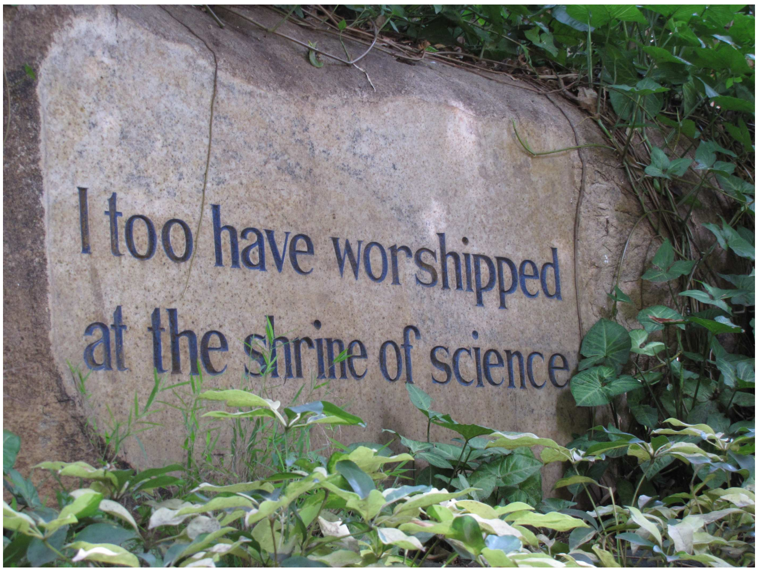
Sela, N. & Goldhirsch, I. 1998 Hydrodynamic equations for rapid flows of smooth inelastic spheres, to burnett order. *J. Fluid Mech.* **361**, 41–74.

Sela, N., Goldhirsch, I. & Noskowicz, S. H. 1996 Kinetic theoretical study of a simply sheared two-dimensional granular gas to burnett order. *Phys. Fluids* **8**, 2337–2353.

Tij, M. & Santos, A. 2004 Poiseuille flow in a heated granular gas. *J. Stat. Phys.* **117**, 901–928.

Trujillo, L., Alam, M. & Herrmann, H. J. 2003 Segregation in a fluidized binary granular mixture: Competition between buoyancy and geometric forces. *Europhys. Lett.* **64**, 190–196.

Umbanhowar, P. B., Melo, F. & Swinney, H. L. 1996 Localized excitations in a vertically vibrated granular layer. *Nature* **382**, 793–796.


I too have worshipped
at the shrine of science